Bartholdi Zeta Functions of Fractal Graphs

Iwao Sato

Oyama National College of Technology,
Oyama, Tochigi 323-0806, Japan

e-mail: isato@oyama-ct.ac.jp

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Abstract

Recently, Guido, Isola and Lapidus [11] defined the Ihara zeta function of a fractal graph, and gave a determinant expression of it. We define the Bartholdi zeta function of a fractal graph, and present its determinant expression.

1 Introduction

Zeta functions of graphs started from p-adic Selberg zeta functions of discrete groups by Ihara [14]. At the beginning, Serre [20] pointed out that the Ihara zeta function is the zeta function of a regular graph. In [14], Ihara showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada [22,23]. Hashimoto [13] treated multivariable zeta functions of bipartite graphs. Bass [3] generalized Ihara's result on zeta functions of regular graphs to irregular graphs. Various proofs of Bass' theorem were given by Stark and Terras [21], Kotani and Sunada [15] and Foata and Zeilberger [5].

Bartholdi [2] extended a result by Grigorchuk [7] relating cogrowth and spectral radius of random walks, and gave an explicit formula determining the number of bumps on paths in a graph. Furthermore, he presented the "circuit series" of the free products and the direct products of graphs, and obtained a generalized form "Bartholdi zeta function" of the Ihara(-Selberg) zeta function.

All graphs in this paper are assumed to be simple. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $R(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$ be the set of oriented edges (or arcs) $(u,v),(v,u)$ directed oppositely for each edge uv of G. For $e=(u,v) \in R(G)$, $u=o(e)$ and $v=t(e)$ are called the *origin* and the *terminal* of e, respectively. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

A *path P of length n* in G is a sequence $P=(e_1,\cdots,e_n)$ of n arcs such that $e_i \in R(G)$, $t(e_i)=o(e_{i+1})(1 \leq i \leq n-1)$. If $e_i=(v_{i-1},v_i)$, $1 \leq i \leq n$, then we also denote P by $(v_0,v_1,\cdots,v_n)$. Set $|P|=n$, $o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called an

$(o(P), t(P))$-*path*. A (v, w)-path is called a v-*closed path* if $v = w$. The *inverse* of a closed path $C = (e_1, \cdots, e_n)$ is the closed path $C^{-1} = (e_n^{-1}, \cdots, e_1^{-1})$.

We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1} = e_i$ for some $i (1 \leq i \leq n-1)$. A path without backtracking is called *proper*. Let B^r be the closed path obtained by going r times around a closed path B. Such a closed path is called a *multiple* of B. Multiples of a closed path without bumps may have a bump. Such a closed path is said to have a *tail*. If its length is n, then the closed path can be written as

$$(e_1, \cdots, e_k, f_1, f_2, \cdots, f_{n-2k}, e_k^{-1}, \cdots, e_1^{-1}),$$

where $(f_1, f_2, \cdots, f_{n-2k})$ is a closed path. A closed path is called *reduced* if C has no backtracking nor tail. Furthermore, a closed path C is *primitive* if it is not a multiple of a strictly shorter closed path. Let $\mathcal{C}$ be the set of closed paths. Furthermore, let $\mathcal{C}^{nontail}$ and $\mathcal{C}^{tail}$ be the set of closed paths without tail, and closed paths with tail, respectively. Note that $\mathcal{C} = \mathcal{C}^{nontail} \cup \mathcal{C}^{tail}$ and $\mathcal{C}^{nontail} \cap \mathcal{C}^{tail} = \phi$.

We introduce an equivalence relation between closed paths. Two closed paths $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if there exists an integer k such that $f_j = e_{j+k}$ for all j, where the subscripts are read modulo n. The inverse of C is not equivalent to C if $|C| \geq 3$. Let $[C]$ be the equivalence class which contains a closed path C. Also, $[C]$ is called a *cycle*.

Let $\mathcal{K}$ be the set of cycles of G. Denote by $\mathcal{R}$, $\mathcal{P} \subset \mathcal{R}$ and $\mathcal{PK} \subset \mathcal{K}$ the set of reduced cycles, primitive, reduced cycles and primitive cycles of G, respectively. Also, primitive, reduced cycles are called *prime cycles*. Let $\mathcal{C}_m, \mathcal{C}_m^{nontail}, \mathcal{C}_m^{tail}, \mathcal{K}_m$ and $\mathcal{PK}_m$ be the subset of $\mathcal{C}, \mathcal{C}^{nontail}, \mathcal{C}^{tail}, \mathcal{K}$ and $\mathcal{PK}$ consisting of elements with length m, respectively. Note that each equivalence class of primitive, reduced closed paths of a graph G passing through a vertex v of G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at v.

The *Ihara zeta function* of a graph G is a function of a complex variable t with $\mid t \mid$ sufficiently small, defined by

$$\mathbf{Z}(G, t) = \mathbf{Z}_G(t) = \prod_{[C] \in \mathcal{P}} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all prime cycles of G.

Let G be a connected graph with n vertices $v_1, \cdots, v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The *degree* of a vertex v_i of G is defined by $\deg v_i = \deg_G v_i = \mid \{v_j \mid v_i v_j \in E(G)\} \mid$. If $\deg_G v = k$(constant) for each $v \in V(G)$, then G is called k-*regular*.

Ihara [14] showed that the reciprocal of the Ihara zeta function of a regular graph is an explicit polynomial. The Ihara zeta function of a regular graph has the following three properties: the rationality; the functional equations; the analogue of the Riemann hypothesis(see [24]). The analogue of the Riemann hypothesis for the zeta function of a graph is given as follows: Let G be any connected $(q+1)$-regular graph($q > 1$) and $s = \sigma + it$ $(\sigma, t \in \mathbf{R})$ a complex number. If $\mathbf{Z}_G(q^{-s}) = 0$ and Re $s \in (0, 1)$, then Re $s = \frac{1}{2}$.

A connected $(q+1)$-regular graph G is called a *Ramanujan graph* if for all eigenvalues λ of the adjacency matrix $\mathbf{A}(G)$ of G such that $\lambda \neq \pm(q+1)$, we have $\mid \lambda \mid \leq 2\sqrt{q}$. This definition was introduced by Lubotzky, Phillips and Sarnak [16]. For a connected $(q+1)$-regular graph G, $\mathbf{Z}_G(q^{-s})$ satisfies the Riemann hypothesis if and only if G is a Ramanujan graph.

Hashimoto [13] treated multivariable zeta functions of bipartite graphs. Bass [3] generalized Ihara's result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial.

Theorem 1 (Bass) *Let G be a connected graph. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G,t)^{-1} = (1-t^2)^{r-1} \det(\mathbf{I} - t\mathbf{A}(G) + t^2(\mathbf{D}-\mathbf{I})),$$

where r is the Betti number of G, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0, i \neq j, (V(G) = \{v_1, \cdots, v_n\})$.

Stark and Terras [21] gave an elementary proof of Theorem 1, and discussed three different zeta functions of any graph. Various proofs of Bass' theorem were known. Kotani and Sunada [15] proved Bass' theorem by using the property of the Perron operator. Foata and Zeilberger [5] presented a new proof of Bass' theorem by using the algebra of Lyndon words.

Let G be a connected graph. Then the *bump count* $bc(P)$ of a path P is the number of bumps in P. Furthermore, the *cyclic bump count* $cbc(C)$ of a closed path $C = (e_1, \cdots, e_n)$ is

$$cbc(C) = \mid \{ i = 1, \cdots, n \mid e_i = e_{i+1}^{-1} \} \mid,$$

where $e_{n+1} = e_1$. An equivalence class of primitive closed paths in G is called a *primitive cycle*. Then the *Bartholdi zeta function* of G is a function of complex variables u, t with $\mid u \mid, \mid t \mid$ sufficiently small, defined by

$$\zeta_G(u,t) = \zeta(G,u,t) = \prod_{[C] \in \mathcal{PK}} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all primitive cycles of G.

If $u = 0$, then the Bartholdi zeta function of G is the Ihara zeta function of G. Because the Bartholdi zeta function $\zeta(G,u,t)$ of a graph is divided into two parts concerned with primitive, non-reduced cycles and primitive, reduced cycles (i.e., prime cycles) of G, respectively:

$$\zeta(G,u,t) = \prod_{[C] \in \mathcal{PK} \setminus \mathcal{P}} (1 - u^{cbc(C)} t^{|C|})^{-1} \times \prod_{[C] \in \mathcal{P}} (1 - t^{|C|})^{-1}.$$

By substituting $u = 0$, we obtain

$$\zeta(G,0,t) = 1 \cdot \prod_{[C] \in \mathcal{P}} (1 - t^{|C|})^{-1} = \mathbf{Z}(G,t).$$

Let n and m be the number of vertices and unoriented edges of G, respectively. Then two $2m \times 2m$ matrices $\mathbf{B} = (\mathbf{B}_{e,f})_{e,f \in R(G)}$ and $\mathbf{J} = (\mathbf{J}_{e,f})_{e,f \in R(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Bartholdi [2] presented a determinant expression for the Bartholdi zeta function of a graph.

Theorem 2 (Bartholdi) *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\begin{aligned} \zeta(G,u,t)^{-1} &= \det(\mathbf{I}_{2m} - (\mathbf{B} - (1-u)\mathbf{J})t) \\ &= (1-(1-u)^2 t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1-u)(\mathbf{D} - (1-u)\mathbf{I})t^2). \end{aligned}$$

In the case of $u = 0$, Theorem 2 implies Theorem 1.

The Ihara zeta function of a finite graph was extended to an infinite graph in [3,4,8,9,10,11], and those determinant expressions were presented. Bass [3] defined the zeta function for a pair of a tree X and a countable group Γ which acts discretely on X with quotient being a graph of finite groups. Clair and Mokhtari-Sharghi [4] extended Ihara zeta functions to infinite graphs on which a group Γ acts isomorphically and with finite quotient. In [8], Grigorchuk and Żuk defined zeta functions of infinite discrete groups, and of some class of infinite periodic graphs.

Guido, Isola and Lapidus [9] defined the Ihara zeta function of a periodic simple graph(i.e., an infinite graph). Let $G = (V(G), E(G))$ be a simple graph which is (countable and) uniformly locally finite, and let Γ be a countable discrete subgroup of automorphisms of G, which acts freely on G, and with finite quotient $B = G/\Gamma$. Then the Ihara zeta function of a periodic simple graph is defined as follows:

$$\mathbf{Z}_{G,\Gamma}(t) = \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - t^{|C|})^{-1/|\Gamma_C|},$$

where $[C]_\Gamma$ runs over all Γ-equivalence classes of prime cycles in G.

Guido, Isola and Lapidus [9] presented a determinant expression for the Ihara zeta function of a periodic simple graph by using Stark and Terras' method [21].

Theorem 3 (Guido, Isola and Lapidus)

$$\mathbf{Z}_{G,\Gamma}(t) = (1-t^2)^{-(m-n)} \det{}_\Gamma(\mathbf{I} - t\mathbf{A}(G) + (\mathbf{D} - \mathbf{I})t^2)^{-1},$$

where $m = \mid E(B) \mid$, $n = \mid V(B) \mid$ and $\det_\Gamma$ is a determinant for bounded operators belonging to a von Neumann algebra with a finite trace.

Also, Guido, Isola and Lapidus [10] presented a determinant expression for the Ihara zeta function of a periodic graph by using Bass' method [3]. Furthermore, Guido, Isola and Lapidus [11] generalized the results of [9,10] to a fractal graph. In [11], they defined the Ihara zeta function of a fractal graph and gave its determinant expression.

In this paper, we define the Bartholdi zeta function of a fractal graph, and present its determinant expression. The proof is an analogue of the method of Guido, Isola and Lapidus [11], and Mizuno and Sato's method [17]. In Section 2, we give a short review on a fractal graph. In Section 3, we present some combinatorial properties on closed paths of a fractal graph. In Section 4, we define the Bartholdi zeta function of a fractal graph, and show that it is holomorphic. In Section 5, we review a determinant for bounded operators acting on an infinite dimensional Hilbert space and belonging to a von Neumann algebra with a finite trace. In Section 6, we present a determinant expression for the Bartholdi zeta function of a fractal graph.

2 Fractal graphs

Let $G = (V(G), E(G))$ be countable and connected. We assume that G has bounded degree, i.e., $d = \sup_{v \in V(G)} \deg_G v < \infty$(see [18,19]). For two vertices $v, w \in V(G)$, the *distance* $d(v, w)$ between v and w is defined as the length of the shortest path between v and w. For $v \in V(G)$ and $r \in \mathbf{N}$, let $B_r(v) = \{w \in V(G) \mid d(v, w) \leq r\}$. For $\Omega \subset V(G)$, let $B_r(\Omega) = \cup_{v \in \Omega} B_r(v)$.

A bounded operator A on $\ell^2(V(G))$ has *finite propagation* $r = r(A) \geq 0$ if, for all $v \in V(G)$, $\mathrm{supp}(Av) \subset B_r(v)$ and $\mathrm{supp}(A^*v) \subset B_r(v)$ S, where A^* is the Hilbert space adjoint of A. Let $B(\ell^2(V(G)))$ be the set of bounded operators on $\ell^2(V(G))$. Note that finite propagation operators forms a $*$-algebra.

A *local isomorphim* of the graph G is a triple $(s(\gamma), r(\gamma), \gamma)$, where $s(\gamma)$, $r(\gamma)$ are subgraphs of G and $\gamma : s(\gamma) \longrightarrow r(\gamma)$ is a graph isomorphism. The local isomorpism γ defines a *partial isometry* $U(\gamma) : \ell^2(V(G)) \longrightarrow \ell^2(V(G))$, by setting

$$U(\gamma)(v) := \begin{cases} \gamma(v) & \text{if } v \in V(s(\gamma)), \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity.

An operator $T \in B(\ell^2(V(G)))$ is called *geometric* if there exists $r \in \mathbf{N}$ such that T has finite propagation r and, for any local isomorphism γ, any vertex $v \in V(G)$ such that $B_r(v) \subset s(\gamma)$ and $B_r(\gamma v) \subset r(\gamma)$, one has

$$TU(\gamma)v = U(\gamma)Tv, \ T^*U(\gamma)v = U(\gamma)T^*v.$$

The *adjacencey matrix* $\mathbf{A}(G) = (a_{vw})$ and the *degree matrix* $\mathbf{D}(G) = (d_{vw})$ are defined by

$$a_{vw} := \begin{cases} 1 & \text{if } (v, w) \in R(G), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_{vw} := \begin{cases} \deg_G v & \text{if } v = w, \\ 0 & \text{otherwise,} \end{cases}$$

For a subgraph K of G, the *frontier* $\mathcal{F}(K)$ is the family of vertices in $V(K)$ having distance 1 from the complement of $V(K)$ in $V(G)$. A countably infinite graph G with bounded degree is *amenable* if it has an *amenable exhaustion*, i.e., an increasing family of finite subgraphs $\{K_n\}_{n\in\mathbf{N}}$ such that $\cup_{n\in\mathbf{N}}K_n = G$ and

$$\frac{\mid \mathcal{F}(K_n) \mid}{\mid V(K_n) \mid} \longrightarrow 0 \ as \ n \to \infty.$$

A countably infinite graph G with bounded degree is called *self-similar* or *fractal* if it has an amenable exhaustion $\{K_n\}$ such that the following conditions (i) and (ii) hold(see [1,12]):

(i) For every $n \in \mathbf{N}$, there is a finite set $\mathcal{I}(n.n+1)$ of local isomorphisms such that, for all $\gamma \in \mathcal{I}(n, n+1)$, one has $s(\gamma) = K_n$,

$$\bigcup_{\gamma\in\mathcal{I}(n,n+1)} \gamma(K_n) = K_{n+1},$$

and moreover, if $\gamma, \gamma' \in \mathcal{I}(n, n+1)$ with $\gamma \neq \gamma'$,

$$V(\gamma K_n) \cap V(\gamma' K_n) = \mathcal{F}(\gamma K_n) \cap \mathcal{F}(\gamma' K_n).$$

(ii) Let $\mathcal{I}(n,m)(n < m)$ be the set of all admissible products $\gamma = \gamma_{m-1}\cdots\gamma_n, \gamma_i \in \mathcal{I}(i, i+1)$, where "admissble" means that, for each term of the product, the range of γ_i is contained in the source of γ_{i+1}. Also, let $\mathcal{I}(n, n) = \{id_{K_n}\}$, and $\mathcal{I}(n) = \cup_{m\geq n}\mathcal{I}(n,m)$.

We define the *$\mathcal{I}$-invariant frontier* of K_n:

$$\mathcal{F}_{\mathcal{I}}(K_n) = \bigcup_{\gamma\in\mathcal{I}(n)} \gamma^{-1}\mathcal{F}(\gamma K_n).$$

and we require that

$$\frac{\mid \mathcal{F}_{\mathcal{I}}(K_n) \mid}{\mid V(K_n) \mid} \longrightarrow 0 \ as \ n \to \infty.$$

Let $\mathcal{I}$ be the family of all local isomorphisms which can be written as admissible products $\gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}\cdots\gamma_k^{\epsilon_k}$, where $\gamma_i \in \cup_{n\in\mathbf{N}}\mathcal{I}(n)$, $\epsilon_i = 1, -1$ for $i = 1, \ldots, k$ and $k \in \mathbf{N}$.

A trace on the algebra of geometric operators on a fractal graph is constructed as follows(see [11]):

Theorem 4 (Guido, Isola and Lapidus) *Let G be a fractal graph, and $\mathcal{A}(G)$ the $*$-algebra defined as the norm closure of the $*$-algebra of geometric operators. Then, on $\mathcal{A}(G)$, there is a well-defined faithful trace state $\mathrm{Tr}_{\mathcal{I}}$ given by*

$$\mathrm{Tr}_{\mathcal{I}}(T) = \lim_n \frac{\mathrm{Tr}(P(K_n)T)}{\mathrm{Tr}(P(K_n))},$$

where $P(K_n)$ is the orthogonal projection of $\ell^2(V(G))$ onto its closed subspace $\ell^2(V(K_n))$.

We use the following result by Guido, Isola and Lapidus [11].

Proposition 1 (Guido, Isola and Lapidus) *Let G be a connected fractal graph with bounded degree $d = \sup_{v \in V(G)} \deg_G v < \infty$. Furthermore, let $\{K_n\}$ be an amenable exhaustion of G such that satisfies the conditions (i) and (ii) in the definition of a fractal graph. Let Ω be any finite subset of $V(G)$. Then the following results hold:*

1. *For any $r \in \mathbf{N}$,*
$$\mid B_r(\Omega) \mid \leq \mid \Omega \mid (d+1)^r .$$

2. *Let $\Omega_{n,r} = V(K_n) \setminus B_r(\mathcal{F}_{\mathcal{I}}(K_n))$. Then, for $n \leq m$,*
$$\mid \mathcal{I}(n,m) \mid\mid \Omega_{n,r} \mid \leq \mid V(K_m) \mid \leq \mid \mathcal{I}(n,m) \mid\mid V(K_n) \mid .$$

3. *Let*
$$\epsilon_n = \frac{\mid \mathcal{F}_{\mathcal{I}}(K_n) \mid}{\mid V(K_n) \mid},$$
where $\epsilon_n \to 0$ as $n \to \infty$ by the definition of a fractal graph. Furthermore, let $\epsilon_n (d+1)^r \leq 1/2$ for all $n > n_0$. Then
$$0 \leq \frac{\mid \mathcal{I}(n,m) \mid\mid V(K_n) \mid}{\mid V(K_m) \mid} - 1 \leq 2\epsilon_n (d+1)^r \leq 1.$$

3 Closed paths in a fractal graph

Let G be a connected fractal graph. Furthermore, let $\{K_n\}$ be an amenable exhaustion of G such that satisfies the conditions (i) and (ii) in the definition of a fractal graph. Let $0 < u < 1$. For $s \geq 1$, the matrix $\mathbf{A}_s = ((\mathbf{A}_s)_{i,j})_{v_i, v_j \in V(G)}$ is defined as follows:

$$(\mathbf{A}_s)_{i,j} = \sum_P u^{bc(P)},$$

where $(\mathbf{A}_s)_{i,j}$ is the (i,j)-component of $\mathbf{A}_s$, and P runs over all paths of length s from v_i to v_j in G. Note that $\mathbf{A}_1 = \mathbf{A}(G)$. Furthermore, let $\mathbf{A}_0 = \mathbf{I}$.

Lemma 1 *Put $\mathbf{Q} = \mathbf{D} - \mathbf{I}$. Then*

$$\mathbf{A}_2 = (\mathbf{A}_1)^2 - (1-u)\mathbf{D} = (\mathbf{A}_1)^2 - (1-u)(\mathbf{Q} + \mathbf{I})$$

and

$$\mathbf{A}_s = \mathbf{A}_{s-1}\mathbf{A}_1 - (1-u)\mathbf{A}_{s-2}(\mathbf{Q} + u\mathbf{I}) \quad \textit{for} \quad s \geq 3.$$

Proof. The first formula is clear. We prove the second formula. The proof is an analogue of the proof of Lemma 1 in [21].

We count the paths of length s from v_i to v_k in G. Let $s \geq 3$ and $\mathbf{A}(G) = (\mathbf{A}_{i,j})$. Then the sum $\sum_j (\mathbf{A}_{s-1})_{i,j} \mathbf{A}_{j,k}$ counts three types of paths P, Q, R in G as follows:

$$
\begin{array}{l}
P = (e_1, \cdots, e_{s-1}, e_s), e_s \neq e_{s-1}^{-1},\ e_s = (v_j, v_k), \\
Q = (e_1, \cdots, e_{s-2}, e_{s-1}, e_s), e_{s-1} \neq e_{s-2}^{-1}, e_s = e_{s-1}^{-1} = (v_j, v_k), \\
R = (e_1, \cdots, e_{s-2}, e_{s-1}, e_s), e_{s-2} = e_{s-1}^{-1} = e_s = (v_j, v_k).
\end{array}
$$

Let $T = (e_1, \cdots, e_{s-2})$. Then the term corresponding to P, Q and R in the sum $\sum_j (\mathbf{A}_{s-1})_{i,j} \mathbf{A}_{j,k}$ is $u^{bc(T)}$, $u^{bc(T)}$ and $u^{bc(T)+1}$, respectively. While, the term corresponding to P, Q and R in $(\mathbf{A}_s)_{i,k}$ is $u^{bc(T)}$, $u^{bc(T)+1}$ and $u^{bc(T)+2}$, respectively. Thus,

$$
(\mathbf{A}_s)_{i,k} = \sum_j (\mathbf{A}_{s-1})_{i,j} \mathbf{A}_{j,k} + (u-1)(\mathbf{A}_{s-2})_{i,k} q_k + (u^2 - u)(\mathbf{A}_{s-2})_{i,k},
$$

where $q_k = \deg v_k - 1$. Therefore, the result follows. Q.E.D.

For $s \geq 1$, let $\mathcal{C}_s^{tail}$ be the set of all closed paths of length s with tails in G, and

$$
a_s = \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum_{x \in V(K_n)} \{ u^{bc(C)} \mid C \in \mathcal{C}_s^{tail} \ and\ o(C) = x \}.
$$

Then $a_1 = 0$.

Lemma 2 *1. For $s \in \mathbf{N}$, a_s exists and is finite.*

2.

$$
a_s = \mathrm{Tr}_{\mathcal{I}}[(\mathbf{Q} - (1-2u)\mathbf{I})\mathbf{A}_{s-2}] + (1-u)^2 a_{s-2} \ for\ s \geq 3.
$$

Proof. 1: For $n \in \mathbf{N}$, let

$$
\Omega_n = V(K_n) \setminus B_1(\mathcal{F}_{\mathcal{I}}(K_n)), \Omega'_n = V(K_n) \cap B_1(\mathcal{F}_{\mathcal{I}}(K_n)).
$$

Then, for all $p \in \mathbf{N}$,

$$
V(K_{n+p}) = \cup_{\gamma \in \mathcal{I}(n,n+p)} \gamma \Omega_n \cup (\cup_{\gamma \in \mathcal{I}(n,n+p)} \gamma \Omega'_n).
$$

Let

$$
a_s(x) = \sum_{x \in V(K_n)} \{ u^{bc(C)} \mid C \in \mathcal{C}_s^{tail} \ and\ o(C) = x \}.
$$

Then $a_s(x) \leq d^{s-1}$. Thus, by 1 and 3 of Proposition 1, we have

$$\begin{aligned}
&\left|\frac{1}{\mid V(K_{n+p}) \mid}\sum_{x\in V(K_{n+p})} a_s(x) - \frac{1}{\mid V(K_n) \mid}\sum_{x\in V(K_n)} a_s(x)\right| \\
&\leq \left|\frac{\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\sum_{x\in \Omega_n} a_s(x) - \frac{1}{\mid V(K_n) \mid}\sum_{x\in V(K_n)} a_s(x)\right| + \frac{\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\sum_{x\in \Omega'_n} \mid a_s(x) \mid \\
&\leq \left|\frac{\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid} - \frac{1}{\mid V(K_n) \mid}\right|\sum_{x\in V(K_n)} \mid a_s(x) \mid + 2\frac{\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\sum_{x\in B_1(\mathcal{F}_{\mathcal{I}}(K_n))} \mid a_s(x) \mid \\
&\leq \left|1 - \frac{\mid V(K_n) \mid\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\right| d^{s-1} + 2\frac{\mid V(K_n) \mid\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\frac{\mid B_1(\mathcal{F}_{\mathcal{I}}(K_n)) \mid}{\mid V(K_n) \mid} d^{s-1} \\
&\leq 2\epsilon_n(d+1)d^{s-1} + 2\frac{\mid V(K_n) \mid\mid \mathcal{I}(n,n+p) \mid}{\mid V(K_{n+p}) \mid}\frac{\mid \mathcal{F}_{\mathcal{I}}(K_n) \mid (d+1)}{\mid V(K_n) \mid} d^{s-1} \\
&\leq 6\epsilon_n(d+1)d^{s-1} \longrightarrow 0 \ as \ n \to \infty,
\end{aligned}$$

where

$$\epsilon_n = \frac{\mid \mathcal{F}_{\mathcal{I}}(K_n) \mid}{\mid V(K_n) \mid} \to 0 \ as \ n \to \infty.$$

2: At first, we have

$$\begin{aligned}
a_s &= \lim_{n\to\infty}\frac{1}{\mid V(K_n) \mid}\sum_{v_i\in V(K_n)}\{u^{bc(C)} \mid C \in \mathcal{C}_s^{tail} \ and \ o(C) = v_i\} \\
&= \lim_{n\to\infty}\frac{1}{\mid V(K_n) \mid}\sum_{v_i\in V(K_n)}\sum_{(v_i,v_j)\in R(G)}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \\
&= \lim_{n\to\infty}\frac{1}{\mid V(K_n) \mid}\sum_{v_j\in V(K_n)}\sum_{(v_i,v_j)\in R(G)}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\}
\end{aligned}$$

The third equality is proved as follows: Let

$$\Omega = \{v \in V(G) \mid v \notin V(K_n), d(v, K_n) = 1\} \subset B_1(\mathcal{F}_{\mathcal{I}}(K_n)).$$

Then we have

$$\begin{aligned}
&\frac{1}{\mid V(K_n) \mid}\sum_{v_i\in V(K_n)}\sum_{(v_i,v_j)\in R(G)}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \\
&\qquad = \frac{1}{\mid V(K_n) \mid}\sum_{v_j\in V(K_n)}\sum_{(v_i,v_j)\in R(G)}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \\
&\qquad\quad + \frac{1}{\mid V(K_n) \mid}\sum_{v_j\in \Omega}\sum_{(v_i,v_j)\in R(G), v_i\in V(K_n)}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \\
&\qquad\quad - \frac{1}{\mid V(K_n) \mid}\sum_{v_j\in V(K_n)}\sum_{(v_i,v_j)\in R(G), v_i\in \Omega}\{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\}.
\end{aligned}$$

But, we have

$$\left| \frac{1}{\mid V(K_n) \mid} \sum_{v_j \in \Omega} \sum_{(v_i,v_j) \in R(G), v_i \in V(K_n)} \{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \right|$$
$$\leq \frac{1}{\mid V(K_n) \mid} \mid \mathcal{F}_{\mathcal{I}}(K_n) \mid d^{s-1} \longrightarrow 0$$

and

$$\left| \frac{1}{\mid V(K_n) \mid} \sum_{v_j \in V(K_n)} \sum_{(v_i,v_j) \in R(G), v_i \in \Omega} \{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \right|$$
$$= \left| \frac{1}{\mid V(K_n) \mid} \sum_{v_j \in \mathcal{F}_{\mathcal{I}}(K_n)} \sum_{(v_i,v_j) \in R(G), v_i \in \Omega} \{u^{bc(C)} \mid C = (v_i, v_j, \ldots) \in \mathcal{C}_s^{tail}\} \right|$$
$$\leq \frac{1}{\mid V(K_n) \mid} \mid \mathcal{F}_{\mathcal{I}}(K_n) \mid d^{s-1} \longrightarrow 0.$$

Thus, the third equality holds.

We want to count closed paths of length s with tails in G. The proof is an analogue of the proof of Lemma 2 in [21].

Let $s \geq 3$ and let v_j be fixed. Furthermore, let $C = (v_i, v_j, v_l, \cdots, v_r, v_j, v_i)$ be any closed path of length s with tails in G, and let $P = (v_j, v_l, \cdots, v_r, v_j)$.

Case 1. P does not have a tail, i.e., $v_l \neq v_r$.

Then the closed path C is divided into two types:

$$\begin{aligned} &C_1 = (v_i, v_j, v_l, \cdots, v_r, v_j, v_i),\ v_i \neq v_l \textit{ and } v_i \neq v_r, \\ &C_2 = (v_i, v_j, v_i, \cdots, v_r, v_j, v_i)(v_l = v_i) \\ &\quad \textit{or } (v_i, v_j, v_l, \cdots, v_i, v_j, v_i)(v_r = v_i). \end{aligned}$$

Case 2. P has a tail, i.e., $v_l = v_r$.

Then the closed path C is divided into two types:

$$\begin{aligned} &C_3 = (v_i, v_j, v_l, \cdots, v_l, v_j, v_i),\ v_i \neq v_l, \\ &C_4 = (v_i, v_j, v_i, \cdots, v_i, v_j, v_i),\ v_i = v_l. \end{aligned}$$

Now, we have

$$u^{bc(C_1)} = u^{bc(C_3)} = u^{bc(P)},\ u^{bc(C_2)} = u^{bc(P)+1},\ u^{bc(C_4)} = u^{bc(P)+2}.$$

Thus,

$$\begin{aligned} b_j = &\sum_{(v_i,v_j) \in R(G)} \{u^{bc(C)} \mid C \supset tail,\ |C| = s,\ C = (v_i, v_j, \cdots)\} \\ = &(q_j - 1) \sum \{u^{bc(P)} \mid P \not\supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path\} \\ &+ 2u \sum \{u^{bc(P)} \mid P \not\supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path\} \\ &+ q_j \sum \{u^{bc(P)} \mid P \supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path\} \\ &+ u^2 \sum \{u^{bc(P)} \mid P \supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path\}. \end{aligned}$$

That is,

$$\begin{aligned} b_j = (q_j - 1) \sum \{ u^{bc(P)} \mid \ |P| = s-2,\ P:\ v_j - closed\ path \} \\ + 2u \sum \{ u^{bc(P)} \mid P \not\supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path \} \\ + (1+u^2) \sum \{ u^{bc(P)} \mid P \supset tail,\ |P| = s-2,\ P:\ v_j - closed\ path \}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} a_s &= \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum_{v_j \in V(K_n)} b_j \\ &= \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum_{v_j \in V(K_n)} (\mathbf{Q}(v_j, v_j) - 1) \sum \left\{ u^{bc(P)} \middle| \ |P| = s-2,\ P:\ v_j - closed\ path \right\} \\ &+ 2u \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum_{v_j} \sum \{ u^{bc(P)} \mid \ |P| = s-2,\ P:\ v_j - closed\ path \} \\ &+ (1 - 2u + u^2) \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum_{v_j} \sum \Big\{ u^{bc(P)} \mid P \supset tail, \\ &\qquad |P| = s-2,\ P:\ v_j - closed\ path \Big\}. \end{aligned}$$

Hence,

$$a_s = \mathrm{Tr}_{\mathcal{I}}[(\mathbf{Q} - \mathbf{I})\mathbf{A}_{s-2}] + 2u \mathrm{Tr}_{\mathcal{I}}[\mathbf{A}_{s-2}] + (1-u)^2 a_{s-2}.$$

Q.E.D.

For $m \geq 1$, let $\mathcal{C}_m$ be the set of all closed paths of length s in G, and

$$N_m = \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum \{ u^{cbc(C)} \mid C \in \mathcal{C}_m \ and\ C \subset K_n \}.$$

Lemma 3 *1. For $m \in \mathbf{N}$, N_m exists and is finite.*

2. $N_m = \mathrm{Tr}_{\mathcal{I}}(\mathbf{A}_m) - (1-u) a_m$.

Proof. 1: At first, we have

$$N_m = \lim_{n \to \infty} \frac{1}{\mid V(K_n) \mid} \sum \{ u^{cbc(C)} \mid C \in \mathcal{C}_m \ and\ o(C) = v \in V(K_n) \}.$$

For, by 2 of Proposition 1,

$$
\begin{aligned}
0 &\leq \Big| \frac{1}{|\, V(K_n) \,|} (\sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m \ and \ o(C) = v \in V(K_n)\} \\
&\quad - \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m \ and \ C \subset K_n\}) \Big| \\
&= \frac{1}{|\, V(K_n) \,|} \Big| \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m, o(C) = v \in V(K_n) \ and \ C \not\subset K_n\} \Big| \\
&\leq \frac{1}{|\, V(K_n) \,|} \Big| \sum \{u^{bc(C)} \mid C \in \mathcal{C}_m, o(C) = v \in B_m(\mathcal{F}_{\mathcal{I}}(K_n))\} \Big| \\
&= \frac{1}{|\, V(K_n) \,|} \Big| \sum_{v \in B_m(\mathcal{F}_{\mathcal{I}}(K_n))} \mathbf{A}_m(v.v) \Big| \\
&\leq \frac{1}{|\, V(K_n) \,|} \Big| \mathrm{Tr}(P(B_m(\mathcal{F}_{\mathcal{I}}(K_n))) \mathbf{A}_m) \Big| \\
&\leq \|\mathbf{A}_m\| \frac{|\, B_m(\mathcal{F}_{\mathcal{I}}(K_n)) \,|}{|\, V(K_n) \,|} \\
&\leq \|\mathbf{A}_m\| (d+1)^m \frac{|\, \mathcal{F}_{\mathcal{I}}(K_n) \,|}{|\, V(K_n) \,|} \longrightarrow 0
\end{aligned}
$$

if $n \to \infty$.

Furthermore, the existence of

$$
\lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m \ and \ o(C) = v \in V(K_n)\}
$$

is proved as 1 of Lemma 2.

Therefore, it follows that

$$
\begin{aligned}
N_m &= \lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m \ and \ o(C) = v \in V(K_n)\} \\
&= \lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum \{u^{bc(C)} \mid C \in \mathcal{C}_m^{nontail} \ and \ o(C) = v \in V(K_n)\} \\
&\quad + \lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m^{tail} \ and \ o(C) = v \in V(K_n)\} \\
&= \lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum_{v \in V(K_n)} \mathbf{A}_m(v.v) \\
&\quad - \lim_{n \to \infty} \frac{1}{|\, V(K_n) \,|} \sum_{v \in V(K_n)} \sum \{u^{bc(C)} - u^{cbc(C)} \mid C \in \mathcal{C}_m^{tail} \ and \ o(C) = v \in V(K_n)\}.
\end{aligned}
$$

Hence, since $cbc(C) = bc(C) + 1$ for each closed path C of length s with tails, we have

$$
N_m = \mathrm{Tr}_{\mathcal{I}}(\mathbf{A}_m) - (1-u)a_m.
$$

Q.E.D.

4 The Bartholdi zeta function of a fractal graph

We define the notion of $\mathcal{I}$-equivalence between cycles. Let G be a connected fractal graph. Furthermore, let $\{K_n\}$ be an amenable exhaustion of G such that satisfies the conditions (i) and (ii) in the definition of a fractal graph. For $[C], [D] \in \mathcal{K}$, $[C]$ and $[D]$ are called *$\mathcal{I}$-equivalent*, denoted $[C] \sim_{\mathcal{I}} [D]$, if there exista a local isomorphism $\gamma \in \mathcal{I}$ such that $D = \gamma(C)$. We denote by $[C]_{\mathcal{I}}$ the set of $\mathcal{I}$-equivalent class containing $[C]$. Note that $[C] \in [C]_{\mathcal{I}}$. Let $[\mathcal{K}]_{\mathcal{I}}$ and $[\mathcal{PK}]_{\mathcal{I}}$ be the set of $\mathcal{I}$-equivalence classes of $\mathcal{K}$ and $\mathcal{PK}$, respectively.

For $[C] \in \mathcal{K}$, the *size* $s(C) \in \mathbf{N}$ of $[C]$ is the least $m \in \mathbf{N}$ such that $C \subset \gamma(K_m)$ for some local isomorphism $\gamma \in \mathcal{I}(m)$. Furthermore, the *effective length* $\ell(C) \in \mathbf{N}$ of $[C]$ is the length of the primitive closed path D underlying C, i.e., such that $C = D^p$ for some $p \in \mathbf{N}$. The *average multiplicity* $\mu(C)$ of $[C]$ is the number in $[0, \infty)$ given by

$$\lim_{n\to\infty} \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid}.$$

Lemma 4 *1. Let $[C] \in \mathcal{K}$. Then the following limit exists and is finite:*

$$\lim_{n\to\infty} \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid}.$$

2. *$s(C)$, $\ell(C)$, $\mu(C)$ only depend on $[C]_{\mathcal{I}} \in [\mathcal{K}]_{\mathcal{I}}$. Furthermore, if $C = D^k$ for some $[D] \in \mathcal{PK}$, $k \in \mathbf{N}$, then $s(C) = s(D)$, $\ell(C) = \ell(D)$, $\mu(C) = \mu(D)$.*

3. *For $m \in \mathbf{N}$,*

$$N_m = \sum_{[C]_{\mathcal{I}} \in [\mathcal{K}_m]_{\mathcal{I}}} \mu(C)\ell(C)u^{cbc(C)}.$$

Proof. 1: At first, we have

$$\mid \mathcal{I}(s(C), n+1) \mid = \mid \mathcal{I}(s(C), n) \mid\mid \mathcal{I}(n, n+1) \mid$$

for any $n \geq s(C)$. By 2 and 3 of Proposition 1, we obtain

$$\begin{aligned}\left| \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid} - \frac{\mid \mathcal{I}(s(C), n+p) \mid}{\mid V(K_{n+p}) \mid} \right| &= \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid} \left| 1 - \frac{\mid V(K_n) \mid\mid \mathcal{I}(n, n+p) \mid}{\mid V(K_{n+p}) \mid} \right| \\ &\leq \frac{1}{\mid \Omega_{n,1} \mid} 2\epsilon_n (d+1).\end{aligned}$$

Furthermore,

$$\frac{\mid \mathcal{I}(s(C), n+1) \mid}{\mid V(K_{n+1}) \mid} = \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid} \frac{\mid V(K_n) \mid\mid \mathcal{I}(n, n+1) \mid}{\mid V(K_{n+1}) \mid} \geq \frac{\mid \mathcal{I}(s(C), n) \mid}{\mid V(K_n) \mid},$$

and so the limit is monotone.

2: Clear.

3: We have

$$
\begin{aligned}
N_m &= \lim_{n\to\infty} \frac{1}{\mid V(K_n)\mid} \sum \{u^{cbc(C)} \mid C \in \mathcal{C}_m \textit{ and } C \subset K_n\} \\
&= \lim_{n\to\infty} \sum_{[C]_{\mathcal{I}} \in [\mathcal{K}_m]_{\mathcal{I}}} \frac{1}{\mid V(K_n)\mid} \sum \{u^{cbc(D)} \mid D \in \mathcal{C}_m, [D] \sim_{\mathcal{I}} [C], D \subset K_n\} \\
&= \lim_{n\to\infty} \sum_{[C]_{\mathcal{I}} \in [\mathcal{K}_m]_{\mathcal{I}}} \frac{1}{\mid V(K_n)\mid} u^{cbc(C)} \ell(C) \mid \mathcal{I}(s(C), n) \mid \\
&= \sum_{[C]_{\mathcal{I}} \in [\mathcal{K}_m]_{\mathcal{I}}} u^{cbc(C)} \ell(C) \mu(C).
\end{aligned}
$$

Q.E.D.

We define the Bartholdi zeta function of a fractal graph as follows:

$$
\zeta_{G,\mathcal{I}}(u,t) = \prod_{[C]_{\mathcal{I}} \in [\mathcal{PK}]_{\mathcal{I}}} (1 - u^{cbc(C)} t^{|C|})^{-\mu(C)},
$$

where $u, t \in \mathbf{C}$ are sufficiently small such that the infinite product converges, and $u > 0$.

Lemma 5

$$
\frac{\partial}{\partial t} \log \zeta_{G,\mathcal{I}}(u,t) = t^{-1} \sum_{s \geq 1} N_s t^s.
$$

Proof. Since

$$
\begin{aligned}
\log \zeta_{G,\mathcal{I}}(u,t) &= -\mu(C) \sum_{[C]_{\mathcal{I}} \in [\mathcal{PK}]_{\mathcal{I}}} \log(1 - u^{cbc(C)} t^{|C|}) \\
&= \mu(C) \sum_{[C]_{\mathcal{I}} \in [\mathcal{PK}]_{\mathcal{I}}} \sum_{s=1}^{\infty} \frac{1}{s} u^{cbc(C)s} t^{|C|s},
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial t} \log \zeta_{G,\mathcal{I}}(u,t) &= t^{-1} \sum_{[C]_{\mathcal{I}} \in [\mathcal{PK}]_{\mathcal{I}}} \sum_{s=1}^{\infty} \mu(C) |C| u^{cbc(C)s} t^{|C|s} \\
&= t^{-1} \sum_{s=1}^{\infty} \sum_{[C]_{\mathcal{I}} \in [\mathcal{PK}]_{\mathcal{I}}} \mu(C) |C| u^{cbc(C)s} t^{|C|s} \\
&= t^{-1} \sum_{[C_1]_{\mathcal{I}} \in [\mathcal{K}]_{\mathcal{I}}} \mu(C_1) \ell(C_1) u^{cbc(C_1)} t^{|C_1|}.
\end{aligned}
$$

Note that $cbc(C^s) = cbc(C)s$. The third equality is obtained by the fact that each closed path of G is a multiple of some primitive closed path of G.

Therefore, by Lemma 4, it follows that

$$\frac{\partial}{\partial t} \log \zeta_{G,\Gamma}(u,t) = t^{-1} \sum_{s \geq 1} N_s t^s. \tag{1}$$

Q.E.D.

5 Analytic determinants for von Neumann algebras with a finite trace

In an excellent paper [6], Fuglede and Kadison defined a positive-valued determinant for von Neumann algebras with trivial center and finite trace. For an invertible operator A with polar decomposition $A = UH$, the Fuglede-Kadison determinant of A is defined by

$$Det(A) = \exp \circ \tau \circ \log H,$$

where $\log H$ may be defined via the functional calculus.

Guido, Isola and Lapidus [9] extended the Fuglede-Kadison determinant to a determinant which is an analytic function. Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace. Then, for $A \in \mathcal{A}$, let

$$\det{}_\tau(A) = \exp \circ \tau \circ \log A,$$

where

$$\log(A) := \frac{1}{2\pi i} \int_\Gamma \log \lambda (\lambda - A)^{-1} d\lambda,$$

and Γ is the boundary of a connected, simply connected region Ω containing the spectrum $\sigma(A)$ of A. Then the following lemma holds(see [9, Lemma 5]).

Lemma 6 (Guido, Isola and Lapidus) *Let $\mathcal{A}, \Omega, \Gamma$ be as above, and ϕ, ψ two branches of the logarithm such that both domains contain Ω. Then*

$$\exp \circ \tau \circ \phi(A) = \exp \circ \tau \circ \psi(A).$$

Next, we consider a determinant on some subset of $\mathcal{A}$. Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace, and $\mathcal{A}_0 = \{A \in \mathcal{A} \mid 0 \notin \text{conv } \sigma(A)\}$. For any $A \in \mathcal{A}_0$, we set

$$\det{}_\tau(A) = \exp \circ \tau \circ (\frac{1}{2\pi i} \int_\Gamma \log \lambda (\lambda - A)^{-1} d\lambda),$$

where Γ is the boundary of a connected, simply connected region Ω containing the spectrum conv $\sigma(A)$, and log is a branch of the logarithm whose domain contains Ω. Then the above determinant is well-defined and analytic on $\mathcal{A}_0$(see [9, Corollary 5.3]). Furthermore, Guido, Isola and Lapidus [9] showed that $\det_\tau$ has the following properties.

Proposition 2 (Guido, Isola and Lapidus) *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace, $A \in \mathcal{A}_0$. Then*

1. $\det_\tau(zA) = z \det_\tau(A)$ *for any* $z \in \mathbf{C} \setminus \{0\}$.

2. *If A is normal, and $A = UH$ is its polar decomposition, then*

$$\det_\tau(A) = \det_\tau(U) \det_\tau(H).$$

3. *If A is positive, then $\det_\tau(A) = Det(A)$, where $Det(A)$ is the Fuglede-Kadison determinant of A.*

6 A determinant expression

In this section, we consider the following determinant:

$$\det_{\mathcal{I}}(A) = \exp \circ \mathrm{Tr}_{\mathcal{I}} \circ \log A$$

for $A \in \mathcal{A}(G)$.

In $(\sum_{s \geq 0} \mathbf{A}_s t^s)(\mathbf{I} - t\mathbf{A}_1 + (1-u)(\mathbf{Q} + u\mathbf{I})t^2)$, the coefficient of t^s for any $s \geq 3$ is 0 by the second formula of Lemma 1. Furthermore, by the first formula of Lemma 1, we have

$$(\sum_{s \geq 0} \mathbf{A}_s t^s)(\mathbf{I} - t\mathbf{A}_1 + (1-u)(\mathbf{Q} + u\mathbf{I})t^2) = (1 - (1-u)^2 t^2)\mathbf{I}. \tag{2}$$

Since $(1 - (1-u)^2 t^2)^{-1} = \sum_{j \geq 0} (1-u)^{2j} t^{2j}$,

$$\begin{aligned}\mathbf{I} &= (\sum_{k \geq 0} \mathbf{A}_k t^k)(\sum_{j \geq 0} (1-u)^{2j} t^{2j})(\mathbf{I} - t\mathbf{A}_1 + (1-u)(\mathbf{Q} + u\mathbf{I})t^2) \\ &= (\sum_{s \geq 0} \sum_{j=0}^{[s/2]} \mathbf{A}_{s-2j} (1-u)^{2j} t^s)(\mathbf{I} - t\mathbf{A}_1 + (1-u)(\mathbf{Q} + u\mathbf{I})t^2).\end{aligned}$$

By Lemmas 2 and 3, we have

$$N_s = \mathrm{Tr}_{\mathcal{I}}[\mathbf{A}_s - (1-u)^{-1}(\mathbf{Q} - (1-2u)\mathbf{I}) \sum_{j=1}^{[(s-1)/2]} (1-u)^{2j} \mathbf{A}_{s-2j}] - \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (1-u)^{s-1} a_2 & \text{if } s \text{ is even.} \end{cases}$$

for $s \geq 3$. Furthermore, $N_1 = \mathrm{Tr}_{\mathcal{I}} \mathbf{A}_1 = 0$, and

$$\begin{aligned}N_2 &= \mathrm{Tr}_{\mathcal{I}} \mathbf{A}_2 - (1-u)a_2 = \lim_{n \to \infty} \frac{2u \mid E(K_n) \mid}{\mid V(K_n) \mid} - (1-u) \lim_{n \to \infty} \frac{2u \mid E(K_n) \mid}{\mid V(K_n) \mid} \\ &= 2u^2 \lim_{n \to \infty} \frac{\mid E(K_n) \mid}{\mid V(K_n) \mid}.\end{aligned}$$

Next, set

$$\begin{aligned}\mathbf{N}^*_s &= \mathbf{A}_s - (1-u)^{-1}(\mathbf{Q}-(1-2u)\mathbf{I})\sum_{j=1}^{[s/2]}(1-u)^{2j}\mathbf{A}_{s-2j} \\ &= \mathbf{A}_s + (1-u)^{-1}(\mathbf{Q}-(1-2u)\mathbf{I})\mathbf{A}_s - (1-u)^{-1}(\mathbf{Q}-(1-2u)\mathbf{I})\sum_{j=0}^{[s/2]}(1-u)^{2j}\mathbf{A}_{s-2j}.\end{aligned}$$

Then (2) and (3) imply that

$$\begin{aligned}&\Big(\sum_{s\geq 0}\mathbf{N}^*_s t^s\Big)(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2) \\ &\quad = (\mathbf{I}+(1-u)^{-1}(\mathbf{Q}-(1-2u)\mathbf{I}))(1-(1-u)^2t^2)\mathbf{I} - (1-u)^{-1}(\mathbf{Q}-(1-2u)\mathbf{I}) \\ &\quad = (1-(1-u)^2t^2)\mathbf{I} - (1-u)t^2(\mathbf{Q}-(1-2u)\mathbf{I}).\end{aligned}$$

Since $\mathbf{N}^*_0 = \mathbf{A}_0 = \mathbf{I}_n$,

$$\begin{aligned}&\Big(\sum_{s\geq 1}\mathbf{N}^*_s t^s\Big)(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2) \\ &\quad = (1-(1-u)^2t^2)\mathbf{I} - (1-u)t^2(\mathbf{Q}-(1-2u)\mathbf{I}) - (\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2) \\ &\quad = t\mathbf{A}_1 - 2(1-u)(\mathbf{Q}+u\mathbf{I})t^2.\end{aligned}$$

Therefore it follows that

$$\sum_{s\geq 1}\mathbf{N}^*_s t^s = (t\mathbf{A}_1 - 2(1-u)(\mathbf{Q}+u\mathbf{I})t^2)(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)^{-1}.$$

Lemma 7 *Let $f : t \in B_\epsilon = \{t \in \mathbf{C} \mid \mid t \mid < \epsilon\} \mapsto f(u,t) \in \mathcal{A}(G)$ be a $\mathbf{C}^1$-function, $f(0,0) = 0$, and $\| f(u,t) \| < 1$ for all $t \in B_\epsilon$, where the absolute value of $u \in \mathbf{C}$ is suffiently small. Then*

$$\mathrm{Tr}_{\mathcal{I}}(-\frac{\partial}{\partial t}\log(\mathbf{I}-f(u,t))) = \mathrm{Tr}_{\mathcal{I}}(\frac{\partial}{\partial t}f(u,t)(\mathbf{I}-f(u,t))^{-1}).$$

Proof. At first, we have

$$-\log(\mathbf{I}-f(u.t)) = \sum_{n\geq 1}\frac{1}{n}f(u,t)^n.$$

Then, the above converges in operator norm, uniformly on compact subsets of B_ϵ, and $\| f(u,t) \| < 1$ for all $t \in B_\epsilon$. Furthermore,

$$\frac{\partial}{\partial t}f(u,t)^n = \sum_{j=0}^{n-1} f(u,t)^j \frac{\partial}{\partial t}f(u,t)f(u,t)^{n-j-1}.$$

Therefore, we have

$$-\frac{\partial}{\partial t}\log(\mathbf{I}-f(u,t)))=\sum_{n\geq 1}\sum_{j=0}^{n-1}\frac{1}{n}f(u,t)^j\frac{\partial}{\partial t}f(u,t)f(u,t)^{n-j-1},$$

and so

$$\begin{aligned}\mathrm{Tr}_{\mathcal{I}}(-\frac{\partial}{\partial t}\log(\mathbf{I}-f(u,t))) &= \sum_{n\geq 1}\frac{1}{n}\sum_{j=0}^{n-1}\mathrm{Tr}_{\mathcal{I}}(f(u,t)^j\frac{\partial}{\partial t}f(u,t)f(u,t)^{n-j-1}) \\ &= \sum_{n\geq 1}\mathrm{Tr}_{\mathcal{I}}(f(u,t)^{n-1}j\frac{\partial}{\partial t}f(u,t)) \\ &= \mathrm{Tr}_{\mathcal{I}}(\frac{\partial}{\partial t}f(u,t)(\mathbf{I}-f(u,t))^{-1}).\end{aligned}$$

Q.E.D.

We state the average Euler-Poincaré characteristic of a fractal graph(see [11]).

Lemma 8 (Guido, Isola and Lapidus) *The following limit exists and is finite:*

$$\chi_{av}(G):=\lim_{n\to\infty}\frac{\chi(K_n)}{\mid V(K_n)\mid}=-\frac{1}{2}\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I}),$$

where $\chi(K_n)=\mid V(K_n)\mid - \mid E(K_n)\mid$.

Theorem 5

$$\zeta_{G,\mathcal{I}}(u,t)^{-1}=(1-(1-u)^2t^2)^{-\chi_{av}(G)}\det{}_{\mathcal{I}}(\mathbf{I}-t\mathbf{A}(G)+(1-u)(\mathbf{D}-(1-u)\mathbf{I})t^2).$$

Proof. By Lemma 7, we have

$$\mathrm{Tr}_{\mathcal{I}}(\sum_{s\geq 1}\mathbf{N}_s^*t^s)=\mathrm{Tr}_{\mathcal{I}}(-t\frac{\partial}{\partial t}\log(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)).$$

By Lemma 8, we have

$$a_2=u\lim_{n\to\infty}\frac{2\mid E(K_n)\mid}{\mid V(K_n)\mid}=u\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}+\mathbf{I}).$$

If s is odd, then $\mathrm{Tr}_{\mathcal{I}}(N_s^*)=N_s$. Otherwise, we have

$$\begin{aligned}\mathrm{Tr}_{\mathcal{I}}(\mathbf{N}_s^*) &= N_s-(1-u)^{s-1}\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-(1-2u)\mathbf{I})+(1-u)^{s-1}a_2 \\ &= N_s-(1-u)^{s-1}\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-(1-2u)\mathbf{I}-u(\mathbf{Q}+\mathbf{I})) \\ &= N_s-(1-u)^{s}\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I}).\end{aligned}$$

Thus, for $s \geq 1$, we have

$$\mathrm{Tr}_{\mathcal{I}}(\mathbf{N}_s^*) = N_s - \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (1-u)^s \mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I}) & \text{if } s \text{ is even.} \end{cases}$$

Thus,

$$\begin{aligned} \mathrm{Tr}_{\mathcal{I}}(\sum_{s\geq 1} \mathbf{N}_s^* t^s) &= \sum_{s\geq 1} N_s t^s - \mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})(\sum_{j\geq 1}(1-u)^{2j}t^{2j}) \\ &= \sum_{s\geq 1} N_s t^s - \mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})\frac{(1-u)^2t^2}{1-(1-u)^2t^2}, \end{aligned}$$

i.e.,

$$\sum_{s\geq 1} N_s t^s = \mathrm{Tr}_{\mathcal{I}}(\sum_{s\geq 1} \mathbf{N}_s^* t^s) + \mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})\frac{(1-u)^2t^2}{1-(1-u)^2t^2}.$$

(1) implies that

$$\begin{aligned} & t\frac{\partial}{\partial t}\log\zeta_{G,\mathcal{I}}(u,t) \\ &\quad = \mathrm{Tr}_{\mathcal{I}}(-t\frac{\partial}{\partial t}\log(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)) + \mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})\frac{(1-u)^2t^2}{1-(1-u)^2t^2} \\ &\quad = \mathrm{Tr}_{\mathcal{I}}(-t\frac{\partial}{\partial t}\log(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)) - t\frac{\partial}{\partial t}\log(1-(1-u)^2t^2)^{\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})/2}. \end{aligned}$$

Both functions are 0 at $t=0$, and so

$$\log\zeta_{G,\mathcal{I}}(u,t) = -\mathrm{Tr}_{\mathcal{I}}(\log(\mathbf{I}-t\mathbf{A}_1+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)) - \log(1-(1-u)^2t^2)^{\mathrm{Tr}_{\mathcal{I}}(\mathbf{Q}-\mathbf{I})/2}.$$

Hence the equality $\mathrm{Tr}_{\mathcal{I}}(\log(\mathbf{I}-\mathbf{B})) = \log\det{}_{\mathcal{I}}(\mathbf{I}-\mathbf{B})$ and Lemma 8 implies that

$$\zeta_{G,\mathcal{I}}(u,t) = (1-(1-u)^2t^2)^{\chi_{av}(G)}\det{}_{\mathcal{I}}(\mathbf{I}-t\mathbf{A}(G)+(1-u)(\mathbf{Q}+u\mathbf{I})t^2)^{-1}.$$

Q.E.D.

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