On $\mathrm{STD}_6[18,3]$'s and $\mathrm{STD}_7[21,3]$'s admitting a semiregular automorphism group of order 9

Kenzi Akiyama
Department of Applied Mathematics
Fukuoka University, Fukuoka 814-0180, Japan
akiyama@sm.fukuoka-u.ac.jp

Masayuki Ogawa
Computer Engineering Inc.
Hikino, Yahatanisi-ku, Kitakyushu-city, Fukuoka 806-0067, Japan
a_meteoric_stream_0521@yahoo.co.jp

Chihiro Suetake*
Department of Mathematics, Faculty of Engineering
Oita University, Oita 870-1192, Japan
suetake@csis.oita-u.ac.jp

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Abstract

In this paper, we characterize symmetric transversal designs $\mathrm{STD}_\lambda[k,u]$'s which have a semiregular automorphism group G on both points and blocks containing an elation group of order u using the group ring $\mathbf{Z}[G]$. Let n_λ be the number of nonisomorphic $\mathrm{STD}_\lambda[3\lambda,3]$'s. It is known that $n_1 = 1,\ n_2 = 1,\ n_3 = 4, n_4 = 1$, and $n_5 = 0$. We classify $\mathrm{STD}_6[18,3]$'s and $\mathrm{STD}_7[21,3]$'s which have a semiregular noncyclic automorphism group of order 9 on both points and blocks containing an elation of order 3 using this characterization. The former case yields exactly twenty nonisomorphic $\mathrm{STD}_6[18,3]$'s and the latter case yields exactly three nonisomorphic $\mathrm{STD}_7[21,3]$'s. These yield $n_6 \geqslant 20$ and $n_7 \geqslant 5$, because B. Brock and A. Murray constructed two other $\mathrm{STD}_7[21,3]$'s in 1991. We used a computer for our research.

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1 Introduction

A *symmetric transversal design* $\mathrm{STD}_\lambda[k,u]$ (STD) is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ satisfying the following three conditions, where $k \geqslant 2$, $u \geqslant 2$, and $\lambda \geqslant 1$:

(i) Each block contains exactly k points.
(ii) The point set $\mathcal{P}$ is partitioned into k point sets $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$ of equal size u such that any two distinct points are incident with exactly λ blocks or no block according as they are contained in different $\mathcal{P}_i$'s or not. $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$ are said to be the *point classes* of $\mathcal{D}$.
(iii) The dual structure of $\mathcal{D}$ also satisfies the above conditions (i) and (ii). The point classes of the dual structure of $\mathcal{D}$ are said to be the *block classes* of $\mathcal{D}$.

We use the notation $\mathrm{STD}_\lambda[k,u]$ in the paper instead of $\mathrm{STD}_\lambda(u)$used by Beth, Jungnickel and Lenz [2], because we want to exhibit the block size k of the design.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an STD with the set of point classes Ω and the set of block classes Δ. Let G be an automorphism group. Then, by definition of STD, G induces a permutation group on $\Omega \cup \Delta$. If G fixes any element of $\Omega \cup \Delta$, then G is said to be an *elation group* and any element of G is said to be an *elation*. In this case, it is known that G acts semiregularly on each point class and on each block class.

Enumerating symmetric transversal designs $\mathrm{STD}_\lambda[k,u]$'s is of interest by itself as well as estimating non equivalent Hadamard matrices of a fixed order and also produces many 2-designs, because $\mathrm{STD}_\lambda[k,u]$'s are powerful tool for constructing 2-designs (for example, see [16]).

In [1], two of the authors classified $\mathrm{STD}_{\frac{k}{3}}[k,3]$'s for $k \leqslant 18$ which have an automorphism group acting regularly on both the set of the point classes and the set of the block classes. They said such automorphism group a GL-regular automorphism group. Especially it was showed that there does not exist an $\mathrm{STD}_6[18,3]$ admitting a GL-regular automorphism group and an $\mathrm{STD}_7[21,3]$ with a relative difference set was constructed.

In this paper, we consider an $\mathrm{STD}_\lambda[k,u]$ $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ satisfying the following condition: $\mathcal{D}$ has a semiregular automorphism group of order su on both points and blocks containing an elation group of order u.

In the first half of the paper, we characterize an $\mathrm{STD}_\lambda[k,u]$ with such automorphism group G using the group ring $\mathbf{Z}[G]$. We remark that a generalized Hadamard matrix over the group U of degree k $\mathrm{GH}(k,U)$ corresponds to $\mathcal{D}$, because $\mathcal{D}$ has an elation group of order u.

In the second half of the paper, we classify $\mathrm{STD}_6[18,3]$'s and $\mathrm{STD}_7[21,3]$'s which have a semiregular noncyclic automorphism group of order 9 on both points and blocks containing an elation of order 3 using this characterization. We show that there are exactly twenty nonisomorphic $\mathrm{STD}_6[18,3]$'s and three nonisomorphic $\mathrm{STD}_7[21,3]$'s with this automorphism group. Two of these $\mathrm{STD}_7[21,3]$'s are new and the remaining one is an STD constructed in [14]. We also investigate the order of the full automorphism group, the action on the point classes, and the block classes for each $\mathrm{STD}_6[18,3]$ or each

$\mathrm{STD}_7[21,3]$ of those.

We remark that the existence of a $\mathrm{STD}_6[18,3]$ is well known, as it can be obtained from a generalized Hadamard matrix of order 18 being the Kronecker product of generalized Hadamard matrices of order 3 and 6 over a group of order 3.

The existence of $\mathrm{STD}_2[2\lambda,2]$'s is equivalent to the existence of Hadamard matrices of order 2λ. The study of Hadamard matrices is one of the major studies in combinatrices. The authors think that $\mathrm{STD}_\lambda[3\lambda,3]$'s, which have the next class size, also is worth studying. Let n_λ be the number of nonisomorphic $\mathrm{STD}_\lambda[3\lambda,3]$'s. It is known that $n_1=1$, $n_2=1$, $n_3=4$([12]), $n_4=1$([13]), and $n_5=0$ ([5]). We can easily check that $n_1=1$. We also checked that $n_2=1$ by a similar manner as in [13] without a computer, but we do not give the proof in this paper. The above results on $\mathrm{STD}_6[18,3]$'s and $\mathrm{STD}_7[21,3]$'s yield $\lambda_6 \geqslant 20$ and $\lambda_7 \geqslant 5$, because B. Brock and A. Murray constructed other two $\mathrm{STD}_7[21,3]$'s in 1991([3]). The authors think that eighteen of these twenty $\mathrm{STD}_6[18,3]$'s are new (see Remark 7.4). We used a computer for our research.

If an $\mathrm{STD}_\lambda[k,u]$ has a relative difference set, since the STD satisfies our assumption, we can expect that the assumption help to look for relative difference sets of STD's. Also, if we assume an appropriate integer s, we can expect that our assumption help to look for new $\mathrm{STD}_\lambda[k,u]$'s or new $\mathrm{GH}(k,U)$'s. Acutually, Y. Hiramine [7] recently generalized our result and constructed $\mathrm{STD}_q[q^2,q]$'s for all prime power q using spreads of $V(2q,GF(q))$. His construction yields class regular $\mathrm{STD}_q[q^2,q]$'s and non class regular $\mathrm{STD}_q[q^2,q]$'s. For example, at least two of four $\mathrm{STD}_3[9,3]$'s found by Mavron and Tonchev [12] have this form.

For general notation and concepts in design theory, we refer the reader to basic textbooks in the subject such as [2], [4], [10], or [15].

2 Definitions of TD, RTD, and STD

Definition 2.1 A *transversal design* $\mathrm{TD}_\lambda[k,u]$ (TD) is an incidence structure $\mathcal{D}=(\mathcal{P},\mathcal{B},I)$ satisfying the following two conditions:

(i) Each block contains exactly k points.
(ii) The point set $\mathcal{P}$ is partitioned into k point sets $\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_{k-1}$ of equal size u such that any two distinct points are incident with exactly λ blocks or no block according as they are contained in different $\mathcal{P}_i$'s or not. $\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_{k-1}$ are said to be the *point classes* of $\mathcal{D}$.

Remark 2.2 In Definition 2.1, we have the following equalities:
(i) $|\mathcal{P}|=uk$.
(ii) $|\mathcal{B}|=u^2\lambda$.

Definition 2.3 A *resolvable transversal design* $\mathrm{RTD}_\lambda[k,u]$ (RTD) is an incidence structure $\mathcal{D}=(\mathcal{P},\mathcal{B},I)$ satisfying the following conditions, where $k\geqslant 2$, $u\geqslant 2$, and $\lambda\geqslant 1$:

(i) $\mathcal{D}$ is a $\mathrm{TD}_\lambda[k,u]$.
(ii) The block set $\mathcal{B}$ is partitioned into r block sets $\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{r-1}$ such that if $B,\ B'$ $(\neq) \in \mathcal{B}_i$, $(B) \cap (B') = \emptyset$ and $\displaystyle\bigcup_{B \in \mathcal{B}_i} (B) = \mathcal{P}$ for $0 \leqslant i \leqslant r-1$.

Remark 2.4 In Definition 2.3, we have $r = u\lambda$.

Definition 2.5 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a $\mathrm{TD}_\lambda[k,u]$. If the dual structure $\mathcal{D}^d$ of $\mathcal{D}$ also is a $\mathrm{TD}_\lambda[k,u]$, $\mathcal{D}$ is said to be a *symmetric transversal design* $\mathrm{STD}_\lambda[k,u]$ (STD). The point classes of $\mathcal{D}^d$ are said to be the *block classes* of $\mathcal{D}$.

Theorem 2.6 ([11]) *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a $\mathrm{TD}_\lambda[k,u]$ and $k = \lambda u$. Then, $\mathcal{D}$ is a $\mathrm{RTD}_\lambda[k,u]$ if and only if $\mathcal{D}$ is an $\mathrm{STD}_\lambda[k,u]$.*

Remark 2.7 If $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ is a $\mathrm{RTD}_\lambda[k,u]$ and $k = \lambda u$, then $\mathcal{B}_0,\ \mathcal{B}_1, \cdots,\ \mathcal{B}_{r-1}\ \ (r = k)$ of Definition 2.3(iii) are block classes of $\mathcal{D}$.

3 Isomorphisms and automorphisms of STD'S

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an $\mathrm{STD}_\lambda[k,u]$. Then $k = \lambda u$. Let $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}\}$ be the set of point classes of $\mathcal{D}$ and $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}\}$ the set of block classes of $\mathcal{D}$. Let $\mathcal{P}_0 = \{p_0, p_1, \cdots, p_{u-1}\}$, $\mathcal{P}_1 = \{p_u, p_{u+1}, \cdots, p_{2u-1}\}, \cdots,\ \mathcal{P}_{k-1} = \{p_{(k-1)u}, p_{(k-1)u+1}, \cdots, p_{ku-1}\}$ and $\mathcal{B}_0 = \{B_0, B_1, \cdots, B_{u-1}\}$, $\mathcal{B}_1 = \{B_u, B_{u+1}, \cdots, B_{2u-1}\}, \cdots,\ \mathcal{B}_{k-1} = \{B_{(k-1)u}, B_{(k-1)u+1}, \cdots, B_{ku-1}\}$.

On the other hand, Let $\mathcal{D}' = (\mathcal{P}', \mathcal{B}', I')$ be an $\mathrm{STD}_\lambda[k;u]$. Let $\Omega' = \{\mathcal{P}_0{}', \mathcal{P}_1{}', \cdots, \mathcal{P}_{k-1}{}'\}$ be the set of point classes of $\mathcal{D}'$ and $\Delta' = \{\mathcal{B}_0{}', \mathcal{B}_1{}', \cdots, \mathcal{B}_{k-1}{}'\}$ the set of block classes of $\mathcal{D}'$. Let $\mathcal{P}_0{}' = \{p_0{}', p_1{}', \cdots, p_{u-1}{}'\}$, $\mathcal{P}_1{}' = \{p_u{}', p_{u+1}{}', \cdots, p_{2u-1}{}'\}, \cdots,\ \mathcal{P}_{k-1}{}' = \{p_{(k-1)u}{}', p_{(k-1)u+1}{}', \cdots,\ p_{ku-1}{}'\}$ and $\mathcal{B}_0{}' = \{B_0{}', B_1{}', \cdots, B_{u-1}{}'\}$, $\mathcal{B}_1{}' = \{B_u{}', B_{u+1}{}', \cdots, B_{2u-1}{}'\}, \cdots,\ \mathcal{B}_{k-1}{}' = \{B_{(k-1)u}{}', B_{(k-1)u+1}{}', \cdots, B_{ku-1}{}'\}$.

Let Λ be the set of permutation matrices of degree u. Let

$$L = \begin{pmatrix} L_{0\ 0} & \cdots & L_{0\ k-1} \\ \vdots & & \vdots \\ L_{k-1\ 0} & \cdots & L_{k-1\ k-1} \end{pmatrix} \quad \text{and} \quad L' = \begin{pmatrix} L_{0\ 0}{}' & \cdots & L_{0\ k-1}{}' \\ \vdots & & \vdots \\ L_{k-1\ 0}{}' & \cdots & L_{k-1\ k-1}{}' \end{pmatrix}$$

be the incidence matrices of $\mathcal{D}$ and $\mathcal{D}'$ corresponding to these numberings of the point sets and the block sets, where $L_{ij},\ L_{ij}{}' \in \Lambda\quad (0 \leqslant i,j \leqslant k-1)$, respectively. Let E be the identity matrix of degree u. Then we may assume that $L_{i\ 0} = L_{i\ 0}{}' = E\quad (0 \leqslant i \leqslant k-1)$ and $L_{0\ j} = L_{0\ j}{}' = E\quad (0 \leqslant j \leqslant k-1)$ after interchanging some rows of (ru)th row, $(ru+1)$th row, $\cdots, ((r+1)u-1)$th row and interchanging some columns of (su)th column, $(su+1)$th column, $\cdots, ((s+1)u-1)$th column of L and L' for $0 \leqslant r,s \leqslant k-1$.

Definition 3.1 Let $S=\{0,1,\cdots,k-1\}$. We denote the symmetric group on S by Sym S. Let $f=\begin{pmatrix} 0 & 1 & \cdots & k-1 \\ f(0) & f(1) & \cdots & f(k-1)\end{pmatrix}\in$ Sym S and $X_0,X_1,\cdots,X_{k-1}\in\Lambda$.

(i) We define $(f,(X_0,X_1,\cdots,X_{k-1}))=\begin{pmatrix} X_{0\ 0} & \cdots & X_{0\ k-1} \\ \vdots & \cdots & \vdots \\ X_{k-1\ 0} & \cdots & X_{k-1\ k-1}\end{pmatrix}$

by $X_{ij}=\begin{cases} X_i & \text{if } j=f(i), \\ O & \text{otherwise}\end{cases}$, where O is the $u\times u$ zero matrix.

(ii) We define $(f,\begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{k-1}\end{pmatrix})=\begin{pmatrix} X_{0\ 0} & \cdots & X_{0\ k-1} \\ \vdots & \cdots & \vdots \\ X_{k-1\ 0} & \cdots & X_{k-1\ k-1}\end{pmatrix}$

by $X_{ij}=\begin{cases} X_j & \text{if } i=f(j), \\ O & \text{otherwise}\end{cases}$, where O is the $u\times u$ zero matrix.

From Lemma 3.2 of [1], it follows that an isomorphism from $\mathcal{D}$ to $\mathcal{D}'$ is given by $f,g\in$ Sym S and $X_0,X_1,\cdots X_{k-1},Y_0,Y_1,\cdots,Y_{k-1}\in\Lambda$ satisfying

$$(f,(X_0,X_1,\cdots,X_{k-1}))L(g,\begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{k-1}\end{pmatrix})=L'.$$

Assume that this equation is satisfied. Then $X_iL_{f(i)\ g(j)}Y_j={L_{ij}}'$ for $0\leqslant i,j\leqslant k-1$. Since $X_iL_{f(i)\ g(0)}Y_0=E$, $X_i={Y_0}^{-1}{L_{f(i)\ g(0)}}^{-1}$ for $0\leqslant i\leqslant k-1$. On the other hand, since $X_0L_{f(0)\ g(j)}Y_j=E$, $Y_j={L_{f(0)\ g(j)}}^{-1}{X_0}^{-1}={L_{f(0)\ g(j)}}^{-1}L_{f(0)\ g(0)}Y_0$ for $1\leqslant j\leqslant k-1$. Therefore, since $X_iL_{f(i)\ g(j)}Y_j={L_{ij}}'$, ${Y_0}^{-1}{L_{f(i)\ g(0)}}^{-1}L_{f(i)\ g(j)}{L_{f(0)\ g(j)}}^{-1}L_{f(0)\ g(0)}Y_0={L_{ij}}'$ for $0\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1$.

Lemma 3.2 *Two* $\mathrm{STD}_\lambda[k,u]$*'s* $\mathcal{D}$ *and* $\mathcal{D}'$ *are isomorphic if and only if there exists* $(f,g,Y_0)\in$ Sym $S\times$ Sym $S\times\Lambda$ *such that*

$${Y_0}^{-1}{L_{f(i)\ g(0)}}^{-1}L_{f(i)\ g(j)}{L_{f(0)\ g(j)}}^{-1}L_{f(0)\ g(0)}Y_0={L_{ij}}'$$

for $0\leqslant i\leqslant k-1,\ 1\leqslant j\leqslant k-1$.

Proof. "only if" part was proved above. "if" part holds, if we follow the converse of the above argument.

Corollary 3.3 *Any automorphism of an* $\mathrm{STD}_\lambda[k,u]$ $\mathcal{D}$ *is given by* $(f,g,Y_0)\in$ Sym $S\times$ Sym $S\times\Lambda$ *such that*

$${Y_0}^{-1}{L_{f(i)\ g(0)}}^{-1}L_{f(i)\ g(j)}{L_{f(0)\ g(j)}}^{-1}L_{f(0)\ g(0)}Y_0=L_{ij}$$

for $0 \leqslant i \leqslant k-1$ *and* $1 \leqslant j \leqslant k-1$*. Actually,*

$$(f, g, Y_0)(f', g', {Y_0}') = (ff', gg', Y_{g'(0)}{Y_0}'),$$

where $Y_{g'(0)} = {L_{f(0)\ g(g'(0))}}^{-1} L_{f(0)\ g(0)} Y_0$*, if* $g'(0) \neq 0$*.*

Set $\Gamma = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \}$.

Corollary 3.4 *Let* $u = 3$ *and* L_{ij}*,* ${L_{ij}}' \in \Gamma$ *for* $0 \leqslant i, j \leqslant k-1$*. Then, two* $\mathrm{STD}_\lambda[3\lambda, 3]$*'s* $\mathcal{D}$ *and* $\mathcal{D}'$ *are isomorphic if and only if there exists* $(f, g) \in \mathrm{Sym}\ S \times \mathrm{Sym}\ S$ *such that*

$${L_{f(i)\ g(0)}}^{-1} L_{f(i)\ g(j)} {L_{f(0)\ g(j)}}^{-1} L_{f(0)\ g(0)} = {L_{ij}}'$$

for $0 \leqslant i \leqslant k-1$ *and* $1 \leqslant j \leqslant k-1$ *or there exists* $(f, g) \in \mathrm{Sym}\ S \times \mathrm{Sym}\ S$ *such that*

$${L_{f(i)\ g(0)}}^{-1} L_{f(i)\ g(j)} {L_{f(0)\ g(j)}}^{-1} L_{f(0)\ g(0)} = {{L_{ij}}'}^{-1}$$

for $0 \leqslant i \leqslant k-1$ *and* $1 \leqslant j \leqslant k-1$*.*

Proof. If $A \in \Gamma$ and $B \in \Lambda - \Gamma$, then $B^{-1}AB = A^{-1}$. From this and Corrolary 3.3 the corollary holds.

Corollary 3.5 *Let* $u = 3$ *and* $L_{ij} \in \Gamma$ *for* $0 \leqslant i, j \leqslant k-1$*. Then any automorphism of* $\mathcal{D}$ *is given* $(f, g, Y) \in \mathrm{Sym}\ S \times \mathrm{Sym}\ S \times \Gamma$ *such that*

$${L_{f(i)\ g(0)}}^{-1} L_{f(i)\ g(j)} {L_{f(0)\ g(j)}}^{-1} L_{f(0)\ g(0)} = L_{ij}$$

for $0 \leqslant i \leqslant k-1$ *and* $1 \leqslant j \leqslant k-1$ *or* $(f, g, Y) \in \mathrm{Sym}\ S \times \mathrm{Sym}\ S \times (\Lambda - \Gamma)$ *such that*

$${L_{f(i)\ g(0)}}^{-1} L_{f(i)\ g(j)} {L_{f(0)\ g(j)}}^{-1} L_{f(0)\ g(0)} = {L_{ij}}^{-1}$$

for $0 \leqslant i \leqslant k-1$ *and* $1 \leqslant j \leqslant k-1$*.*

4 A semiregular automorphism group of order su of an $\mathrm{STD}_\lambda[k, u]$

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an $\mathrm{STD}_\lambda[k, u]$ and $s \in \mathbf{N}$ such that s divides k. Set $t = \dfrac{k}{s}$. Then $k = u\lambda = ts$. Let $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}\}$ be the set of point classes of $\mathcal{D}$ and $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}\}$ the set of block classes of $\mathcal{D}$. Let $\mathcal{P}_0 = \{p_0, p_1, \cdots, p_{u-1}\}$, $\mathcal{P}_1 = \{p_u, p_{u+1}, \cdots, p_{2u-1}\}$, $\mathcal{P}_2 = \{p_{2u}, p_{2u+1}, \cdots, p_{3u-1}\}$, $\cdots$, $\mathcal{P}_{k-1} = \{p_{(k-1)u}, p_{(k-1)u+1}, \cdots, p_{ku-1}\}$ and $\mathcal{B}_0 = \{B_0, B_1, \cdots, B_{u-1}\}$, $\mathcal{B}_1 = \{B_u, B_{u+1}, \cdots, B_{2u-1}\}$, $\mathcal{B}_2 = \{B_{2u}, B_{2u+1}, \cdots, B_{3u-1}\}$, $\cdots$, $\mathcal{B}_{k-1} = \{B_{(k-1)u}, B_{(k-1)u+1}, \cdots, B_{ku-1}\}$.

Throughout this section we assume the following.

Hypothesis 4.1 Let G be an automorphism group of order su of $\mathcal{D}$ and we assume that G acts semiregularly on $\mathcal{P}$ and $\mathcal{B}$. Moreover we assume that the order of the kernel U of

$$G \ni \varphi \longmapsto \begin{pmatrix} \mathcal{P}_i \\ \mathcal{P}_i{}^{\varphi} \end{pmatrix} \in \mathrm{Sym}\Omega$$

is u and U coincides with the kernel of

$$G \ni \varphi \longmapsto \begin{pmatrix} \mathcal{B}_j \\ \mathcal{B}_j{}^{\varphi} \end{pmatrix} \in \mathrm{Sym}\Delta.$$

Remark 4.2 (Hine and Mavron [8]) The kernel U of the two homomorphisms of Hypothesis 4.1 acts regularly on each $\mathcal{P}_i$ and on each $\mathcal{B}_j$. Therefore a generalized Hadamard matrix $\mathrm{GH}(k, U)$ of degree k over U corresponds to $\mathcal{D}$.

The terminology *elation* will be used in §6, §7 and §8.

Definition 4.3 Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be an STD with the set of point classes Ω and the set of block classes Δ. Let G be an automorphism group. If G fixes any element of $\Omega \cup \Delta$, then G is said to be an *elation group* and any element of G is said to be an *elation*.

From now, we describe $\mathcal{D}$ satisfying Hypothesis 4.1 by elements of the group ring $\mathbf{Z}[G]$. Let $\{\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{s-1}\}$, $\{\mathcal{P}_s, \mathcal{P}_{s+1}, \cdots, \mathcal{P}_{2s-1}\}$, $\{\mathcal{P}_{2s}, \mathcal{P}_{2s+1}, \cdots, \mathcal{P}_{3s-1}\}$, $\cdots$, $\{\mathcal{P}_{(t-1)s}, \mathcal{P}_{(t-1)s+1}, \cdots, \mathcal{P}_{ts-1}\}$ be the orbits of $(G/U, \Omega)$ and $\{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{s-1}\}$, $\{\mathcal{B}_s, \mathcal{B}_{s+1}, \cdots, \mathcal{B}_{2s-1}\}$, $\{\mathcal{B}_{2s}, \mathcal{B}_{2s+1}, \cdots, \mathcal{B}_{3s-1}\}$, $\cdots$, $\{\mathcal{B}_{(t-1)s}, \mathcal{B}_{(t-1)s+1}, \cdots, \mathcal{B}_{ts-1}\}$ the orbits of $(G/U, \Delta)$.

Set G-orbits on $\mathcal{P}$ and $\mathcal{B}$ as follows: $\mathcal{Q}_i = \mathcal{P}_{is} \cup \mathcal{P}_{is+1} \cup \cdots \cup \mathcal{P}_{(i+1)s-1}$ for $0 \leqslant i \leqslant t-1$ and $\mathcal{C}_j = \mathcal{B}_{js} \cup \mathcal{B}_{js+1} \cup \cdots \cup \mathcal{B}_{(j+1)s-1}$ for $0 \leqslant j \leqslant t-1$. Set $q_i = p_{isu}$ for $0 \leqslant i \leqslant t-1$, $C_j = B_{jsu}$ for $0 \leqslant j \leqslant t-1$ and $D_{ij} = \{\alpha \in G | q_i{}^{\alpha} \in (C_j)\}$ for $0 \leqslant i, j \leqslant t-1$. Then $|D_{ij}| = |\mathcal{Q}_i \cap (C_j)| = s$.

For a subset H of G, we denote $\displaystyle\sum_{h \in H} h \in \mathbf{Z}[G]$ by H for simplicity and $\displaystyle\sum_{h \in H} h^{-1} \in \mathbf{Z}[G]$ by $H^{(-1)}$.

Lemma 4.4 *For $0 \leqslant i, i' \leqslant t-1$ set $A(i, i') = \displaystyle\sum_{0 \leqslant j \leqslant t-1} D_{ij} D_{i'j}{}^{(-1)}$. Then*

$$A(i, i') = \begin{cases} \lambda G & \text{if } \ i \neq i', \\ k + \lambda(G - U) & \text{if } \ i = i' \end{cases} .$$

Proof. Let $0 \leqslant i, i' \leqslant t-1$. For a fixed element $\alpha \in G$, we want to know the number of (β, γ)'s in $D_{ij} \times D_{i'j}$ satisfying $\alpha = \beta\gamma^{-1}$. Since $\alpha\gamma = \beta \in D_{ij}$ and $\gamma \in D_{i'j}$, $q_i{}^{\alpha} \in (C_j{}^{\gamma^{-1}})$ and $q_{i'} \in (C_j{}^{\gamma^{-1}})$.

(i) Assume that $i \neq i'$.

Since $q_i{}^{\alpha}$ and $q_{i'}$ are distinct points, there exist λ these blocks $C_j{}^{\gamma^{-1}}$'s and therefore $A(i, i') = \lambda G$.

(ii) Assume that $i = i'$.
If $\alpha = 1$, then there exist k these blocks ${C_j}^{\gamma^{-1}}$'s. If $\alpha \notin U$, then since ${q_i}^{\alpha}$ and q_i are contained in distinct point classes respectively, there exist λ these blocks ${C_j}^{\gamma^{-1}}$'s. If $\alpha \in U - \{1\}$, then since ${q_i}^{\alpha}$ and q_i are contained in a same point class, there is no such ${C_j}^{\gamma^{-1}}$'s. Therefore $A(i, i) = k + \lambda(G - U)$.

Lemma 4.5 *For $0 \leqslant j, j' \leqslant t-1$ set $B(j, j') = \sum_{0 \leqslant i \leqslant t-1} {D_{ij'}}^{(-1)} D_{ij}$. Then*

$$B(j, j') = \begin{cases} \lambda G & \text{if } j \neq j', \\ k + \lambda(G - U) & \text{if } j = j' \end{cases} .$$

Proof. Let $0 \leqslant j, j' \leqslant t-1$. For a fixed element $\alpha \in G$, we want to know the number of (γ, β)'s in $D_{ij'} \times D_{ij}$ satisfying $\alpha = \gamma^{-1}\beta$. Since $\gamma\alpha = \beta \in D_{ij}$ and $\gamma \in D_{ij'}$, ${q_i}^{\gamma} \in ({C_j}^{\alpha^{-1}})$ and ${q_i}^{\gamma} \in (C_{j'})$.
(i) Assume that $j \neq j'$.
Since ${C_j}^{\alpha^{-1}}$ and $C_{j'}$ are contained in distinct block classes respectively, there exist λ these points ${q_i}^{\gamma}$'s and therefore $B(j, j') = \lambda G$.
(ii) Assume that $j = j'$.
If $\alpha = 1$, then there exist k these points ${q_i}^{\gamma}$'s. If $\alpha \notin U$, then since ${C_j}^{\alpha^{-1}}$ and C_j are contained in distinct block classes respectively, there exist λ these points ${q_i}^{\gamma}$'s. If $\alpha \in U - \{1\}$, then since ${C_j}^{\alpha^{-1}}$ and C_j are contained in a same block class, there is no such point ${q_i}^{\gamma}$. Therefore $B(j, j) = k + \lambda(G - U)$.

5 An $\mathrm{STD}_\lambda[k, u]$ constructed from a group of order su

In this section we show that the converse of Lemma 4.4 holds.

Theorem 5.1 *Let λ and u be positive integers with $u \geqslant 2$ and set $k = \lambda u$. Let s be a positive integer such that s divides k and set $t = \dfrac{k}{s}$. Let G be a group of order su and U a normal subgroup of G of order u. For $0 \leqslant i, j \leqslant t-1$ let D_{ij} be a subset of G with $|D_{ij}| = s$. For $0 \leqslant i, i' \leqslant t-1$ let*

$$\sum_{0 \leqslant j \leqslant t-1} D_{ij} {D_{i'j}}^{(-1)} = \begin{cases} \lambda G & \text{if } i \neq i', \\ k + \lambda(G - U) & \text{if } i = i' \end{cases} .$$

Let $G/U = \{U\tau_0,\ U\tau_1, \cdots, U\tau_{s-1}\}$. Set $\mathcal{P}_{is+r} = \{(i, \varphi\tau_r) |\ \varphi \in U\}$, $\mathcal{B}_{is+r} = \{[i, \varphi\tau_r] |\ \varphi \in U\}$ for $0 \leqslant i \leqslant t-1,\ 0 \leqslant r \leqslant s-1$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k-1}$, $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{k-1}$. We define an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ by

$$(i, \alpha) I [j, \beta] \Longleftrightarrow \alpha\beta^{-1} \in D_{ij} \quad \textit{for } 0 \leqslant i, j \leqslant t-1 \textit{ and } \alpha, \beta \in G.$$

Then $\mathcal{D}$ is an $\mathrm{STD}_\lambda[k,u]$ *with point classes $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$, block classes $\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}$ and the group G acts semiregularly on $\mathcal{P}$ and on $\mathcal{B}$. Also, if we set $\Omega = \{\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}\}$, $\Delta = \{\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}\}$, these kernels coincide with U, and G/U acts semiregularly on Ω and Δ.*

Proof. (i) Let $0 \leqslant j \leqslant t-1$ and $\beta \in G$. First we show that the number of (i,α)'s with $(i,\alpha)I[j,\beta]$ is k. By definition, $(i,\alpha)I[j,\beta]$ if and only if $\alpha\beta^{-1} \in D_{ij}$. Since $|D_{ij}| = s$, there are s α's satisfying $\alpha\beta^{-1} \in D_{ij}$ for each $0 \leqslant i \leqslant t-1$. Thus the number of (i,α)'s with $(i,\alpha)I[j,\beta]$ is exactly $ts = k$. Therefore the block size of $\mathcal{B}$ is constant and it is k.
(ii) For $0 \leqslant i \leqslant k-1$, $|\mathcal{P}_i| = u$ and $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$ give a partition of $\mathcal{P}$.
(iii) Let $0 \leqslant i \leqslant t-1$ and α, α' be distinct elements of U. Suppose that $(i, \alpha\tau_r)I[j,\beta]$, $(i,\alpha'\tau_r)I[j,\beta]$. Then $\alpha\tau_r\beta^{-1} \in D_{ij}$, $\alpha'\tau_r\beta^{-1} \in D_{ij}$ and therefore $1 \neq \alpha\alpha'^{-1} =$
$(\alpha\tau_r\beta^{-1})(\alpha'\tau_r\beta^{-1})^{-1} \in D_{ij}{D_{ij}}^{(-1)}$. But $\alpha\alpha'^{-1} \in U$. This is contradict to the assumption. Hence there is no block through the distinct points $(i,\alpha\tau_r)$, $(i,\alpha'\tau_r) \in \mathcal{P}_{is+r}$ for $0 \leqslant r \leqslant s-1$.

Let $0 \leqslant i \leqslant t-1$, $\alpha, \alpha' \in U$, and $0 \leqslant r_1 \neq r_2 \leqslant s-1$. Suppose that $(i,\alpha\tau_{r_1})I[j,\beta]$, $(i,\alpha'\tau_{r_2})I[j,\beta]$. Since $\alpha\tau_{r_1}\beta^{-1} \in D_{ij}$, $\alpha'\tau_{r_2}\beta^{-1} \in D_{ij}$, we have $(\alpha\tau_{r_1}\beta^{-1})(\alpha'\tau_{r_2}\beta^{-1})^{-1} = \alpha\tau_{r_1}{\tau_{r_2}}^{-1}\alpha'^{-1} \in D_{ij}{D_{ij}}^{(-1)}$. If $\alpha\tau_{r_1}{\tau_{r_2}}^{-1}\alpha'^{-1} \in U$, $\tau_{r_1}{\tau_{r_2}}^{-1} \in U$. But this is contradict to $r_1 \neq r_2$. Therefore $\alpha\tau_{r_1}{\tau_{r_2}}^{-1}\alpha'^{-1} \notin U$ and hence there are exactly λ these $[j,\beta]$'s by the assumption.

Let $0 \leqslant i \neq i' \leqslant t-1$ and $\alpha, \alpha' \in G$. Suppose that $(i,\alpha)I[j,\beta]$ and $(i',\alpha')I[j,\beta]$. Then since $\alpha\beta^{-1} \in D_{ij}$ and $\alpha'\beta^{-1} \in D_{i'j}$, we have $(\alpha\beta^{-1})(\alpha'\beta^{-1})^{-1} = \alpha\alpha'^{-1} \in D_{ij}{D_{i'j}}^{(-1)}$. There are λ these $[j,\beta]$'s by the assumption.
(i)$'$ By a similar argument as in stated in the proof of (i), we can show that the number of blocks through a point is constant and it is k.
(ii)$'$ For $0 \leqslant j \leqslant k-1$ $|\mathcal{B}_j| = u$ and $\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}$ give a partition of $\mathcal{B}$. Therefore $\mathcal{D}$ is a $\mathrm{TD}_\lambda[k,u]$ with point classes $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_{k-1}$. By definition of $\mathcal{B}_j$'s $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{k-1}$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $0 \leqslant i \neq j \leqslant k-1$. Let $0 \leqslant j \leqslant t-1$, $0 \leqslant r \leqslant s-1$, and $\varphi, \varphi'(\neq) \in U$. Suppose that $(i,\alpha)I[j,\varphi\tau_r]$ and $(i,\alpha)I[j,\varphi'\tau_r]$. Then $\alpha{\tau_r}^{-1}\varphi^{-1} \in D_{ij}$ and $\alpha{\tau_r}^{-1}\varphi'^{-1} \in D_{ij}$. But $1 \neq (\alpha{\tau_r}^{-1}\varphi^{-1})(\alpha{\tau_r}^{-1}\varphi'^{-1})^{-1} = \alpha{\tau_r}^{-1}(\varphi^{-1}\varphi')(\alpha{\tau_r}^{-1})^{-1} \in D_{ij}{D_{ij}}^{(-1)} \cap U$. This is contradict to the assumption. Therefore $[j,\varphi\tau_r]$ and $[j,\varphi'\tau_r]$ do not intersect. This yields that for distinct blocks B, $B' \in \mathcal{B}_i$ $\quad (0 \leqslant i \leqslant k-1)$ $(B) \cap (B') = \emptyset$ and $\bigcup_{B\in\mathcal{B}_i}(B) = \mathcal{P}$. Hence $\mathcal{D}$ is a $\mathrm{RTD}_\lambda[k,u]$. Since $k = \lambda u$, $\mathcal{D}$ is an $\mathrm{STD}_\lambda[k,u]$ with block classes $\mathcal{B}_0, \mathcal{B}_1, \cdots, \mathcal{B}_{k-1}$ by Theorem 2.6. Any element μ of G induces an automorphism

$$\mathcal{P} \ni (i,\xi) \longrightarrow (i,\xi\mu) \in \mathcal{P} \quad (0 \leqslant i \leqslant t-1,\ \xi \in G)$$

of $\mathcal{D}$. This satisfies the assertion of the theorem.

Lemma 5.2 *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be the* $\mathrm{STD}_\lambda[k,u]$ *defined in Theorem* 5.1. *Then we have the following statements.*

(i) *Let $\alpha_0, \alpha_1, \cdots, \alpha_{t-1}, \beta_0, \beta_1, \cdots, \beta_{t-1} \in G$. Set ${D_{ij}}' = \alpha_i D_{ij} \beta_j$ for $0 \leqslant i,j \leqslant t-1$.*

Then for $0 \leqslant i, l \leqslant t-1$

$$\sum_{0 \leqslant j \leqslant t-1} {D_{ij}}' {D_{lj}}'^{(-1)} = \begin{cases} \lambda G & \text{if } i \neq l, \\ k + \lambda(G-U) & \text{if } i = l \end{cases}.$$

If for this $\{{D_{ij}}' |\ 0 \leqslant i, j \leqslant t-1\}$ *we define an incidence structure* $\mathcal{D}' = (\mathcal{P}', \mathcal{B}', I')$ *using Theorem* 5.1*, then it follows that* $\mathcal{D} \cong \mathcal{D}'$.
(ii) *Let* $p, q \in \mathrm{Sym}\{0, 1, \cdots, t-1\}$*. Set* ${D_{ij}}'' = D_{i^p, j^q}$ *for* $0 \leqslant i, j \leqslant t-1$.
Then for $0 \leqslant i, l \leqslant t-1$

$$\sum_{0 \leqslant j \leqslant t-1} {D_{ij}}'' {D_{lj}}''^{(-1)} = \begin{cases} \lambda G & \text{if } i \neq l, \\ k + \lambda(G-U) & \text{if } i = l \end{cases}.$$

If for this $\{{D_{ij}}'' |\ 0 \leqslant i, j \leqslant t-1\}$ *we define an incidence structure* $\mathcal{D}'' = (\mathcal{P}'', \mathcal{B}'', I'')$ *using Theorem* 5.1*, then it follows that* $\mathcal{D} \cong \mathcal{D}''$.

Proof. (i) Let $0 \leqslant i, l \leqslant t-1$. Since U is a normal subgroup of G,

$$\begin{aligned} \sum_{0 \leqslant j \leqslant t-1} {D_{ij}}' {D_{lj}}'^{(-1)} &= \sum_{0 \leqslant j \leqslant t-1} \alpha_i D_{ij} \beta_j {\beta_j}^{-1} {D_{lj}}^{(-1)} {\alpha_i}^{-1} \\ &= \alpha_i \Big(\sum_{0 \leqslant j \leqslant t-1} D_{ij} {D_{lj}}^{(-1)} \Big) {\alpha_i}^{-1} = \begin{cases} \lambda G & \text{if } i \neq l, \\ k + \lambda(G-U) & \text{if } i = l. \end{cases} \end{aligned}$$

Let $\mathcal{D}' = (\mathcal{P}', \mathcal{B}', I')$ be the $\mathrm{STD}_\lambda[k, u]$ corresponding to $\{{D_{ij}}' |\ 0 \leqslant i, j \leqslant t-1\}$, where $\mathcal{P}' = \{(i, \alpha)' |\ 0 \leqslant i \leqslant t-1,\ \alpha \in G\}$ and $\mathcal{B}' = \{[j, \beta]' |\ 0 \leqslant j \leqslant t-1,\ \beta \in G\}$. We define a bijection from $\mathcal{P} \cup \mathcal{B}$ to $\mathcal{P}' \cup \mathcal{B}'$ by $(i, \alpha)^f = (i, \alpha_i \alpha)'$ and $[j, \beta]^f = [j, {\beta_j}^{-1} \beta]'$. Since $(i, \alpha) I [j, \beta] \iff \alpha \beta^{-1} \in D_{ij} \iff \alpha_i \alpha \beta^{-1} \beta_j \in \alpha_i D_{ij} \beta_j \iff (\alpha_i \alpha)({\beta_j}^{-1} \beta)^{-1} \in {D_{ij}}'$ $\iff (i, \alpha_i \alpha)' I' [j, {\beta_j}^{-1} \beta]' \iff (i, \alpha)^f I' [j, \beta]^f$, we have $\mathcal{D} \cong \mathcal{D}'$.
(ii) Let $0 \leqslant i, l \leqslant t-1$. Then

$$\sum_{0 \leqslant j \leqslant t-1} {D_{ij}}'' {D_{lj}}''^{(-1)} = \sum_{0 \leqslant j \leqslant t-1} D_{i^p j^q} {D_{l^p j^q}}^{(-1)} = \begin{cases} \lambda G & \text{if } i \neq l, \\ k + \lambda(G-U) & \text{if } i = l. \end{cases}$$

Let $\mathcal{D}'' = (\mathcal{P}'', \mathcal{B}'', I'')$ be the $\mathrm{STD}_\lambda[k, u]$ corresponding to $\{{D_{ij}}'' |\ 0 \leqslant i, j \leqslant t-1\}$, where $\mathcal{P}'' = \{(i, \alpha)'' |\ 0 \leqslant i \leqslant t-1,\ \alpha \in G\}$ and $\mathcal{B}'' = \{[j, \beta]'' |\ 0 \leqslant j \leqslant t-1,\ \beta \in G\}$. We define a bijection g from $\mathcal{P} \cup \mathcal{B}$ to $\mathcal{P}'' \cup \mathcal{B}''$ by $(i, \alpha)^g = (i^{p^{-1}}, \alpha)''$, $[j, \beta]^g = [j^{q^{-1}}, \beta]''$. Since $(i, \alpha) I [j, \beta] \iff \alpha \beta^{-1} \in D_{ij} \iff \alpha \beta^{-1} \in D_{(i^{p^{-1}})^p, (j^{q^{-1}})^q} \iff (i^{p^{-1}}, \alpha)'' I'' [j^{q^{-1}}, \beta]''$ $\iff (i, \alpha)^g I'' [j, \beta]^g$, we have $\mathcal{D} \cong \mathcal{D}''$

6 $\mathrm{STD}_\lambda[3\lambda, 3]$'s

In this section, we consider an $\mathrm{STD}_\lambda[3\lambda, 3]$ which has a semiregular noncyclic automorphism group G on both points and blocks containing an elation of order 3. For that, we use notations and the construction of an STD stated in Theorem 5.1. Then $k = 3\lambda,\ u = 3,\ s = 3$, and $t = \lambda$.

Let G be an elementary abelian group of order 9 and U a subgroup of G of order 3. Set $G = \{(x, y) |\ x, y \in GF(3)\}$ and $U = \{(x, 0) |\ x \in GF(3)\}$.

Definition 6.1 Let Φ be the set of subsets of G with the form $D = \{(a_0, 0), (a_1, 1), (a_2, 2)\}$. Let $D,\ D' \in \Phi$. We define a binary relation on Φ as follows.

$$D \sim D' \Longleftrightarrow D' = (a, b) + D \ \text{ for some } (a, b) \in G.$$

Lemma 6.2 *$\sim$ is an equivalence relation on Φ and a complete system of representatives of $\Phi/\sim$ are the following five sets.*

$D_1 = \{(0, 0), (0, 1), (0, 2)\}$, $D_2 = \{(0, 0), (0, 1), (1, 2)\}$, $D_3 = \{(0, 0), (2, 1), (0, 2)\}$, $D_4 = \{(0, 0), (1, 1), (2, 2)\}$, $D_5 = \{(0, 0), (2, 1), (1, 2)\}$.

Proof. A straightforward calculation yields the lemma.

Lemma 6.3 *Let $D_{ij} \subseteq G$ such that $|D_{ij}| = 3$ for $0 \leqslant i, j \leqslant \lambda-1$. Let for $0 \leqslant i, i' \leqslant \lambda-1$*

$$\sum_{0\leqslant j\leqslant \lambda-1} D_{ij}{D_{i'j}}^{(-1)} = \begin{cases} \lambda G & \text{if } \ i \neq i', \\ 3\lambda + \lambda(G-U) & \text{if } \ i = i' \end{cases}.$$

Here we remark that ${D_{i'j}}^{(-1)} = \sum_{\alpha\in D_{i'j}} (-\alpha)$. Then we have the following statements.

(i) *For $0 \leqslant i, j \leqslant \lambda - 1$*

$$D_{ij} = \{(a_0, 0),\ (a_1, 1),\ (a_2, 2)\} \text{ for some } a_0, a_1, a_2 \in GF(3).$$

(ii) *We may assume that $D_{0\ 0} = D_{j_0},\ D_{0\ 1} = D_{j_1}, \cdots, D_{0\ \lambda-1} = D_{j_{\lambda-1}}, D_{1\ 0} = D_{i_1}$, $D_{2\ 0} = D_{i_2}, \cdots, D_{\lambda-1\ 0} = D_{i_{\lambda-1}}$ for some $1 \leqslant j_0 \leqslant j_1 \leqslant \cdots \leqslant j_{\lambda-1} \leqslant 5$ and for some $1 \leqslant j_0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{\lambda-1} \leqslant 5$.*

Proof. (i) holds by the definition of D_{ij}'s. (ii) holds from Lemma 5.2.

7 $\text{STD}_6[18, 3]$'s

In this section we consider the case of $\lambda = 6$ in §6. That is, we will classify $\text{STD}_6[18, 3]$'s which have a semiregular noncyclic automorphism group of order 9 on both points and blocks containing an elation of order 3.

Lemma 7.1 *The possibilities of $(D_{0,0},\ D_{0,1}, \cdots, D_{0,5})$ and $(D_{0,0},\ D_{1,0}, \cdots, D_{5,0})$ are the following 12 cases respectively.*
(1) $(D_1, D_1, D_4, D_4, D_5, D_5)$,
(2) $(D_1, D_2, D_2, D_2, D_4, D_5)$,

(3) $(D_1, D_2, D_2, D_3, D_4, D_5)$,
(4) $(D_1, D_2, D_3, D_3, D_4, D_5)$,
(5) $(D_1, D_3, D_3, D_3, D_4, D_5)$,
(6) $(D_2, D_2, D_2, D_2, D_2, D_2)$,
(7) $(D_2, D_2, D_2, D_2, D_2, D_3)$,
(8) $(D_2, D_2, D_2, D_2, D_3, D_3)$,
(9) $(D_2, D_2, D_2, D_3, D_3, D_3)$,
(10) $(D_2, D_2, D_3, D_3, D_3, D_3)$,
(11) $(D_2, D_3, D_3, D_3, D_3, D_3)$,
(12) $(D_3, D_3, D_3, D_3, D_3, D_3)$.

Proof. The lemma holds by Lemma 4.4, Lemma 4.5, and Lemma 6.3 using a computer.

We follow the following procedure.

(i) All desired $D = (D_{ij})_{0 \leqslant i,j \leqslant 5}$'s are determined.
(ii) Generalized Hadamard matrices $GH(18, GF(3))$'s corresponding to these D's are determined.
(iii) These generalized Hadamard matrices are normalised.
(iv) All generalized Hadamard matrices of (iii) which correspond to non isomorphic $\mathrm{STD}_6[18, 3]$'s are chosen using Corollary 3.4.

We do not state the details of the calculation, because it requires a tedious explanation. If the reader wants the information, we can offer a note about this.

Example 7.2 $D = (D_{ij})_{0 \leqslant i,j \leqslant 5}$

$$= \left(\begin{array}{ccc} \{(0,0),(0,1),(0,2)\} & \{(0,0),(0,1),(0,2)\} & \{(0,0),(1,1),(2,2)\} \\ \{(0,0),(0,1),(1,2)\} & \{(1,0),(2,1),(2,2)\} & \{(0,0),(0,1),(1,2)\} \\ \{(0,0),(0,1),(1,2)\} & \{(2,0),(1,1),(2,2)\} & \{(1,0),(1,1),(2,2)\} \\ \{(0,0),(0,1),(1,2)\} & \{(2,0),(2,1),(1,2)\} & \{(2,0),(2,1),(0,2)\} \\ \{(0,0),(1,1),(2,2)\} & \{(0,0),(2,1),(1,2)\} & \{(1,0),(0,1),(2,2)\} \\ \{(0,0),(2,1),(1,2)\} & \{(0,0),(1,1),(2,2)\} & \{(0,0),(0,1),(0,2)\} \end{array} \right|$$

$$\left| \begin{array}{ccc} \{(0,0),(1,1),(2,2)\} & \{(0,0),(2,1),(1,2)\} & \{(0,0),(2,1),(1,2)\} \\ \{(1,0),(1,1),(0,2)\} & \{(1,0),(2,1),(2,2)\} & \{(2,0),(1,1),(1,2)\} \\ \{(2,0),(0,1),(0,2)\} & \{(0,0),(2,1),(0,2)\} & \{(1,0),(0,1),(0,2)\} \\ \{(2,0),(1,1),(2,2)\} & \{(1,0),(1,1),(0,2)\} & \{(0,0),(2,1),(2,2)\} \\ \{(0,0),(0,1),(0,2)\} & \{(0,0),(1,1),(2,2)\} & \{(2,0),(2,1),(2,2)\} \\ \{(0,0),(2,1),(1,2)\} & \{(0,0),(0,1),(0,2)\} & \{(0,0),(1,1),(2,2)\} \end{array} \right)$$

satisfies the assumption of lemma 6.3. Thus we can get an $\mathrm{STD}_6[18, 3]$ corresponding to D. We state how to make a normalized generalized Hadamard matrix with D. The generalized Hadamard matrix $GH(18, GF(3))$ corresponding to D is

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&1&2&0&1&2&0&2&1&0&2&1\\
0&0&0&0&0&0&2&0&1&2&0&1&1&0&2&1&0&2\\
0&0&0&0&0&0&1&2&0&1&2&0&2&1&0&2&1&0\\
\hline
0&0&1&1&2&2&0&0&1&1&1&0&1&2&2&2&1&1\\
1&0&0&2&1&2&1&0&0&0&1&1&2&1&2&1&2&1\\
0&1&0&2&2&1&0&1&0&1&0&1&2&2&1&1&1&2\\
\hline
0&0&1&2&1&2&1&1&2&2&0&0&0&2&0&1&0&0\\
1&0&0&2&2&1&2&1&1&0&2&0&0&0&2&0&1&0\\
0&1&0&1&2&2&1&2&1&0&0&2&2&0&0&0&0&1\\
\hline
0&0&1&2&2&1&2&2&0&2&1&2&1&1&0&0&2&2\\
1&0&0&1&2&2&0&2&2&2&2&1&0&1&1&2&0&2\\
0&1&0&2&1&2&2&0&2&1&2&2&1&0&1&2&2&0\\
\hline
0&1&2&0&2&1&1&0&2&0&0&0&0&1&2&2&2&2\\
2&0&1&1&0&2&2&1&0&0&0&0&2&0&1&2&2&2\\
1&2&0&2&1&0&0&2&1&0&0&0&1&2&0&2&2&2\\
\hline
0&2&1&0&1&2&0&0&0&0&2&1&0&0&0&0&1&2\\
1&0&2&2&0&1&0&0&0&1&0&2&0&0&0&2&0&1\\
2&1&0&1&2&0&0&0&0&2&1&0&0&0&0&1&2&0
\end{array}\right).$$

Let H be the normalized generalized Hadamard matrix obtained from this matrix. Then

$$H = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&1&2&2&0&2&2&1&0&1&1&0&1&2&2&0\\
0&2&2&1&0&1&0&1&0&2&2&1&1&1&0&0&2&2\\
0&1&0&2&2&1&0&0&1&1&2&2&2&0&0&1&2&1\\
\hline
0&0&1&2&1&2&1&0&0&2&2&1&0&0&2&1&1&2\\
0&2&2&1&1&0&1&2&1&2&0&0&2&0&0&2&1&1\\
0&1&0&1&2&2&1&1&2&0&2&0&2&1&2&0&1&0\\
\hline
0&0&1&2&2&1&2&1&1&2&0&0&1&2&2&0&0&1\\
0&2&2&0&1&1&2&0&2&1&0&1&2&1&2&1&0&0\\
0&1&0&2&1&2&2&2&0&1&1&0&1&1&0&2&0&2\\
\hline
0&1&2&0&2&1&1&2&0&0&2&1&0&2&1&2&0&1\\
0&1&2&2&1&0&0&1&2&1&0&2&0&2&1&0&1&2\\
0&1&2&1&0&2&2&0&1&2&1&0&0&2&1&1&2&0\\
\hline
0&2&1&0&1&2&0&2&1&0&1&2&0&1&2&0&2&1\\
0&2&1&1&2&0&2&1&0&0&1&2&2&0&1&1&0&2\\
0&2&1&2&0&1&1&0&2&0&1&2&1&2&0&2&1&0
\end{array}\right).$$

Let $L = (L_{ij})_{0 \leqslant i,j \leqslant 17}$ be the 54×54 matrix by replacing entries 0,1,2 of H with $\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, $\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}$, $\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}$, respectively. Then L is a normalized incidence matrix of an $\mathrm{STD}_6[18,3]$.

We denote the STD corresponding to a generalized Hadamard matrix $GH(16, GF(3))$ H by $\mathcal{D}(H)$. We have the following result.

Theorem 7.3 *There are exactly* 20 *nonisomorphic* $\mathrm{STD}_6[18,3]$*'s which have a semiregular noncyclic automorphism group of order* 9 *on both points and blocks containing an elation of order* 3*. These are* $\mathcal{D}(H_i)$ $\quad (i = 1, 2, \cdots, 11)$ *and* $\mathcal{D}(H_j)^d$ $(j = 1, 2, 3, 4, 5, 7, 8, 9, 10)$*, where* H_i $\quad (i = 1, 2, \cdots, 11)$ *are generalized Hadamard matrices of degree* 18 *on* $GF(3)$ *given in Appendix* A*. Let* $\Omega_i = \Omega(\mathcal{D}(H_i))$ *and* $\Delta_i = \Delta(\mathcal{D}(H_i))$ *be a set of the point classes and a set of the block classes of* $\mathcal{D}(H_i)$*, respectively. Then we* *also have the following table.*

i	$\lvert\mathrm{Aut}\mathcal{D}(H_i)\rvert$	sizes of orbits on Ω_i	sizes of orbits on Δ_i
1	54×3	(3,6,9)	(18)
2	54×3	(3,6,9)	(9,9)
3	54×3	(3,6,9)	(9,9)
4	54×3	(3,6,9)	(9,9)
5	108×3	(3,6,9)	(9,9)
6	324×3	(9,9)	(9,9)
7	432×3	(6,12)	(18)
8	432×3	(6,12)	(18)
9	648×3	(9,9)	(18)
10	1080×3	(3,15)	(18)
11	12960×3	(18)	(18)

Remark 7.4 (i) For any prime power q, it is known that there exist $\mathrm{STD}_2[2q, q]$'s (see Theorem 6.33 in [6]). In particular, when $q = 9$, we can construct $\mathrm{STD}_2[18, 9]$'s and we get $\mathrm{STD}_6[18, 3]$'s by reducing these $\mathrm{STD}_2[18, 9]$'s (see [6] or [9]). We checked that all STD's of these are isomorphic each other and this STD is isomorphic to $\mathcal{D}(H_6)$.
(ii) We also checked that the tensor product of the $\mathrm{STD}_2[6, 3]$ and the $\mathrm{STD}_1[3, 3]$ yields an $\mathrm{STD}_6[18, 3]$, but this STD is isomorphic to $\mathcal{D}(H_{11})$. Therefore (i) and all STD's of Theorem 7.3 except $\mathcal{D}(H_6)$ and $\mathcal{D}(H_{11})$ are new. If n_6 is the number of nonisomorphic $\mathrm{STD}_6[18, 3]$'s, $n_6 \geqslant 20$.
(iii) $\mathcal{D}(H_{11})$ does not have a regular automorphism group on both the point set and the block set.
(iv) It is known that a transversal design $\mathrm{TD}_\lambda[k, u]$ is precisely the same as an orthogonal array $OA(\lambda u^2, k, u, 2)$. Therefore, a symmetric transversal design $\mathrm{STD}_\lambda[k; u]$ yields $OA(\lambda u^2, \lambda u, u, 2)$ (see page 242 of [6]). If we can know the orbit structure of the full automorphism group of a symmetric transversal design $\mathrm{STD}_\lambda[k, u]$ $\mathcal{D}$, we can express more clearly the orthogonal array $OA(\lambda u^2, \lambda u, u, 2)$ A corresponding to $\mathcal{D}$.

8 $\mathrm{STD}_7[21, 3]$'s

In this section we consider the case of $\lambda = 7$ in §6. That is, we will classify $\mathrm{STD}_7[21, 3]$'s which have a semiregular noncyclic automorphism group of order 9 on both points and blocks containing an elation of order 3.

Lemma 8.1 *The possibilities of* $(D_{0,0},\ D_{0,1}, \cdots, D_{0,6})$ *and* $(D_{0,0},\ D_{1,0}, \cdots, D_{6,0})$ *are the following* 15 *cases respectively.*
(1) $(D_1, D_1, D_2, D_4, D_4, D_5, D_5)$,
(2) $(D_1, D_1, D_3, D_4, D_4, D_5, D_5)$,
(3) $(D_1, D_2, D_2, D_2, D_2, D_4, D_5)$,

(4) $(D_1, D_2, D_2, D_2, D_3, D_4, D_5)$,
(5) $(D_1, D_2, D_2, D_3, D_3, D_4, D_5)$,
(6) $(D_1, D_2, D_3, D_3, D_3, D_4, D_5)$,
(7) $(D_1, D_3, D_3, D_3, D_3, D_4, D_5)$,
(8) $(D_2, D_2, D_2, D_2, D_2, D_2, D_2)$,
(9) $(D_2, D_2, D_2, D_2, D_2, D_2, D_3)$,
(10) $(D_2, D_2, D_2, D_2, D_2, D_3, D_3)$,
(11) $(D_2, D_2, D_2, D_2, D_3, D_3, D_3)$,
(12) $(D_2, D_2, D_2, D_3, D_3, D_3, D_3)$,
(13) $(D_2, D_2, D_3, D_3, D_3, D_3, D_3)$,
(14) $(D_2, D_3, D_3, D_3, D_3, D_3, D_3)$,
(15) $(D_3, D_3, D_3, D_3, D_3, D_3, D_3)$.

Proof. The lemma holds by Lemma 4.4, Lemma 4.5, and Lemma 6.3 using a computer.

By a similar computation as in §7, we have the following theorem.

Theorem 8.2 *There are exactly* 3 *nonisomorphic* $\mathrm{STD}_7[21,3]$*'s which have a semiregular noncyclic automorphism group of order* 9 *on both points and blocks containing an elation of order* 3*. These are* $\mathcal{D}(K_1)$, $\mathcal{D}\ (K_2)$, *and* $\mathcal{D}(K_1)^d$, *where* K_1 *and* K_2 *are generalized Hadamard matrices of degree* 21 *on* $GF(3)$ *given in Appendix* B*. Let* $\Omega_i = \Omega(\mathcal{D}(K_i))$ *and* $\Delta_i = \Delta(\mathcal{D}(K_i))$ *be a set of the point classes and a set of the block classes of* $\mathcal{D}(K_i)$*, respectively. Then we also have the following table.*

i	$\lvert\mathrm{Aut}\mathcal{D}(K_i)\rvert$	sizes of orbits on Ω_i	sizes of orbits on Δ_i
1	18×3	(3,9,9)	(3,9,9)
2	336×3	(21)	(21)

Remark 8.3 (i) $\mathcal{D}(K_1)$ and $\mathcal{D}(K_1)^d$ are new two STD's.
(ii) $\mathcal{D}(K_2)$ have a regular automorphism group on both the point set and the block set. $\mathcal{D}(K_2)$ was constructed in [14].
(iii) B. Brock and A. Murray [3] constructed other two generalized Hadamard matrices K_3 and K_4 given in Appendix C. Let $\mathcal{D}(K_i)$ be the $\mathrm{STD}_7[21,3]$ corresponding to K_i for $i = 3, 4$. Then both $\mathcal{D}(K_3)$ and $\mathcal{D}(K_4)$ are selfdual and we have the following table.

i	$\lvert\mathrm{Aut}\mathcal{D}(K_i)\rvert$	sizes of orbits on Ω_i	sizes of orbits on Δ_i
3	12×3	(1,2,3,3,12)	(1,2,3,3,12)
4,	16×3	(1,4,8,8)	(1,4,8,8)

(iv) Therefore, if n_7 is the number of nonisomorphic $\mathrm{STD}_7[21,3]$'s, $n_7 \geqslant 5$.

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Appendix A

$$H_1 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\ \hline
0&0&1&1&2&2&0&2&2&1&0&1&1&0&1&2&2&0\\
0&2&2&1&0&1&0&1&0&2&2&1&1&1&0&0&2&2\\
0&1&0&2&2&1&0&0&1&1&2&2&2&0&0&1&2&1\\ \hline
0&0&1&2&1&2&1&0&0&2&2&1&0&0&2&1&1&2\\
0&2&2&1&1&0&1&2&1&2&0&0&2&0&0&2&1&1\\
0&1&0&1&2&2&1&1&2&0&2&0&2&1&2&0&1&0\\ \hline
0&0&1&2&2&1&2&1&1&2&0&0&1&2&2&0&0&1\\
0&2&2&0&1&1&2&0&2&1&0&1&2&1&2&1&0&0\\
0&1&0&2&1&2&2&2&0&1&1&0&1&1&0&2&0&2\\ \hline
0&1&2&0&2&1&1&2&0&0&2&1&0&2&1&2&0&1\\
0&1&2&2&1&0&0&1&2&1&0&2&0&2&1&0&1&2\\
0&1&2&1&0&2&2&0&1&2&1&0&0&2&1&1&2&0\\ \hline
0&2&1&0&1&2&0&2&1&0&1&2&0&1&2&0&2&1\\
0&2&1&1&2&0&2&1&0&0&1&2&2&0&1&1&0&2\\
0&2&1&2&0&1&1&0&2&0&1&2&1&2&0&2&1&0
\end{array}\right)$$

$$H_2 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\ \hline
0&0&1&1&2&2&0&2&2&1&0&1&1&0&1&2&2&0\\
0&2&2&1&0&1&0&1&0&2&2&1&1&1&0&0&2&2\\
0&1&0&2&2&1&0&0&1&1&2&2&2&0&0&1&2&1\\ \hline
0&0&1&2&1&2&1&0&0&2&2&1&0&0&2&1&1&2\\
0&2&2&1&1&0&1&2&1&2&0&0&2&0&0&2&1&1\\
0&1&0&1&2&2&1&1&2&0&2&0&2&1&2&0&1&0\\ \hline
0&0&1&2&2&1&2&1&1&2&0&0&1&2&2&0&0&1\\
0&2&2&0&1&1&2&0&2&1&0&1&2&1&2&1&0&0\\
0&1&0&2&1&2&2&2&0&1&1&0&1&1&0&2&0&2\\ \hline
0&1&2&0&2&1&1&2&0&1&0&2&0&2&1&0&1&2\\
0&1&2&2&1&0&0&1&2&2&1&0&0&2&1&1&2&0\\
0&1&2&1&0&2&2&0&1&0&2&1&0&2&1&2&0&1\\ \hline
0&2&1&0&1&2&0&2&1&0&1&2&0&1&2&0&2&1\\
0&2&1&1&2&0&2&1&0&0&1&2&2&0&1&1&0&2\\
0&2&1&2&0&1&1&0&2&0&1&2&1&2&0&2&1&0
\end{array}\right)$$

$$H_3 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\ \hline
0&0&1&1&2&2&0&2&2&1&0&1&2&2&1&0&1&0\\
0&2&2&1&0&1&0&1&0&2&2&1&1&2&2&0&0&1\\
0&1&0&2&2&1&0&0&1&1&2&2&1&0&1&0&2&2\\ \hline
0&0&1&2&1&2&1&0&0&2&2&1&0&1&1&2&0&2\\
0&2&2&1&1&0&1&2&1&2&0&0&1&0&1&2&2&0\\
0&1&0&1&2&2&1&1&2&0&2&0&0&0&2&2&1&1\\ \hline
0&0&1&2&2&1&2&1&1&2&0&0&0&2&0&1&2&1\\
0&2&2&0&1&1&2&0&2&1&0&1&0&0&2&1&1&2\\
0&1&0&2&1&2&2&2&0&1&1&0&1&2&2&1&0&0\\ \hline
0&1&2&0&2&1&1&2&0&0&2&1&2&1&0&1&2&0\\
0&1&2&2&1&0&0&1&2&1&0&2&2&1&0&2&0&1\\
0&1&2&1&0&2&2&0&1&2&1&0&2&1&0&0&1&2\\ \hline
0&2&1&0&1&2&0&2&1&0&1&2&0&1&2&0&2&1\\
0&2&1&1&2&0&2&1&0&0&1&2&2&0&1&1&0&2\\
0&2&1&2&0&1&1&0&2&0&1&2&1&2&0&2&1&0
\end{array}\right)$$

$$H_4 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 2 & 1 & 1 \\
0 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\
0 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 2 & 0 \\
\hline
0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 \\
0 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 0 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \\
\hline
0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 2 & 0 \\
0 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 1 \\
\hline
0 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 2 & 1 \\
0 & 2 & 1 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 0
\end{array}\right)$$

$$H_5 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\
0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 2 \\
0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 1 & 1 & 0 \\
\hline
0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 1 \\
0 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 2 \\
\hline
0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 0 & 0 \\
0 & 2 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 \\
0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 1 \\
\hline
0 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\
0 & 1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 \\
\hline
0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\
0 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 2 \\
0 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 1 & 0
\end{array}\right)$$

$$H_6 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
\hline
0 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 0 \\
0 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 2 \\
\hline
0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 0 & 2 \\
0 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 0 \\
0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \\
\hline
0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 \\
0 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\
\hline
0 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 0 \\
\hline
0 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 2 & 1 \\
0 & 2 & 1 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 0
\end{array}\right)$$

$$H_7 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&2&1&2&0&2&2&1&1&0&0&1&0&1&2&2\\
0&2&2&1&1&0&0&1&0&1&2&2&0&0&1&2&1&2\\
0&1&0&1&2&2&0&0&1&2&1&2&0&2&2&1&1&0\\
\hline
0&0&1&2&1&2&1&0&0&2&2&1&2&0&2&0&1&1\\
0&2&2&1&1&0&1&2&1&2&0&0&2&2&0&1&0&1\\
0&1&0&1&2&2&1&1&2&0&2&0&2&1&1&0&0&2\\
\hline
0&0&1&2&1&2&2&1&1&0&0&2&1&2&1&2&0&0\\
0&2&2&1&1&0&2&0&2&0&1&1&1&1&2&0&2&0\\
0&1&0&1&2&2&2&2&0&1&0&1&1&0&0&2&2&1\\
\hline
0&1&2&2&0&1&0&1&2&2&0&1&2&0&1&1&2&0\\
0&1&2&2&0&1&2&0&1&1&2&0&0&1&2&2&0&1\\
0&1&2&2&0&1&1&2&0&0&1&2&1&2&0&0&1&2\\
\hline
0&2&1&0&2&1&0&2&1&0&2&1&0&2&1&0&2&1\\
0&2&1&0&2&1&2&1&0&2&1&0&1&0&2&1&0&2\\
0&2&1&0&2&1&1&0&2&1&0&2&2&1&0&2&1&0
\end{array}\right)$$

$$H_8 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&2&1&2&0&2&2&1&1&0&0&2&2&1&1&0\\
0&2&2&1&1&0&0&1&0&1&2&2&2&0&2&0&1&1\\
0&1&0&1&2&2&0&0&1&2&1&2&1&1&2&0&2&0\\
\hline
0&0&1&2&1&2&1&0&0&2&2&1&2&1&1&0&0&2\\
0&2&2&1&1&0&1&2&1&2&0&0&1&2&1&2&0&0\\
0&1&0&1&2&2&1&1&2&0&2&0&0&0&1&2&1&2\\
\hline
0&0&1&2&1&2&2&1&1&0&0&2&1&0&0&2&2&1\\
0&2&2&1&1&0&2&0&2&0&1&1&0&1&0&1&2&2\\
0&1&0&1&2&2&2&2&0&1&0&1&2&2&0&1&0&1\\
\hline
0&1&2&2&0&1&0&1&2&2&0&1&0&1&2&2&0&1\\
0&1&2&2&0&1&2&0&1&1&2&0&1&2&0&0&1&2\\
0&1&2&2&0&1&1&2&0&0&1&2&2&0&1&1&2&0\\
\hline
0&2&1&0&2&1&0&2&1&0&2&1&0&2&1&0&2&1\\
0&2&1&0&2&1&2&1&0&2&1&0&1&0&2&1&0&2\\
0&2&1&0&2&1&1&0&2&1&0&2&2&1&0&2&1&0
\end{array}\right)$$

$$H_9 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&1&2&2&1&0&0&2&2&1&2&1&2&0&1&0\\
0&2&2&1&0&1&1&2&1&2&0&0&2&2&1&0&0&1\\
0&1&0&2&2&1&1&1&2&0&2&0&0&1&1&0&2&2\\
\hline
0&0&1&2&1&2&2&1&1&2&0&0&1&1&0&2&0&2\\
0&2&2&1&1&0&2&0&2&1&0&1&0&1&1&2&2&0\\
0&1&0&1&2&2&2&2&0&1&1&0&0&2&0&2&1&1\\
\hline
0&0&1&2&2&1&0&2&2&1&0&1&2&0&0&1&2&1\\
0&2&2&0&1&1&0&1&0&2&2&1&0&2&0&1&1&2\\
0&1&0&2&1&2&0&0&1&1&2&2&2&2&1&1&0&0\\
\hline
0&1&2&0&2&1&2&0&1&2&1&0&1&0&2&1&2&0\\
0&1&2&2&1&0&1&2&0&0&2&1&1&0&2&2&0&1\\
0&1&2&1&0&2&0&1&2&1&0&2&1&0&2&0&1&2\\
\hline
0&2&1&0&1&2&0&2&1&0&1&2&0&1&2&0&2&1\\
0&2&1&1&2&0&2&1&0&0&1&2&2&0&1&1&0&2\\
0&2&1&2&0&1&1&0&2&0&1&2&1&2&0&2&1&0
\end{array}\right)$$

$$
H_{10} = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&2&1&2&0&2&2&1&1&0&0&2&2&1&1&0\\
0&2&2&1&1&0&0&1&0&1&2&2&2&0&2&0&1&1\\
0&1&0&1&2&2&0&0&1&2&1&2&1&1&2&0&2&0\\
\hline
0&0&1&2&1&2&1&0&0&2&2&1&2&1&1&0&0&2\\
0&2&2&1&1&0&1&2&1&2&0&0&1&2&1&2&0&0\\
0&1&0&1&2&2&1&1&2&0&2&0&0&0&1&2&1&2\\
\hline
0&0&1&2&1&2&2&1&1&0&0&2&1&0&0&2&2&1\\
0&2&2&1&1&0&2&0&2&0&1&1&0&1&0&1&2&2\\
0&1&0&1&2&2&2&2&0&1&0&1&2&2&0&1&0&1\\
\hline
0&1&2&2&0&1&0&1&2&2&0&1&2&0&1&1&2&0\\
0&1&2&2&0&1&2&0&1&1&2&0&0&1&2&2&0&1\\
0&1&2&2&0&1&1&2&0&0&1&2&1&2&0&0&1&2\\
\hline
0&2&1&0&2&1&0&2&1&0&2&1&0&2&1&0&2&1\\
0&2&1&0&2&1&2&1&0&2&1&0&1&0&2&1&0&2\\
0&2&1&0&2&1&1&0&2&1&0&2&2&1&0&2&1&0
\end{array}\right)
$$

$$
H_{11} = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&2&2&2&2&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2\\
\hline
0&0&1&2&1&2&0&2&2&1&1&0&1&2&1&2&0&0\\
0&2&2&1&1&0&0&1&0&1&2&2&1&1&2&0&2&0\\
0&1&0&1&2&2&0&0&1&2&1&2&1&0&0&2&2&1\\
\hline
0&0&1&2&1&2&1&0&0&2&2&1&0&1&0&1&2&2\\
0&2&2&1&1&0&1&2&1&2&0&0&0&0&1&2&1&2\\
0&1&0&1&2&2&1&1&2&0&2&0&0&2&2&1&1&0\\
\hline
0&0&1&2&1&2&2&1&1&0&0&2&2&0&2&0&1&1\\
0&2&2&1&1&0&2&0&2&0&1&1&2&2&0&1&0&1\\
0&1&0&1&2&2&2&2&0&1&0&1&2&1&1&0&0&2\\
\hline
0&1&2&2&0&1&0&1&2&2&0&1&0&1&2&2&0&1\\
0&1&2&2&0&1&2&0&1&1&2&0&1&2&0&0&1&2\\
0&1&2&2&0&1&1&2&0&0&1&2&2&0&1&1&2&0\\
\hline
0&2&1&0&2&1&0&2&1&0&2&1&0&2&1&0&2&1\\
0&2&1&0&2&1&2&1&0&2&1&0&1&0&2&1&0&2\\
0&2&1&0&2&1&1&0&2&1&0&2&2&1&0&2&1&0
\end{array}\right)
$$

Appendix B

$$
K_1 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&2&1&0&2&1&0&2&1&0&2&2&2&2&1&1&1\\
0&0&0&0&1&2&0&1&2&0&1&2&0&1&2&1&1&1&2&2&2\\
\hline
0&0&1&0&0&1&2&2&2&2&2&0&1&2&0&1&0&1&2&1&1\\
0&2&2&1&2&1&2&1&0&0&1&0&0&0&0&2&2&1&1&2&1\\
0&1&0&0&2&2&2&0&1&2&1&1&2&1&0&1&2&2&0&0&1\\
\hline
0&0&1&0&2&1&1&1&1&1&2&1&2&0&2&2&0&0&0&2&2\\
0&2&2&1&2&0&1&0&2&1&0&0&2&1&1&1&0&1&2&0&2\\
0&1&0&2&2&2&1&2&0&2&2&0&0&0&1&1&1&0&1&1&2\\
\hline
0&0&1&1&2&0&2&2&1&0&1&2&2&0&1&0&1&2&2&1&0\\
0&2&2&0&0&0&1&1&0&2&2&2&1&1&1&0&1&2&0&2&1\\
0&1&0&2&1&0&2&2&1&1&0&2&0&2&1&2&0&1&0&2&1\\
\hline
0&0&1&2&1&1&0&0&0&2&2&2&2&1&1&2&2&1&1&0&0\\
0&2&2&1&1&2&0&2&1&2&1&0&1&1&2&2&0&0&0&1&0\\
0&1&0&1&2&1&0&1&2&2&0&1&1&2&1&0&2&0&2&2&0\\
\hline
0&1&2&0&0&0&0&2&0&1&1&1&2&2&2&0&2&1&1&1&2\\
0&1&2&2&1&0&2&1&2&0&2&1&1&0&2&1&0&2&1&0&0\\
0&1&2&2&0&1&1&0&1&0&1&2&1&2&0&2&1&0&2&0&2\\
\hline
0&2&1&1&1&2&2&0&0&1&2&1&0&2&2&0&1&0&2&0&1\\
0&2&1&2&0&2&0&1&1&1&0&0&2&2&0&1&1&2&1&2&0\\
0&2&1&2&1&1&1&2&2&0&0&1&0&1&0&0&2&2&0&1&2
\end{array}\right)
$$

$$
K_2 = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
\hline
0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 \\
0 & 2 & 2 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 0 \\
0 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 0 \\
\hline
0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 2 & 2 & 0 \\
0 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 2 & 2 \\
0 & 1 & 0 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 \\
\hline
0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 2 & 2 & 1 & 1 \\
0 & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
0 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 0 & 2 & 0 \\
\hline
0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 1 & 2 \\
0 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 2 \\
\hline
0 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & 2 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 2 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 0 \\
0 & 2 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 1 & 2
\end{array}\right)
$$

Appendix C

$$
K_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 2 & 2 & 1 \\
0 & 0 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 2 \\
0 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 2 & 1 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 1 & 2 & 0 \\
0 & 1 & 2 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 0 \\
0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\
0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 0 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 0 & 1 \\
0 & 2 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 1 & 1 \\
0 & 2 & 2 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 0 \\
0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 \\
0 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 0 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
0 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 & 0 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 1 & 1 & 0 & 2 & 0 \\
0 & 1 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 \\
0 & 2 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 2 & 1
\end{pmatrix}
$$

$$
K_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 \\
0 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 2 \\
0 & 1 & 2 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 2 & 1 & 1 \\
0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 \\
0 & 1 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 2 & 0 \\
0 & 1 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\
0 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 2 & 1 & 1 & 0 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 \\
0 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 1 \\
0 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 2 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\
0 & 2 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\
0 & 2 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 1 & 2 & 2 \\
0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & 2 & 0 & 1 \\
0 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & 0
\end{pmatrix}
$$