The valuations of the near polygon $\mathbb{G}_n$

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Abstract

We show that every valuation of the near $2n$-gon $\mathbb{G}_n$, $n \geqslant 2$, is induced by a unique classical valuation of the dual polar space $DH(2n-1,4)$ into which $\mathbb{G}_n$ is isometrically embeddable.

1 Basic definitions and main results

A *near polygon* is a connected partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point x and every line L, there exists a unique point on L nearest to x. Here, distances $\mathrm{d}(\cdot,\cdot)$ are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles. If X_1 and X_2 are two nonempty sets of points of $\mathcal{S}$, then $\mathrm{d}(X_1, X_2)$ denotes the smallest distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x\}$, then we will also write $\mathrm{d}(x, X_2)$ instead of $\mathrm{d}(\{x\}, X_2)$. For every $i \in \mathbb{N}$ and every nonempty set X of points of $\mathcal{S}$, $\Gamma_i(X)$ denotes the set of all points $x \in X$ for which $\mathrm{d}(x, X) = i$. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

Let $\mathcal{S}$ be a near polygon. A set X of points of $\mathcal{S}$ is called a *subspace* if every line of $\mathcal{S}$ having two of its points in X has all its points in X. If X is a subspace, then we denote by $\widetilde{X}$ the subgeometry of $\mathcal{S}$ induced on the point set X by those lines of $\mathcal{S}$ which have all their points in X. A set X of points of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X. If X is a non-empty convex subspace of $\mathcal{S}$, then $\widetilde{X}$ is also a near polygon. Clearly, the intersection of any number of (convex) subspaces is again a (convex) subspace. If $*_1, *_2, \ldots, *_k$ are $k \geqslant 1$ objects (i.e., points or nonempty sets of points) of $\mathcal{S}$, then $\langle *_1, *_2, \ldots, *_k \rangle$ denotes the smallest convex subspace

*Postdoctoral Fellow of the Research Foundation - Flanders of $\mathcal{S}$ containing $*_1, *_2, \ldots, *_k$. The set $\langle *_1, *_2, \ldots, *_k \rangle$ is well-defined since it equals the intersection of all convex subspaces containing $*_1, *_2, \ldots, *_k$.

A near polygon $\mathcal{S}$ is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. If x and y are two points of a dense near polygon $\mathcal{S}$ at distance δ from each other, then by Brouwer and Wilbrink [6, Theorem 4], $\langle x, y \rangle$ is the unique convex subspace of diameter δ containing x and y. The convex subspace $\langle x, y \rangle$ is called a *quad* if $\delta = 2$, a *hex* if $\delta = 3$ and a *max* if $\delta = n-1$. We will now describe two classes of dense near polygons.

(I) Let $n \geqslant 2$, let $\mathbb{K}'$ be a field with involutory automorphism ψ and let $\mathbb{K}$ denote the fix field of ψ. Let V be a $2n$-dimensional vector space over $\mathbb{K}'$ equipped with a nondegenerate skew-ψ-Hermitian form f_V of maximal Witt index n. The subspaces of V which are totally isotropic with respect to f_V define a Hermitian polar space $H(2n-1, \mathbb{K}'/\mathbb{K})$. We denote the corresponding Hermitian dual polar space by $DH(2n-1, \mathbb{K}'/\mathbb{K})$. So, $DH(2n-1, \mathbb{K}'/\mathbb{K})$ is the point-line geometry whose points, respectively lines, are the n-dimensional, respectively $(n-1)$-dimensional, subspaces of V which are totally isotropic with respect to f_V, with incidence being reverse containment. The dual polar space $DH(2n-1, \mathbb{K}'/\mathbb{K})$ is a dense near $2n$-gon. In the finite case, we have $\mathbb{K} \cong \mathbb{F}_q$ and $\mathbb{K}' \cong \mathbb{F}_{q^2}$ for some prime power q. In this case, we will denote $DH(2n-1, \mathbb{K}'/\mathbb{K})$ also by $DH(2n-1, q^2)$. The dual polar space $DH(3, q^2)$ is isomorphic to the generalized quadrangle $Q^-(5, q)$ described in Payne and Thas [24, Section 3.1].

(II) Let $n \geqslant 2$, let V be a $2n$-dimensional vector space over $\mathbb{F}_4$ with basis $B = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n}\}$. The *support* of a vector $\bar{x} = \sum_{i=1}^{2n} \lambda_i \bar{e}_i$ of V is the set of all $i \in \{1, \ldots, 2n\}$ satisfying $\lambda_i \neq 0$; the cardinality of the support of $\bar{x}$ is called the *weight* of $\bar{x}$. Now, we can define the following point-line geometry $\mathbb{G}_n(V, B)$. The points of $\mathbb{G}_n(V, B)$ are the n-dimensional subspaces of V which are generated by n vectors of weight 2 whose supports are two by two disjoint. The lines of $\mathbb{G}_n(V, B)$ are of two types:

(a) *Special lines*: these are $(n-1)$-dimensional subspaces of V which are generated by $n-1$ vectors of weight 2 whose supports are two by two disjoint.

(b) *Ordinary lines:* these are $(n-1)$-dimensional subspaces of V which are generated by $n-2$ vectors of weight 2 and 1 vector of weight 4 such that the $n-1$ supports associated with these vectors are mutually disjoint.

Incidence is reverse containment. By De Bruyn [10] (see also [11, Section 6.3]), the geometry $\mathbb{G}_n(V, B)$ is a dense near $2n$-gon with three points on each line. The isomorphism class of the geometry $\mathbb{G}_n(V, B)$ is independent from the vector space V and the basis B of V. We will denote by $\mathbb{G}_n$ any suitable element of this isomorphism class. The near polygon $\mathbb{G}_2$ is isomorphic to the generalized quadrangle $Q^-(5, 2)$.

Now, endow the vector space V with the (skew-)Hermitian form f_V which is linear in the first argument, semi-linear in the second argument and which satisfies $f_V(\bar{e}_i, \bar{e}_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, 2n\}$. With the pair (V, f_V), there is associated a Hermitian dual polar space $DH(V, B) \cong DH(2n-1, 4)$, and every point of $\mathbb{G}_n(V, B)$ is also a point of $DH(V, B)$. By [10] or [11, Section 6.3], the set X of points of $\mathbb{G}_n(V, B)$ is a subspace of $DH(V, B)$ and the following two properties hold:

(1) $\widetilde{X} = \mathbb{G}_n(V, B)$;

(2) If x and y are two points of X, then the distance between x and y in $\widetilde{X}$ equals the distance between x and y in $DH(V, B)$.

Properties (1) and (2) imply that the near polygon $\mathbb{G}_n$ admits a full and isometric embedding into the dual polar space $DH(2n-1, 4)$. It can be shown that there exists up to isomorphism a unique such isometric embedding, see De Bruyn [16].

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon. A function $f : \mathcal{P} \to \mathbb{N}$ is called a *valuation* of $\mathcal{S}$ if it satisfies the following properties:

(V1) $f^{-1}(0) \neq \emptyset$.

(V2) Every line L contains a unique point x_L with smallest f-value and $f(x) = f(x_L) + 1$ for every point $x \in L \setminus \{x_L\}$.

(V3) Through every point x of $\mathcal{S}$, there exists a (necessarily unique) convex subspace F_x such that the following holds for any point y of F_x: (i) $f(y) \leqslant f(x)$; (ii) if z is a point collinear with y such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [18] and are a very important tool for classifying dense near polygons. For several classes of dense near polygons, see De Bruyn [14, Corollary 1.4], it can be shown that Property (V3) is a consequence of Property (V2). This is also the case for the Hermitian dual polar space $DH(2n-1, \mathbb{K}'/\mathbb{K})$ and the dense near polygon $\mathbb{G}_n$ ($n \geqslant 2$). We now describe two classes of valuations of a dense near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ which were also mentioned in [18].

(1) For every point x of $\mathcal{S}$, the map $\mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x, y)$ is a valuation of $\mathcal{S}$. This valuation is called the *classical valuation of* $\mathcal{S}$ *with center* x.

(2) Suppose F is a (not necessarily convex) subspace of $\mathcal{S}$ satisfying the following properties: (i) $\widetilde{F}$ is a dense near polygon; (ii) if x and y are two points of F, then the distance between x and y in $\widetilde{F}$ equals the distance between x and y in $\mathcal{S}$. If f is a valuation of $\mathcal{S}$ and if $m = \min\{f(y) \,|\, y \in F\}$, then the map $F \to \mathbb{N}; x \mapsto f(x) - m$ is a valuation of $\widetilde{F}$. This valuation is called the *valuation of* $\widetilde{F}$ *induced by* f.

By Theorem 6.8 of De Bruyn [11], every valuation of the dual polar space $DH(2n-1, 4)$, $n \geqslant 2$, is classical. What about valuations of the near polygon $\mathbb{G}_n$? If we regard $\mathbb{G}_n$ as a subgeometry of $DH(2n-1, 4)$ which is isometricaly embedded into $DH(2n-1, 4)$, then we know by the above discussion that every (classical) valuation of $DH(2n-1, 4)$ will induce a valuation of $\mathbb{G}_n$. Is the converse also true: is every valuation of $\mathbb{G}_n$ induced by some valuation of $DH(2n-1, 4)$? The main result of this paper gives a positive answer to this question.

Theorem 1.1 *Regard $\mathbb{G}_n$, $n \geqslant 2$, as a subgeometry of $DH(2n-1, 4)$ which is isometrically embedded into $DH(2n-1, 4)$. Then every valuation of $\mathbb{G}_n$ is induced by a unique (classical) valuation of $DH(2n-1, 4)$.*

We will prove Theorem 1.1 by induction on n. The case $n = 2$ is trivial since $\mathbb{G}_2 \cong Q^-(5,2) \cong DH(3,4)$. The cases $n = 3$ and $n = 4$ were respectively treated in De Bruyn & Vandecasteele [19, Proposition 7.7] and [21, Proposition 6.13]. We will make use of the results of [21] to obtain a proof of Theorem 1.1 for any $n \geqslant 5$.

Definition. Two valuations f_1 and f_2 of a dense near polygon $\mathcal{S}$ are called *neighboring valuations* if there exists an $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ for every point x of $\mathcal{S}$. If this condition holds, then we necessarily have $\epsilon \in \{-1, 0, 1\}$, see Proposition 2.6.

We will also prove the following.

Theorem 1.2 *Regard $\mathbb{G}_n$, $n \geqslant 2$, as a subgeometry of $DH(2n-1,4)$ which is isometrically embedded into $DH(2n-1,4)$. Let f_1 and f_2 be two distinct valuations of $\mathbb{G}_n$ and let x_i, $i \in \{1,2\}$, denote the unique point of $DH(2n-1,4)$ such that the valuation f_i of $\mathbb{G}_n$ is induced by the classical valuation of $DH(2n-1,4)$ with center x_i. Then the following are equivalent:*

(1) *f_1 and f_2 are neighboring valuations of $\mathbb{G}_n$;*

(2) *x_1 and x_2 are collinear.*

2 (Semi-)Valuations

2.1 Semi-valuations of general point-line geometries

Throughout this subsection, we suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a connected partial linear space.

Definitions. (1) A semi-valuation of $\mathcal{S}$ is a map $f : \mathcal{P} \to \mathbb{Z}$ such that for every line L of $\mathcal{S}$, there exists a unique point x_L on L such that $f(x) = f(x_L) + 1$ for every point x of L distinct from x_L.

(2) It is possible to define an equivalence relation on the set of all semi-valuations of $\mathcal{S}$: two semi-valuations f_1, f_2 of $\mathcal{S}$ are called *equivalent* if there exists an $\epsilon \in \mathbb{Z}$ such that $f_2(x) = f_1(x) + \epsilon$ for every point x of $\mathcal{S}$. The equivalence class containing the semi-valuation f of $\mathcal{S}$ will be denoted by $[f]$.

(3) A hyperplane of $\mathcal{S}$ is a proper subspace meeting each line of $\mathcal{S}$. If f is a semi-valuation of $\mathcal{S}$ attaining a maximal value, then the set of points of $\mathcal{S}$ with non-maximal f-value is a hyperplane H_f of $\mathcal{S}$. If f_1 and f_2 are two equivalent semi-valuations of $\mathcal{S}$ attaining a maximal value, then $H_{f_1} = H_{f_2}$.

(4) Two semi-valuations f_1 and f_2 of $\mathcal{S}$ are called *neighboring semi-valuations* if there exists an $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ for every point x of $\mathcal{S}$.

Lemma 2.1 *Suppose f_1 and f_2 are two neighboring semi-valuations of $\mathcal{S}$ and let $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ for every point x of $\mathcal{S}$. Then the following holds:*

(1) *If the set $\{f_1(x) \,|\, x \in \mathcal{P}\}$ has a minimal element m_1, then the set $\{f_2(x) \,|\, x \in \mathcal{P}\}$ has a minimal element m_2 and $|m_1 - m_2 + \epsilon| \leqslant 1$.*

(2) *If the set* $\{f_1(x) \mid x \in \mathcal{P}\}$ *has a maximal element* M_1*, then the set* $\{f_2(x) \mid x \in \mathcal{P}\}$ *has a maximal element* M_2 *and* $|M_1 - M_2 + \epsilon| \leqslant 1$*.*

(3) *If* L *is a line of* $\mathcal{S}$ *such that the unique point* x_1 *of* L *with smallest* f_1*-value is distinct from the unique point* x_2 *of* L *with smallest* f_2*-value, then* $\epsilon = f_2(x_2) - f_1(x_1)$*.*

Proof. Clearly, $f_1(x) + \epsilon - 1 \leqslant f_2(x) \leqslant f_1(x) + \epsilon + 1$ for every point x of $\mathcal{S}$. So, if the set $\{f_1(x) \mid x \in \mathcal{P}\}$ has a minimal (respectively maximal) element, then also the set $\{f_2(x) \mid x \in \mathcal{P}\}$ has a minimal (respectively maximal) element.

(1) If $m_1 - m_2 + \epsilon \leqslant -2$, then for every point x with f_1-value m_1, we have $f_1(x) - f_2(x) + \epsilon = m_1 - f_2(x) + \epsilon \leqslant m_1 - m_2 + \epsilon \leqslant -2$, a contradiction. If $m_1 - m_2 + \epsilon \geqslant 2$, then for every point x with f_2-value m_2, we have $f_1(x) - f_2(x) + \epsilon = f_1(x) - m_2 + \epsilon \geqslant m_1 - m_2 + \epsilon \geqslant 2$, a contradiction. Hence, $|m_1 - m_2 + \epsilon| \leqslant 1$.

(2) If $M_1 - M_2 + \epsilon \geqslant 2$, then for every point x with f_1-value M_1, we have $f_1(x) - f_2(x) + \epsilon = M_1 - f_2(x) + \epsilon \geqslant M_1 - M_2 + \epsilon \geqslant 2$, a contradiction. If $M_1 - M_2 + \epsilon \leqslant -2$, then for every point x with f_2-value M_2, we have $f_1(x) - f_2(x) + \epsilon = f_1(x) - M_2 + \epsilon \leqslant M_1 - M_2 + \epsilon \leqslant -2$, a contradiction. Hence, $|M_1 - M_2 + \epsilon| \leqslant 1$.

(3) Since $f_1(x_1) - f_2(x_1) = f_1(x_1) - f_2(x_2) - 1$ and $f_1(x_2) - f_2(x_2) = f_1(x_1) - f_2(x_2) + 1$, we necessarily have that $\epsilon = f_2(x_2) - f_1(x_1)$. ∎

Lemma 2.2 *Let* f_1 *and* f_2 *be two semi-valuations of* $\mathcal{S}$ *satisfying the following property:*

> $(*)$ *For every line* L *of* $\mathcal{S}$*, the unique point of* L *with smallest* f_1*-value coincides with the unique point of* L *with smallest* f_2*-value.*

Then f_1 *and* f_2 *are equivalent.*

Proof. Let x^* be an arbitrary point of $\mathcal{S}$ and put $\epsilon := f_2(x^*) - f_1(x^*)$. We prove by induction on the distance $\mathrm{d}(x^*, x)$ that $f_2(x) = f_1(x) + \epsilon$ for every point x of $\mathcal{S}$. Obviously, this holds if $x = x^*$. So, suppose $\mathrm{d}(x^*, x) \geqslant 1$ and let y be a point collinear with x at distance $\mathrm{d}(x^*, x) - 1$ from x^*. By the induction hypothesis, $f_2(y) = f_1(y) + \epsilon$. Applying property $(*)$ to the line xy, we find that $f_2(x) = f_1(x) + \epsilon$. ∎

The following is an immediate corollary of Lemma 2.1(3) and Lemma 2.2.

Corollary 2.3 *The following holds for two neighboring semi-valuations* f_1 *and* f_2 *of* $\mathcal{S}$*.*

(1) *If* f_1 *and* f_2 *are equivalent, then there exist precisely three* $\epsilon \in \mathbb{Z}$ *such that* $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ *for every point* x *of* $\mathcal{S}$*. These three possible values of* ϵ *are consecutive integers.*

(2) *Suppose* f_1 *and* f_2 *are not equivalent. Then there exists a unique* $\epsilon \in \mathbb{Z}$ *such that* $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ *for every point* x *of* $\mathcal{S}$*. There also exists a line* L *of* $\mathcal{S}$ *such that the unique point* x_1 *of* L *with smallest* f_1*-value is distinct from the unique point* x_2 *of* L *with smallest* f_2*-value. Moreover,* $\epsilon = f_2(x_2) - f_1(x_1)$*.*

For the remainder of this subsection, we suppose that every line of $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is incident with precisely 3 points.

Definition. Suppose $f_1 : \mathcal{P} \to \mathbb{Z}$ and $f_2 : \mathcal{P} \to \mathbb{Z}$ are two maps such that $|f_1(x) - f_2(x)| \leqslant 1$ for every point $x \in \mathcal{P}$. If $f_1(x) = f_2(x)$, then we define $f_1 \diamond f_2(x) := f_1(x) - 1 = f_2(x) - 1$. If $|f_1(x) - f_2(x)| = 1$, then we define $f_1 \diamond f_2(x) := \max\{f_1(x), f_2(x)\}$. Clearly, $f_2 \diamond f_1 = f_1 \diamond f_2$. Notice also that $|f_1(x) - f_1 \diamond f_2(x)|, |f_2(x) - f_1 \diamond f_2(x)| \leqslant 1$ for every point x of $\mathcal{S}$. Moreover $(f_1 \diamond f_2) \diamond f_1 = f_2$ and $(f_1 \diamond f_2) \diamond f_2 = f_1$.

Proposition 2.4 *If f_1 and f_2 are two semi-valuations of $\mathcal{S}$ such that $|f_1(u) - f_2(u)| \leqslant 1$ for every point u of $\mathcal{S}$, then also $f_3 := f_1 \diamond f_2$ is a semi-valuation of $\mathcal{S}$. If two semi-valuations of the set $\{f_1, f_2, f_3\}$ are equivalent, then all of them are equivalent. If this occurs, then two of them, say f_{i_1} and f_{i_2}, are equal and the third one f_{i_3} satisfies $f_{i_3}(x) = f_{i_1}(x) - 1 = f_{i_2}(x) - 1$ for every point x of $\mathcal{S}$.*

Proof. Let $L = \{x, y, z\}$ be an arbitrary line of $\mathcal{S}$. Without loss of generality, we may suppose that one of the following cases occurs:

(1) x is the unique point of L with smallest f_1-value and smallest f_2-value. If $f_1(x) = f_2(x)$, then $f_3(x) = f_1(x) - 1$ and $f_3(y) = f_3(z) = f_1(x)$. If $f_1(x) \neq f_2(x)$, then $f_3(x) = \max\{f_1(x), f_2(x)\}$ and $f_3(y) = f_3(z) = \max\{f_1(x) + 1, f_2(x) + 1\} = f_3(x) + 1$.

(2) x is the unique point of L with smallest f_1-value and y is the unique point of L with smallest f_2-value. The fact that $|f_1(u) - f_2(u)| \leqslant 1$ for every $u \in L$ implies that $f_1(x) = f_2(y)$. Since $f_2(x) = f_2(y) + 1 = f_1(x) + 1$, we have $f_3(x) = f_1(x) + 1$. Since $f_1(y) = f_1(x) + 1$ and $f_2(y) = f_1(x)$, we have $f_3(y) = f_1(x) + 1$. Since $f_1(z) = f_1(x) + 1$ and $f_2(z) = f_2(y) + 1 = f_1(x) + 1$, we have $f_3(z) = f_1(x)$.

In both cases, L contains a unique point with smallest f_3-value. So, f_3 is a semi-valuation. From the definition of the map $f_1 \diamond f_2$, it follows that if f_1 and f_2 are equivalent, then $f_3 = f_1 \diamond f_2$ is equivalent with f_1 and f_2. So, if f_1 and f_3 are equivalent, then $f_3 \diamond f_1 = (f_1 \diamond f_2) \diamond f_1 = f_2$ is equivalent with f_1 and f_3, and if f_2 and f_3 are equivalent, then $f_3 \diamond f_2 = (f_1 \diamond f_2) \diamond f_2 = f_1$ is equivalent with f_2 and f_3. ■

Definition. Suppose f_1 and f_2 are two neighboring semi-valuations of $\mathcal{S}$. Then we define $[f_1] * [f_2] := [g_1 \diamond g_2]$ where $g_1 \in [f_1]$ and $g_2 \in [f_2]$ are chosen such that $|g_1(x) - g_2(x)| \leqslant 1$ for every point x of $\mathcal{S}$. Using Corollary 2.3, it is straightforward to verify that $[g_1 \diamond g_2]$ is independent from the chosen $g_1 \in [f_1]$ and $g_2 \in [f_2]$ satisfying $|g_1(x) - g_2(x)| \leqslant 1$, $\forall x \in \mathcal{P}$. Notice also that f_1, f_2 and $g_1 \diamond g_2$ are three mutually neighboring semi-valuations of $\mathcal{S}$. For every semi-valuation f of $\mathcal{S}$, we have $[f] * [f] = [f]$.

Notice that if H_1 and H_2 are two distinct hyperplanes of $\mathcal{S}$, then the complement of the symmetric difference of H_1 and H_2 is again a hyperplane of $\mathcal{S}$.

Proposition 2.5 *Suppose f_1, f_2 and f_3 are three mutually neighboring semi-valuations of $\mathcal{S}$ such that $[f_3] = [f_1] * [f_2]$. Suppose also that at least one (and hence all) of f_1, f_2, f_3 attains a maximal value. Then precisely one of the following cases occurs:*

(1) *$H_{f_1} \neq H_{f_2}$ and H_{f_3} is the complement of the symmetric difference $H_{f_1} \Delta H_{f_2}$ of H_{f_1} and H_{f_2}.*

(2) *One of H_{f_1}, H_{f_2} is properly contained in the other, and H_{f_3} is the larger of the two.*

(3) *H_{f_3} is (properly or improperly) contained in $H_{f_1} = H_{f_2}$.*

Proof. Without loss of generality, we may suppose that $|f_1(x) - f_2(x)| \leqslant 1$ for every point x of $\mathcal{S}$ and $f_3 = f_1 \diamond f_2$. Let M_i, $i \in \{1,2,3\}$, denote the maximal value attained by f_i. By Lemma 2.1(2), $|M_1 - M_2| \leqslant 1$. Without loss of generality, we may suppose that $M_2 \geqslant M_1$.

(a) Suppose that $M_1 = M_2$. If $x \in H_{f_1} \cap H_{f_2}$, then since $f_1(x), f_2(x) \leqslant M_1 - 1$, we have $f_3(x) \leqslant M_1 - 1$. If $x \in H_{f_1} \setminus H_{f_2}$, then since $f_1(x) \leqslant M_1 - 1$ and $f_2(x) = M_1$, we have $f_1(x) = M_1 - 1$ and $f_3(x) = M_1$. Similarly, if $x \in H_{f_2} \setminus H_{f_1}$, then $f_3(x) = M_1$. Finally, if $x \notin H_{f_1} \cup H_{f_2}$, then since $f_1(x) = f_2(x) = M_1$, we have $f_3(x) = M_1 - 1$. If $H_{f_1} \neq H_{f_2}$, then $M_3 = M_1$ and H_{f_3} is the complement of the symmetric difference of H_{f_1} and H_{f_2}. If $H_{f_1} = H_{f_2}$, then $M_3 = M_1 - 1$ and H_{f_3} is contained in $H_{f_1} = H_{f_2}$.

(b) Suppose that $M_2 = M_1 + 1$. Then $H_{f_1} \subseteq H_{f_2}$ since every point of H_{f_1} has f_1-value at most $M_1 - 1$ and hence f_2-value at most $M_1 < M_2$. If $x \in H_{f_2}$, then since $f_1(x), f_2(x) \leqslant M_1$, we have $f_3(x) \leqslant M_1$. If $x \notin H_{f_2}$, then since $f_1(x) = M_1$ and $f_2(x) = M_2 = M_1 + 1$, we have $f_3(x) = M_1 + 1$. So, $M_3 = M_1 + 1$ and $H_{f_3} = H_{f_2}$. If $H_{f_1} \neq H_{f_2}$, then case (2) of the proposition occurs. If $H_{f_1} = H_{f_2}$, then case (3) occurs. ■

2.2 Valuations of dense near polygons

In this section, we suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near $2n$-gon. Since every valuation of $\mathcal{S}$ is also a semi-valuation, the definitions and results of Section 2.1 also apply to valuations of $\mathcal{S}$.

Proposition 2.6 *If f_1 and f_2 are two neighboring valuations of $\mathcal{S}$ and if $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leqslant 1$ for every point x of $\mathcal{S}$, then $\epsilon \in \{-1, 0, 1\}$.*

Proof. This is a special case of Lemma 2.1(1). ■

Proposition 2.7 *If f_1 and f_2 are two valuations of $\mathcal{S}$, then $f_1 = f_2$ if and only if $H_{f_1} = H_{f_2}$.*

Proof. Obviously, $H_{f_1} = H_{f_2}$ if $f_1 = f_2$. We will now also prove that $f_1 = f_2$ if $H_{f_1} = H_{f_2}$.

Let $i \in \{1,2\}$. Let M_i denote the maximal value attained by f_i. Then the complement $\overline{H_{f_i}}$ of H_{f_i} consists of those points of $\mathcal{S}$ with f_i-value M_i. By Property (V2), $\mathrm{d}(x, \overline{H_{f_i}}) \geqslant M_i - f_i(x)$ for every point x of $\mathcal{S}$ (consider a shortest path between x and $\overline{H_{f_i}}$). We will now prove by induction on $M_i - f_i(x)$ that $\mathrm{d}(x, \overline{H_{f_i}}) = M_i - f_i(x)$ for every point x of $\mathcal{S}$. Obviously, this holds if $M_i - f_i(x) = 0$ since $x \in \overline{H_{f_i}}$ in this case. So, suppose that $M_i - f_i(x) > 0$. Let F_x denote the convex subspace through x as mentioned in Property (V3). Then $f_i(y) \leqslant f_i(x) \leqslant M_i - 1$ for every point y of F_x. So, $F_x \neq \mathcal{S}$ and there exists a line L through x not contained in F_x. By Property (V3), L contains a point x' with f_i-value $f_i(x) + 1$. By the induction hypothesis, $\mathrm{d}(x', \overline{H_{f_i}}) = M_i - f_i(x') = M_i - f_i(x) - 1$. Hence, $\mathrm{d}(x, \overline{H_{f_i}}) \leqslant M_i - f_i(x)$. Together with $\mathrm{d}(x, \overline{H_{f_i}}) \geqslant M_i - f_i(x)$, this implies that $\mathrm{d}(x, \overline{H_{f_i}}) = M_i - f_i(x)$.

Now, suppose $H_{f_1} = H_{f_2}$. Then $M_1 = \max\{\mathrm{d}(y, \overline{H_{f_1}}) \,|\, y \in \mathcal{P}\} = \max\{\mathrm{d}(y, \overline{H_{f_2}}) \,|\, y \in \mathcal{P}\} = M_2$ and $f_1(x) = M_1 - \mathrm{d}(x, \overline{H_{f_1}}) = M_2 - \mathrm{d}(x, \overline{H_{f_2}}) = f_2(x)$ for every point x of $\mathcal{S}$. ■

The proof of the following proposition is straightforward.

Proposition 2.8 *Let F be a subspace of $\mathcal{S}$, isometrically embedded in $\mathcal{S}$, such that $\widetilde{F}$ is a dense near polygon. Let f_1 and f_2 be two neighboring valuations of $\mathcal{S}$ and let f_i', $i \in \{1,2\}$, denote the valuation of $\widetilde{F}$ induced by f_i. Then f_1' and f_2' are neighboring valuations of $\widetilde{F}$.*

Definitions. (1) If F is a convex subspace of $\mathcal{S}$, then for every point x of $\mathcal{S}$ satisfying $\mathrm{d}(x,F) \leqslant 1$, there exists a unique point in F nearest to x. We will denote this point by $\pi_F(x)$. By Theorem 1.5 of [11], if $\mathrm{d}(x,F) \leqslant 1$, then $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_F(x)) + \mathrm{d}(\pi_F(x),y)$ for every point $y \in F$.

(2) Two convex subspaces F_1 and F_2 of $\mathcal{S}$ are called *parallel* if for every $i \in \{1,2\}$ and every point $x \in F_i$, there exists a unique point $x' \in F_{3-i}$ at distance $\mathrm{d}(F_1,F_2)$ from x and $\mathrm{d}(x,y) = \mathrm{d}(x,x') + \mathrm{d}(x',y) = \mathrm{d}(F_1,F_2) + \mathrm{d}(x',y)$ for every point y of F_{3-i}. The following proposition is precisely Theorem 1.10 of De Bruyn [11].

Proposition 2.9 *Let F_1 and F_2 be two parallel convex subspaces of $\mathcal{S}$. Then the map $\pi_{i,3-i} : F_i \to F_{3-i}$, $i \in \{1,2\}$, which maps a point x of F_i to the unique point of F_{3-i} nearest to x, is an isomorphism from $\widetilde{F_i}$ to $\widetilde{F_{3-i}}$. Moreover, $\pi_{2,1} = \pi_{1,2}^{-1}$.*

Proposition 2.10 *Let f be a valuation of $\mathcal{S}$, let F_1 and F_2 be two parallel convex subspaces at distance 1 from each other, and let f_i, $i \in \{1,2\}$, denote the valuation of $\widetilde{F_i}$ induced by f. For every point x of F_1, put $f_1'(x) := f_2(\pi_{F_2}(x))$. Then f_1 and f_1' are neighboring valuations of $\widetilde{F_1}$.*

Proof. Observe first that f_1' is a valuation of $\widetilde{F_1}$ by Proposition 2.9. Let δ_i, $i \in \{1,2\}$, be the unique element of $\mathbb{N}$ such that $f(x) = f_i(x) + \delta_i$ for every $x \in F_i$. For every point x of F_1, we have $|f_1(x) - f_1'(x) + \delta_1 - \delta_2| = |f(x) - f_2(\pi_{F_2}(x)) - \delta_2| = |f(x) - f(\pi_{F_2}(x))| \leqslant 1$. So, f_1 and f_1' are neighboring valuations of $\widetilde{F_1}$. ∎

Definition. (1) Let O be an *ovoid* of $\mathcal{S}$, i.e. a set of points of $\mathcal{S}$ intersecting each line of $\mathcal{S}$ in a singleton. For a point x of $\mathcal{S}$, define $f(x) := 0$ if $x \in O$ and $f(x) := 1$ if $x \notin O$. Then f is a so-called *ovoidal valuation* of $\mathcal{S}$.

(2) Let $\delta \in \{0,\ldots,n-1\}$, let x be a point of $\mathcal{S}$ and let O be a set of points of $\mathcal{S}$ at distance at least $\delta+2$ from x such that every line at distance at least $\delta+1$ from x has a unique point in common with O. For a point y of $\mathcal{S}$, we define

$$\begin{cases} f(y) := \mathrm{d}(x,y) & \text{if } \mathrm{d}(x,y) \leqslant \delta+1; \\ f(y) := \delta+1 & \text{if } \mathrm{d}(x,y) \geqslant \delta+2 \text{ and } y \notin O; \\ f(y) := \delta & \text{if } \mathrm{d}(x,y) \geqslant \delta+2 \text{ and } y \in O. \end{cases}$$

By [18, Section 3.1] or [11, Section 5.6.1], f is a (so-called *hybrid*) valuation of $\mathcal{S}$. We denote f also by $f_{x,\delta,O}$. If $\delta = 0$, then f is an ovoidal valuation of $\mathcal{S}$ with associated ovoid $O \cup \{x\}$. If $\delta = n-1$, then f is a classical valuation of $\mathcal{S}$. If $\delta = n-2$, then f is called a *semi-classical* valuation of $\mathcal{S}$.

Proposition 2.11 *Let $\delta \in \{0, \ldots, n-1\}$, let L be a line of $\mathcal{S}$, let x_1 and x_2 be two (not necessarily distinct) points of L and let O_i, $i \in \{1,2\}$, be a set of points of $\mathcal{S}$ at distance at least $\delta + 2$ from x_i such that every line at distance at least $\delta+1$ from x_i has a unique point in common with O_i. Then $f_1 := f_{x_1,\delta,O_1}$ and $f_2 := f_{x_2,\delta,O_2}$ are neighboring valuations of $\mathcal{S}$.*

Proof. Let y be an arbitrary point of $\mathcal{S}$.

If $\mathrm{d}(y, L) \leqslant \delta$, then $\mathrm{d}(x_1, y), \mathrm{d}(x_2, y) \leqslant \delta+1$ and $|f_1(y)-f_2(y)| = |\,\mathrm{d}(x_1,y)-\mathrm{d}(x_2,y)| \leqslant \mathrm{d}(x_1, x_2) \leqslant 1$ by the triangle inequality.

Suppose $\mathrm{d}(y, L) \geqslant \delta+1$. Then $\mathrm{d}(y, x_1), \mathrm{d}(y, x_2) \geqslant \delta+1$. It follows that $f_1(y), f_2(y) \in \{\delta, \delta+1\}$ and $|f_1(y) - f_2(y)| \leqslant 1$. ■

In the following corollary, we collect two special cases of Proposition 2.11.

Corollary 2.12 (1) *Every two ovoidal valuations of $\mathcal{S}$ are neighboring valuations.*

(2) *If f_1 and f_2 are two classical valuations whose centers lie at distance at most* 1 *from each other, then f_1 and f_2 are neighboring valuations.*

Definition. Suppose that every line of $\mathcal{S}$ is incident with precisely three points. If f_1 and f_2 are two neighboring valuations of $\mathcal{S}$, then we denote by $f_1 * f_2$ the unique element of $[f_1] * [f_2]$ whose minimal value is equal to 0. By Proposition 2.4, we know that $f_1 * f_2$ is a semi-valuation of $\mathcal{S}$.

Proposition 2.13 *Suppose every line of $\mathcal{S}$ is incident with precisely three points. Let F_1 and F_2 be two parallel convex subspaces at distance* 1 *from each other and let F_3 denote the set of all points of $\mathcal{S}$ not contained in $F_1 \cup F_2$ which are contained in a line joining a point of F_1 with a point of F_2. Suppose moreover that F_3 is also a convex subspace of $\mathcal{S}$. Let f be a valuation of $\mathcal{S}$ and let f_i, $i \in \{1,2,3\}$, denote the valuation of $\widetilde{F_i}$ induced by f. For every point x of F_1, we define $f_1'(x) = f_2(\pi_{F_2}(x))$ and $f_1''(x) = f_3(\pi_{F_3}(x))$. Then $f_1'' = f_1 * f_1'$.*

Proof. Notice first that f_1 and f_1' are neighboring valuations of $\widetilde{F_1}$ by Proposition 2.10. For every point x of F_1, we put $g_1(x) := f(x)$, $g_2(x) := f(\pi_{F_2}(x))$ and $g_3(x) := f(\pi_{F_3}(x))$. Then g_1, g_2 and g_3 are semi-valuations of $\widetilde{F_1}$. Since every line meeting F_1, F_2 and F_3 contains a unique point with smallest f-value (recall (V2)), we necessarily have $g_3 = g_1 \diamond g_2$. It follows that $f_1'' = f_1 * f_1'$. ■

Proposition 2.14 *Suppose that every line of $\mathcal{S}$ is incident with precisely three points. If f_1 and f_2 are distinct neighboring valuations of $\mathcal{S}$, then $H_{f_1 * f_2}$ is the complement of the symmetric difference of H_{f_1} and H_{f_2}.*

Proof. By Proposition 2.7, $H_{f_1} \neq H_{f_2}$. By Blok and Brouwer [1, Theorem 7.3] or Shult [26, Lemma 6.1], every hyperplane of a dense near polygon is also a maximal subspace. In particular, H_{f_1}, H_{f_2} and $H_{f_1 * f_2}$ are maximal subspaces of $\mathcal{S}$. It is now clear that case (1) of Proposition 2.5 must occur. So, $H_{f_1 * f_2}$ is the complement of the symmetric difference of H_{f_1} and H_{f_2}. ∎

Suppose again that every line of $\mathcal{S}$ is incident with precisely three points. If f_1 and f_2 are distinct neighboring valuations of $\mathcal{S}$, then $f_1 * f_2$ satisfies properties (V1) and (V2) in the definition of valuation. The following question can now be considered: does $f_1 * f_2$ also satisfy Property (V3)? If this is the case, then $f_1 * f_2$ is a valuation of $\mathcal{S}$. We will demonstrate below that the claim that $f_1 * f_2$ is a valuation is false in general, but true for a large class of dense near polygons. We will construct counter examples with the aid of the following lemma. Recall that by Corollary 2.12(1) any two ovoidal valuations of a given dense near polygon are neighboring valuations.

Lemma 2.15 *Suppose every line of $\mathcal{S}$ is incident with precisely three points and that f_1 and f_2 are two distinct ovoidal valuations of $\mathcal{S}$ for which $|H_{f_1} \cap H_{f_2}| \geqslant 2$ (so, $n \geqslant 3$). If $f_1 * f_2$ is a valuation of $\mathcal{S}$, then $f_1 * f_2$ is neither classical nor ovoidal.*

Proof. Since H_{f_1} and H_{f_2} are two distinct maximal subspaces of $\mathcal{S}$, $H_{f_1} \setminus H_{f_2} \neq \emptyset \neq H_{f_2} \setminus H_{f_1}$. So, $H_{f_1} \Delta H_{f_2} \neq \emptyset$.

Put $f_3 := f_1 \diamond f_2$. If $x \in H_{f_1} \cap H_{f_2}$, then $f_3(x) = -1$. If $x \in H_{f_1} \Delta H_{f_2}$, then $f_3(x) = 1$. If $x \notin H_{f_1} \cup H_{f_2}$, then $f_3(x) = 0$. So, $f_1 * f_2(x)$ is equal to 0 if $x \in H_{f_1} \cap H_{f_2}$, equal to 2 if $x \in H_{f_1} \Delta H_{f_2}$ and equal to 1 if $x \notin H_{f_1} \cup H_{f_2}$. Since $|H_{f_1} \cap H_{f_2}| \geqslant 2$, $f_1 * f_2$ is not a classical valuation of $\mathcal{S}$. Since $f_1 * f_2$ can take the value 2, it cannot be an ovoidal valuation of $\mathcal{S}$. ∎

We will now apply Lemma 2.15 to two particular cases.

Example 1. By Brouwer [2], there exists up to isomorphism a unique dense near hexagon $\mathcal{S}$ which satisfies the following properties: (1) every line of $\mathcal{S}$ is incident with precisely 3 points; (2) every point of $\mathcal{S}$ is incident with precisely 12 lines; (3) every quad of $\mathcal{S}$ is a (3×3)-grid. This near hexagon is related to the extended ternary Golay code, see Shult and Yanushka [27, p. 30]. Using the notation of [11] we will denote this near hexagon by $\mathbb{E}_1$. The ovoids of the near hexagon $\mathbb{E}_1$ have been classified in De Bruyn [9, Theorem 4.2]. There are 36 distinct ovoids (all of size 243) and any two distinct ovoids intersect in either 0 or 81 points. The valuations of the near hexagon $\mathbb{E}_1$ have been classified in De Bruyn and Vandecasteele [20]. Every valuation of $\mathbb{E}_1$ is either classical or ovoidal. Now, suppose f_1 and f_2 are two ovoidal valuations of $\mathbb{E}_1$ for which $|H_{f_1} \cap H_{f_2}| = 81$. Then Lemma 2.15 implies that $f_1 * f_2$ is not a valuation of $\mathbb{E}_1$. So, the map $f_1 * f_2$ satisfies properties (V1) and (V2), but not (V3). Such maps (for $\mathbb{E}_1$) were already constructed in De Bruyn [14, Section 4.1].

Example 2. By Brouwer [3], there exists up to isomorphism a unique dense near hexagon $\mathcal{S}$ which satisfies the following properties: (1) every line of $\mathcal{S}$ is incident with precisely 3 points; (2) every point of $\mathcal{S}$ is incident with precisely 15 lines; (3) every quad of $\mathcal{S}$ is isomorphic to the symplectic generalized quadrangle $W(2)$. This near hexagon is related to the Steiner system $S(5, 8, 24)$, see Shult and Yanushka [27, p. 40]. Using the notation of [11] we will denote this near hexagon by $\mathbb{E}_2$. The ovoids of the near hexagon $\mathbb{E}_2$ have been classified by Brouwer and Lambeck [5, p. 105], see also De Bruyn [11, Section 6.6.2] for an alternative proof. There are 24 distinct ovoids (all of size 253) and any two distinct ovoids intersect in precisely 77 points. The valuations of the near hexagon $\mathbb{E}_2$ have been classified in De Bruyn and Vandecasteele [20]. Every valuation of $\mathbb{E}_2$ is either classical or ovoidal. Now, suppose f_1 and f_2 are two distinct ovoidal valuations of $\mathbb{E}_2$. Then Lemma 2.15 implies that $f_1 * f_2$ is not a valuation of $\mathbb{E}_2$. So, the map $f_1 * f_2$ satisfies properties (V1) and (V2), but not (V3). Such maps (for $\mathbb{E}_2$) were already constructed in De Bruyn [14, Section 4.2].

The above two examples allow is to draw the following conclusion.

> If f_1 and f_2 are two distinct neighboring valuations of a general dense near polygon $\mathcal{S}$ with three points per line, then $f_1 * f_2$ is not necessarily a valuation of $\mathcal{S}$.

Definition. For every point x of $\mathcal{S}$, the following point-line geometry $\mathcal{L}(\mathcal{S}, x)$ can be defined. The points of $\mathcal{L}(\mathcal{S}, x)$ are the lines of $\mathcal{S}$ through x, the lines of $\mathcal{L}(\mathcal{S}, x)$ are the quads of $\mathcal{S}$ through x, and incidence is containment. The point-line geometry $\mathcal{L}(\mathcal{S}, x)$ is a linear space and is called the *local space at* x. If F is a convex subspace through x, then the set of all lines of F through x is a subspace of $\mathcal{L}(\mathcal{S}, x)$. The local space $\mathcal{L}(\mathcal{S}, x)$ is called *regular* if every subspace of $\mathcal{L}(\mathcal{S}, x)$ arises from a convex subspace through x in the above-described way.

In De Bruyn [14, Theorem 1.3 + Corollary 1.4], we proved the following:

Proposition 2.16 (1) *If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a dense near polygon, every local space of which is regular, then every map $f : \mathcal{P} \to \mathbb{N}$ which satisfies properties $(V1)$ and $(V2)$ also satisfies property $(V3)$.*

(2) *If $\mathcal{S}$ is a thick dual polar space, then every local space of $\mathcal{S}$ is regular.*

(3) *If $\mathcal{S}$ is a known dense near polygon without hexes isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$, then every local space of $\mathcal{S}$ is regular.*

By Propositions 2.4 and 2.16, we have

Corollary 2.17 *Let $\mathcal{S}$ be a dense near polygon with three points on each line, every local space of which is regular. If f_1 and f_2 are two neighboring valuations of $\mathcal{S}$, then $f_1 * f_2$ is also a valuation of $\mathcal{S}$. In particular, this holds if $\mathcal{S}$ is a known dense near hexagon with three points on each line which does not contain hexes isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$.*

The following special case of Corollary 2.17 will be of importance in this paper. (The regularity of the local spaces of $\mathbb{G}_n$ was demonstrated in [14, Section 3 (IV)]; also, no hex of $\mathbb{G}_n$ is isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_2$, see [11, Section 6.3.2]).

Corollary 2.18 *If f_1 and f_2 are two neighboring valuations of the near polygon $\mathbb{G}_n$, $n \geqslant 2$, then also $f_1 * f_2$ is a valuation of $\mathbb{G}_n$.*

3 Projective embeddings

3.1 Embeddings of general point-line geometries

A *full (projective) embedding* of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ into a projective space Σ is an injective mapping e from $\mathcal{P}$ to the point-set of Σ satisfying: (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $e(L) := \{e(x) \,|\, x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. The dimensions $\dim(\Sigma)$ and $\dim(\Sigma) + 1$ are respectively called the *projective dimension* and the *vector dimension* of e. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ into the projective space Σ, then for every hyperplane α of Σ, $H(\alpha) := e^{-1}(\alpha \cap e(\mathcal{P}))$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $H(\alpha)$ of $\mathcal{S}$ *arises from the embedding* e. If H is a hyperplane of $\mathcal{S}$ which is also a maximal subspace of $\mathcal{S}$ (as it is always the case if $\mathcal{S}$ is a dense near polygon), then $\langle e(H) \rangle_\Sigma$ is either Σ or a hyperplane of Σ. Moreover, if $\langle e(H) \rangle_\Sigma$ is a hyperplane of Σ, then $H = e^{-1}(\langle e(H) \rangle_\Sigma \cap e(\mathcal{P}))$, i.e. H arises from the embedding e.

Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if U is a subspace of Σ satisfying (C1): $\langle U, e(p) \rangle_\Sigma \neq U$ for every point p of $\mathcal{S}$, (C2): $\langle U, e(p_1) \rangle_\Sigma \neq \langle U, e(p_2) \rangle_\Sigma$ for any two distinct points p_1 and p_2 of $\mathcal{S}$, then there exists a full embedding e/U of $\mathcal{S}$ into the quotient space Σ/U mapping each point p of $\mathcal{S}$ to $\langle U, e(p) \rangle_\Sigma$. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two full embeddings of $\mathcal{S}$, then we say that $e_1 \geqslant e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then by Ronan [25], there exists (up to isomorphism) a unique full embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ satisfying (i) $\widetilde{e} \geqslant e$, (ii) if $e' \geqslant e$ for some embedding e' of $\mathcal{S}$, then $\widetilde{e} \geqslant e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e} \cong e$ for some full embedding e of $\mathcal{S}$, then we say that e is *relatively universal*. A full embedding e of $\mathcal{S}$ is called *absolutely universal* if it is universal relative to any full embedding of $\mathcal{S}$ defined over the same division ring as e.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a fully embeddable point-line geometry with three points on each line. Then by Ronan [25], $\mathcal{S}$ admits the absolutely universal embedding and every hyperplane of $\mathcal{S}$ arises from this embedding. We now give a description of the absolutely universal embedding of $\mathcal{S}$. Let V be a vector space over the field $\mathbb{F}_2$ with a basis B whose vectors are indexed by the elements of $\mathcal{P}$, e.g. $B = \{\overline{v}_p \,|\, p \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors $\overline{v}_{p_1} + \overline{v}_{p_2} + \overline{v}_{p_3}$ where $\{p_1, p_2, p_3\}$ is a line of $\mathcal{S}$. Then the map $p \in \mathcal{P} \mapsto \{\overline{v}_p + W, W\}$ defines a full embedding of $\mathcal{S}$ into the projective space $\mathrm{PG}(V/W)$ which is isomorphic to the absolutely universal embedding of $\mathcal{S}$.

3.2 The Grassmann embedding of the Hermitian dual polar space $DH(2n-1, \mathbb{K}'/\mathbb{K})$

Let $n \geqslant 2$, let $\mathbb{K}'$ be a field with involutory automorphism ψ and let $\mathbb{K}$ denote the fix field of ψ. Then $\mathbb{K}'$ can be regarded as a two-dimensional vector space over $\mathbb{K}$. Let V be a $2n$-dimensional vector space over $\mathbb{K}'$ equipped with a nondegenerate skew-ψ-Hermitian form f_V of maximal Witt index n. With the pair (V, f_V), there is associated a Hermitian dual polar space $DH(2n-1, \mathbb{K}'/\mathbb{K})$.

For every point $p = \langle \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n \rangle$ of $DH(2n-1, \mathbb{K}'/\mathbb{K})$, let $e_1(p)$ denote the point $\langle \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_n \rangle$ of $\mathrm{PG}(\bigwedge^n V)$. By Cooperstein [8] and De Bruyn [13], there exists a (necessarily unique) Baer-$\mathbb{K}$-subgeometry Σ of $\mathrm{PG}(\bigwedge^n V)$ containing the image of e_1. Moreover, e_1 defines a full embedding of $DH(2n-1, \mathbb{K}'/\mathbb{K})$ into Σ. This embedding is called the *Grassmann embedding* of $DH(2n-1, \mathbb{K}'/\mathbb{K})$. By results of Cooperstein [8], De Bruyn & Pasini [17], Kasikova & Shult [22] and Tits [28], we know that the Grassmann embedding of $DH(2n-1, \mathbb{K}'/\mathbb{K})$ is absolutely universal if $n = 2$ or $|\mathbb{K}'| > 4$. The same conclusion cannot be drawn in the case $n \geqslant 3$, $\mathbb{K} \cong \mathbb{F}_2$ and $\mathbb{K}' \cong \mathbb{F}_4$. Li [23] proved that the absolutely universal embedding of $DH(2n-1, 2)$ has vector dimension $\frac{4^n+2}{3}$ (which is bigger than $\binom{2n}{n}$ if $n \geqslant 3$).

Now, let B be a set of $\binom{2n}{n}$ vectors of $\bigwedge^n V$ such that $\Sigma = \mathrm{PG}(W)$, where W is the $\binom{2n}{n}$-dimensional vector space over $\mathbb{K}$ whose vectors consist of all $\mathbb{K}$-linear combinations of the elements of B. By De Bruyn [15, Section 4], there exists a nondegenerate bilinear form f_W on W satisfying the following properties:

(1) f_W is symplectic (or alternating) if either n is odd or $\mathrm{char}(\mathbb{K}) = 2$ and orthogonal if n is even and $\mathrm{char}(\mathbb{K}) \neq 2$.

(2) If ζ is the polarity of $\Sigma = \mathrm{PG}(W)$ associated with f_W, then for every point x of $DH(2n-1, \mathbb{K}'/\mathbb{K})$, $e_1(x)^\zeta = \langle e_1(H_x) \rangle_\Sigma$, where H_x is the hyperplane of $DH(2n-1, \mathbb{K}'/\mathbb{K})$ consisting of all points of $DH(2n-1, \mathbb{K}'/\mathbb{K})$ at distance at most $n-1$ from x.

Lemma 3.1 *Let $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ be a set of n linearly independent vectors of V. Let A denote the set of all vectors $\bar{v} \in V$ for which $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \wedge \bar{v} = 0$. Then $A = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$. As a consequence, A has dimension n.*

Proof. Clearly, $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \wedge \bar{v} = 0$ if and only if $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n, \bar{v}\}$ is linearly dependent, i.e. if and only if $\bar{v} \in \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$. ■

Lemma 3.2 *Let $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ and $\{\bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_n\}$ be two sets of n linearly independent vectors of V such that $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle \neq \langle \bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_n \rangle$. Let $\delta \in \mathbb{K}' \setminus \{0\}$ and put $\chi := \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n + \delta \cdot \bar{v}'_1 \wedge \bar{v}'_2 \wedge \cdots \wedge \bar{v}'_n$. Let A denote the set of all $\bar{v} \in V$ for which $\chi \wedge \bar{v} = 0$. Then A is an n-dimensional subspace of V if and only if the subspace $I := \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle \cap \langle \bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_n \rangle$ has dimension $n-1$. Moreover, if $\dim(I) = n-1$, then $\chi = \bar{v}''_1 \wedge \bar{v}''_2 \wedge \cdots \wedge \bar{v}''_n$ where $\bar{v}''_1, \bar{v}''_2, \ldots, \bar{v}''_n$ are n linearly independent vectors of V such that $\langle \bar{v}''_1, \bar{v}''_2, \ldots, \bar{v}''_n \rangle$ is an n-dimensional subspace of V through I distinct from $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ and $\langle \bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_n \rangle$.*

Proof. Put $k := \dim(I)$. Without loss of generality, we may suppose that $\bar{v}_i = \bar{v}'_i$ for every $i \in \{1, \ldots, k\}$. Extend $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ to a basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2n}\}$ of V such that $\bar{v}_{n+i} = \bar{v}'_{k+i}$ for every $i \in \{1, \ldots, n-k\}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n} \in \mathbb{K}$. Then $\chi \wedge (\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 + \cdots + \lambda_{2n} \bar{v}_{2n})$ is equal to

$$(\lambda_{n+1} \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \wedge \bar{v}_{n+1}) + (\lambda_{n+2} \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \wedge \bar{v}_{n+2}) + \cdots +$$

$$(\lambda_{2n} \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \wedge \bar{v}_{2n}) + ((-1)^{n-k} \lambda_{k+1} \delta \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k \wedge \bar{v}_{k+1} \wedge \bar{v}_{n+1} \wedge \bar{v}_{n+2} \wedge \cdots$$

$$\wedge\bar{v}_{2n-k}) + ((-1)^{n-k}\lambda_{k+2}\delta\cdot\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_k\wedge\bar{v}_{k+2}\wedge\bar{v}_{n+1}\wedge\bar{v}_{n+2}\wedge\cdots\wedge\bar{v}_{2n-k})$$
$$+\cdots+((-1)^{n-k}\lambda_n\delta\cdot\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_k\wedge\bar{v}_n\wedge\bar{v}_{n+1}\wedge\bar{v}_{n+2}\wedge\cdots\wedge\bar{v}_{2n-k})$$
$$+(\lambda_{2n-k+1}\delta\cdot\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_k\wedge\bar{v}_{n+1}\wedge\bar{v}_{n+2}\wedge\cdots\wedge\bar{v}_{2n-k}\wedge\bar{v}_{2n-k+1})$$
$$+(\lambda_{2n-k+2}\delta\cdot\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_k\wedge\bar{v}_{n+1}\wedge\bar{v}_{n+2}\wedge\cdots\wedge\bar{v}_{2n-k}\wedge\bar{v}_{2n-k+2})$$
$$+\cdots+(\lambda_{2n}\delta\cdot\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_k\wedge\bar{v}_{n+1}\wedge\bar{v}_{n+2}\wedge\cdots\wedge\bar{v}_{2n-k}\wedge\bar{v}_{2n}).$$

If $k\leqslant n-2$, then the $2n$ vectors of the form $\bar{v}_{i_1}\wedge\bar{v}_{i_2}\wedge\cdots\wedge\bar{v}_{i_{n+1}}$ occurring in the above sum are distinct and linearly independent. So, in this case $\chi\wedge(\lambda_1\bar{v}_1+\cdots+\lambda_{2n}\bar{v}_{2n})=0$ if and only if $\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{2n}=0$. It follows that $\dim(A)=k<n$.

If $k=n-1$, then $\chi=\bar{v}_1\wedge\bar{v}_2\wedge\cdots\wedge\bar{v}_{n-1}\wedge(\bar{v}_n+\delta\bar{v}'_n)$. By Lemma 3.1, it then follows that $\dim(A)=n$. Notice also that $\langle\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{n-1},\bar{v}_n+\delta\bar{v}'_n\rangle$ is an n-dimensional subspace of V through $I=\langle\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{n-1}\rangle$ distinct from $\langle\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{n-1},\bar{v}_n\rangle$ and $\langle\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{n-1},\bar{v}'_n\rangle=\langle\bar{v}'_1,\bar{v}'_2,\ldots,\bar{v}'_{n-1},\bar{v}'_n\rangle$. ■

The following corollary to Lemmas 3.1 and 3.2 will be useful later.

Corollary 3.3 *Any line of Σ containing at least three points of the image of e_1 is of the form $e_1(L)$ for some line L of $DH(2n-1,\mathbb{K}'/\mathbb{K})$.*

3.3 Embeddings of the dense near $2n$-gon $\mathbb{G}_n$

Let $n\geqslant 2$. In Section 1, we mentioned that there exists a subspace X of $DH(2n-1,4)$ satisfying: (i) $\widetilde{X}\cong\mathbb{G}_n$; (ii) if $x_1,x_2\in X$, then the distance between x_1 and x_2 in the geometry $\widetilde{X}$ is equal to the distance between x_1 and x_2 in the dual polar space $DH(2n-1,4)$. It can be proved, see De Bruyn [16], that there exists up to isomorphism a unique set of points of $DH(2n-1,4)$ satisfying (i) and (ii).

Since X is a subspace in $DH(2n-1,4)$, the Grassmann embedding $e_1: DH(2n-1,4)\to\Sigma$ of $DH(2n-1,4)$ will induce an embedding e_2 of $\widetilde{X}\cong\mathbb{G}_n$ into a subspace Σ' of Σ. In De Bruyn [12], we proved that $\Sigma'=\Sigma$ and that e_2 is the absolutely universal embedding of $\widetilde{X}\cong\mathbb{G}_n$. The latter implies (recall Ronan [25]) that every hyperplane of $\widetilde{X}$ arises from the embedding e_2. Since every hyperplane of $\widetilde{X}\cong\mathbb{G}_n$ is also a maximal subspace of $\widetilde{X}$, we can say more: if H is a hyperplane of $\widetilde{X}$, then $\Pi=\langle e_2(H)\rangle_\Sigma$ is a hyperplane of Σ and $H=e_2^{-1}(e_2(X)\cap\Pi)$. The embedding e_1 is not absolutely universal if $n\geqslant 3$[1]. However, since every hyperplane of $DH(2n-1,4)$ is a maximal subspace of $DH(2n-1,4)$, a similar property as above holds: if H is a hyperplane of $DH(2n-1,4)$ arising from e_1, then $\Pi=\langle e_1(H)\rangle_\Sigma$ is a hyperplane of Σ and $H=e_1^{-1}(e_1(\mathcal{P})\cap\Pi)$, where $\mathcal{P}$ denotes the point-set of $DH(2n-1,4)$.

Let ζ denote the polarity of Σ as defined in Section 3.2. Recall that for every point $x\in\mathcal{P}$, H_x denotes the hyperplane of $DH(2n-1,4)$ consisting of all points of $DH(2n-1,4)$

[1]The reader might be puzzled by the fact that e_2 is absolutely universal, while e_1 is not. This happens because, when you lift e_1 to the absolutely universal embedding $\widetilde{e_1}$ of $DH(2n-1,4)$, the image of $\mathbb{G}_n$ lifts to a set of points that spans a complement of the kernel of the projection of $\widetilde{e_1}$ onto e_1.

at distance at most $n-1$ from x. For every point x of $DH(2n-1,4)$, we have $\langle e_1(H_x)\rangle_\Sigma = e_1(x)^\zeta$.

Lemma 3.4 *Let x be a point of $DH(2n-1,4)$ and let f denote the valuation of $\widetilde{X} \cong \mathbb{G}_n$ induced by the classical valuation of $DH(2n-1,4)$ with center x. Then $\langle e_2(H_f)\rangle_\Sigma = \langle e_1(H_x)\rangle_\Sigma$. Hence, $e_1(x) = \langle e_1(H_x)\rangle_\Sigma^\zeta = \langle e_2(H_f)\rangle_\Sigma^\zeta$.*

Proof. Since both $\langle e_2(H_f)\rangle_\Sigma = \langle e_1(H_f)\rangle_\Sigma$ and $\langle e_1(H_x)\rangle_\Sigma = e_1(x)^\zeta$ are hyperplanes of Σ and $H_f \subseteq H_x$, we necessarily have $\langle e_2(H_f)\rangle_\Sigma = \langle e_1(H_x)\rangle_\Sigma$. ■

The last claim of Lemma 3.4 says that the point x is uniquely determined by the hyperplane H_f of $\mathbb{G}_n$. So, we have:

Corollary 3.5 *For every valuation f of $\widetilde{X} \cong \mathbb{G}_n$, there exists at most one point x of $DH(2n-1,4)$ such that f is induced by the classical valuation of $DH(2n-1,4)$ with center x.*

Lemma 3.6 *Let f_1 and f_2 be two distinct neighboring valuations of $\widetilde{X} \cong \mathbb{G}_n$ and let f_3 be the valuation $f_1 * f_2$ of $\widetilde{X}$. Suppose that for every $i \in \{1,2,3\}$, there exists a (necessarily unique) point x_i of $DH(2n-1,4)$ such that the valuation f_i of $\widetilde{X}$ is induced by the classical valuation of $DH(2n-1,4)$ with center x_i. Then $\{x_1,x_2,x_3\}$ is a line of $DH(2n-1,4)$.*

Proof. By Proposition 2.14, H_{f_3} is the complement of the symmetric difference of H_{f_1} and H_{f_2}. This implies that $\langle e_2(H_{f_1})\rangle_\Sigma$, $\langle e_2(H_{f_2})\rangle_\Sigma$ and $\langle e_2(H_{f_3})\rangle_\Sigma$ are the three hyperplanes of Σ through a given subspace of Σ of co-dimension 2. It follows that $e_1(x_1) = \langle e_2(H_{f_1})\rangle_\Sigma^\zeta$, $e_1(x_2) = \langle e_2(H_{f_2})\rangle_\Sigma^\zeta$ and $e_1(x_3) = \langle e_2(H_{f_3})\rangle_\Sigma^\zeta$ determine a line of Σ. By Corollary 3.3, $\{x_1,x_2,x_3\}$ is a line of $DH(2n-1,4)$. ■

4 Several useful lemmas

A max M of a dense near polygon $\mathcal{S}$ is called *big* if every point of $\mathcal{S}$ has distance at most 1 from M. If M is a big max of $\mathcal{S}$, then by Theorem 2.30 of [11], every quad of $\mathcal{S}$ which meets M is either contained in M or intersects M in a line.

If M_1 and M_2 are two disjoint big maxes of a dense near polygon $\mathcal{S}$, then M_1 and M_2 are parallel convex subspaces at distance 1 from each other. Proposition 2.9 tells us that there exist a natural isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$. If F is a convex subspace of diameter δ of M_1, then $\langle F, \pi_{M_2}(F)\rangle$ is a convex subspace of diameter $\delta+1$ of $\mathcal{S}$.

Suppose $\mathcal{S}$ is a dense near polygon with three points on each line and that M is a big max of $\mathcal{S}$. For every point x of M, we define $\mathcal{R}_M(x) := x$. For every point x of $\mathcal{S}$ not contained in M, let $\mathcal{R}_M(x)$ denote the unique point of the line $x\pi_M(x)$ distinct from x and $\pi_M(x)$. By Theorem 1.11 of [11], $\mathcal{R}_M$ is an automorphism of $\mathcal{S}$. So, if M' is a (big) max of $\mathcal{S}$, then $\mathcal{R}_M(M')$ is also a (big) max of $\mathcal{S}$.

Every max of the dual polar space $DH(2n-1,4)$, $n \geqslant 2$, is big. If F is a convex subspace of the dual polar space $DH(2n-1,4)$, $n \geqslant 2$, then for every point x of $DH(2n-$

$1,4)$, there exists a unique point $\pi_F(x) \in F$ nearest to x. Moreover, $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every $y \in F$. If F has diameter $\delta \in \{2, \ldots, n\}$, then $\widetilde{F} \cong DH(2\delta - 1, 4)$.

Let V be a $2n$-dimensional vector space ($n \geqslant 2$) with basis B. We will now collect several properties of the near polygon $\mathbb{G}_n := \mathbb{G}_n(V, B)$. We refer to [11, Section 6.3] for proofs.

If $\bar{x}$ is a vector of weight 2 of V, then the set of all points of $\mathbb{G}_n$ which, regarded as n-dimensional subspaces of V, contain the vector $\bar{x}$ is a big max of $\mathbb{G}_n$. In the sequel, we will say that M is the big max of $\mathbb{G}_n$ corresponding to $\bar{x}$. If $n \geqslant 3$, then every big max of $\mathbb{G}_n$ arises from a vector of weight 2 of V. If M is a big max of $\mathbb{G}_n$, $n \geqslant 3$, then $\widetilde{M} \cong \mathbb{G}_{n-1}$. Suppose M is a big max of $\mathbb{G}_n$ corresponding to a vector $\bar{x}$ of weight 2 of V. The set of points of $DH(V, B) \cong DH(2n - 1, 4)$ which, regarded as n-dimensional subspaces of V, contain the vector $\bar{x}$ is a max $\overline{M}$ of $DH(V, B)$. $\overline{M}$ is the unique max of $DH(V, B)$ containing M.

Let $\bar{x}_1$ and $\bar{x}_2$ be two linearly independent vectors of weight 2 of V and let M_i, $i \in \{1, 2\}$, denote the big max of $\mathbb{G}_n$ corresponding to $\bar{x}_i$. If $\bar{x}_1$ and $\bar{x}_2$ have disjoint supports, then M_1 and M_2 meet. If the supports of $\bar{x}_1$ and $\bar{x}_2$ are not disjoint, then M_1 and M_2 are disjoint.

Suppose the supports of $\bar{x}_1$ and $\bar{x}_2$ are not disjoint. Then the two-space $\langle \bar{x}_1, \bar{x}_2 \rangle$ contains a unique vector $\bar{x}_3$ of weight 2 distinct from $\bar{x}_1$ and $\bar{x}_2$, and we denote by M_3 the big max of $\mathbb{G}_n$ corresponding to $\bar{x}_3$. We have $M_3 = \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. If $\overline{M_i}$, $i \in \{1, 2, 3\}$, denotes the unique max of $DH(V, B)$ containing M_i, then $\overline{M_3} = \mathcal{R}_{\overline{M_1}}(\overline{M_2}) = \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. So, every line meeting M_1 ($\overline{M_1}$) and M_2 ($\overline{M_2}$) also meets M_3 ($\overline{M_3}$). If the supports of $\bar{x}_1$ and $\bar{x}_2$ are equal, then every line meeting M_1, M_2 (and M_3) is special. If the supports of $\bar{x}_1$ and $\bar{x}_2$ intersect in a singleton, then every line meeting M_1, M_2 (and M_3) is an ordinary line.

Every quad of $\mathbb{G}_n$, $n \geqslant 3$, is isomorphic to either the (3×3)-grid, the generalized quadrangle $W(2)$ or the generalized quadrangle $Q^-(5, 2)$.

If $n \geqslant 3$, then the automorphism group of $\mathbb{G}_n$ has two orbits on the set of lines of $\mathbb{G}_n$, namely the set of ordinary lines and the set of special lines. A line of $\mathbb{G}_n$, $n \geqslant 3$, is an ordinary line if and only if it is contained in a $W(2)$-quad. An ordinary line of $\mathbb{G}_n$, $n \geqslant 3$, is contained in a unique $Q^-(5, 2)$-quad. The automorphism group of $\mathbb{G}_n$, $n \geqslant 3$, acts transitively on the set of $W(2)$-quads of $\mathbb{G}_n$ and the set of $Q^-(5, 2)$-quads of $\mathbb{G}_n$. A grid-quad of $\mathbb{G}_n$, $n \geqslant 3$, is said to be of *Type I* if it contains a special line, otherwise it is called a *grid-quad of Type II*. Every grid-quad of $\mathbb{G}_3$ has Type I and the automorphism group of $\mathbb{G}_3$ acts transitively on the set of its grid-quads. The automorphism group of $\mathbb{G}_n$, $n \geqslant 4$, has two orbits on the set of grid-quads of $\mathbb{G}_n$, namely the set of grid-quads of Type I and the set of grid-quads of Type II.

Every point of $\mathbb{G}_n$, $n \geqslant 3$, is contained in precisely n special lines. If $L_1, \ldots, L_k$ are $k \geqslant 2$ special lines through a given point of $\mathbb{G}_n$, then $\widetilde{\langle L_1, \ldots, L_k \rangle} \cong \mathbb{G}_k$. Conversely, if F is a convex subspace of $\mathbb{G}_n$, $n \geqslant 3$, such that $\widetilde{F} \cong \mathbb{G}_k$ for some $k \geqslant 2$, then through every point of F, there are precisely k special lines of $\mathbb{G}_n$ which are contained in F. If F is a convex subspace of $\mathbb{G}_n$, $n \geqslant 3$, such that $\widetilde{F} \cong \mathbb{G}_k$ for some $k \geqslant 3$, then a line contained in F is a special line of $\widetilde{F}$ if and only if it is a special line of $\mathbb{G}_n$.

The following lemma was proved in De Bruyn [11, Section 6.3.3].

Lemma 4.1 *Let Q be a quad of $\mathbb{G}_n$, $n \geqslant 3$, containing a special line. Then there are two possibilities:*

(1) *Q is a grid-quad of Type I. Then Q contains precisely three special lines. These three lines partition the point-set of Q.*

(2) *Q is a $Q^-(5,2)$-quad of $\mathbb{G}_n$. Then Q can be partitioned into three subgrids G_1, G_2, G_3. A line of Q is special if and only if it is contained in one of the grids G_1, G_2 and G_3.*

Lemma 4.2 (1) *Every grid-quad Q of Type I of $\mathbb{G}_n$, $n \geqslant 3$, is contained in a unique hex isomorphic to $\mathbb{G}_3$.*

(2) *Every $Q^-(5,2)$-quad Q of $\mathbb{G}_n$, $n \geqslant 3$, is contained in precisely $n-2$ hexes isomorphic to $\mathbb{G}_3$.*

(3) *Let M_1 and M_2 be two disjoint maxes of $\mathbb{G}_n$, $n \geqslant 3$, such that every line meeting M_1 and M_2 is special. Let F be a convex subspace of M_1 such that $\widetilde{F} \cong \mathbb{G}_k$ for some $k \geqslant 2$. Then $\langle F, \pi_{M_2}(F)\rangle \cong \mathbb{G}_{k+1}$.*

Proof. (1) Let x be an arbitrary point of Q, let L_1 denote the unique special line of Q through x and let M denote the unique ordinary line of Q through x. Then M is contained in a unique $Q^-(5,2)$-quad R of $\mathbb{G}_n$. Let L_2 and L_3 denote the unique special lines of R through x. Then $\langle L_1, L_2, L_3\rangle = \langle Q, R\rangle$ is a $\mathbb{G}_3$-hex containing Q. Conversely, if F is a $\mathbb{G}_3$-hex through Q, then there exists a $Q^-(5,2)$-quad of $\widetilde{F}$ containing the line M. This $Q^-(5,2)$-quad necessarily coincides with R. So, $F = \langle Q, R\rangle$.

(2) Let x be an arbitrary point of Q, and let L_1 and L_2 be the two special lines of Q through x. If F is a $\mathbb{G}_3$-hex through Q, then there exists a unique special line $L_3 \notin \{L_1, L_2\}$ through x contained in F. Conversely, if L_3 is one of the $n-2$ special lines of $\mathbb{G}_n$ through x distinct from L_1 and L_2, then $\langle L_1, L_2, L_3\rangle$ is a $\mathbb{G}_3$-hex containing Q. It follows that there are precisely $n-2$ $\mathbb{G}_3$-hexes containing Q.

(3) Recall that since F has diameter δ, the convex subspace $\langle F, \pi_{M_2}(F)\rangle$ has diameter $\delta+1$. Let x be an arbitrary point of F and let $L_1, \ldots, L_k$ denote the k special lines of F through x. Let L_{k+1} denote the unique line through x meeting M_2. Then L_{k+1} is a special line. So, $\langle F, \pi_{M_2}(F)\rangle = \langle F, L_{k+1}\rangle = \langle L_1, L_2, \ldots, L_{k+1}\rangle \cong \mathbb{G}_{k+1}$. ■

Lemma 4.1 implies the following.

Corollary 4.3 *Let L_1 and L_2 be two disjoint special lines of $\mathbb{G}_n$, $n \geqslant 3$, which are contained in a quad Q, let G denote the unique (3×3)-subgrid of Q containing L_1, L_2 and let L_3 denote the unique line of G disjoint from L_1 and L_2. Then also L_3 is a special line of $\mathbb{G}_n$.*

Let $\mathcal{S}_n$, $n \geqslant 3$, be the following point-line geometry:

- The points of $\mathcal{S}_n$ are the special lines of $\mathbb{G}_n$;
- The lines of $\mathcal{S}_n$ are all the triples $\{L_1, L_2, L_3\}$, where L_1, L_2 and L_3 are three mutually disjoint special lines which are contained in some (3×3)-subgrid of $\mathbb{G}_n$.

- Incidence is containment.

Lemma 4.4 *The complement of a proper subspace of $\mathcal{S}_n$, $n \geqslant 3$, is connected.*

Proof. Let S be a subspace of $\mathcal{S}_n$ and let L_1, L_2 be two distinct special lines contained in the complement of S. We will prove by induction on $\mathrm{d}(L_1, L_2)$ that L_1 and L_2 are connected by a path which entirely consists of points of $\mathcal{S}_n$ not contained in S. Here, $\mathrm{d}(L_1, L_2)$ denotes the distance between L_1 and L_2 in the near polygon $\mathbb{G}_n$.

First, suppose that $\mathrm{d}(L_1, L_2) = 0$. Then L_1 and L_2 are contained in a unique $Q^-(5,2)$-quad Q. By Lemma 4.1(2), there exist special lines L_1' and L_1'' of Q such that: (i) $\{L_1, L_1', L_1''\}$ is a line of $\mathcal{S}_n$; (ii) the unique (3×3)-subgrid of Q containing L_1, L_1' and L_1'' does not contain L_2. Since $L_1 \notin S$, at least one of L_1', L_1'' does not belong to S. Hence, L_1, L_1', L_2 or L_1, L_1'', L_2 is a path of $\mathcal{S}_n$ contained in the complement of S.

Suppose now that $\mathrm{d}(L_1, L_2) > 0$. Let $x_1 \in L_1$ and $x_2 \in L_2$ be points such that $\mathrm{d}(x_1, x_2) = \mathrm{d}(L_1, L_2)$. Let M_1 denote a line through x_1 containing a unique point y_1 at distance $\mathrm{d}(x_1, x_2) - 1$ from x_2 and let M_2 denote a line of $\langle x_1, x_2 \rangle$ through x_2 which is not contained in $\langle x_2, y_1 \rangle$. Then M_2 contains a unique point y_2 at distance $\mathrm{d}(x_1, x_2) - 1$ from x_1. Let z_i, $i \in \{1, 2\}$, denote the unique point of M_i distinct from x_i and y_i.

Since $\mathrm{d}(x_2, y_1) = \mathrm{d}(x_1, x_2) - 1$, we have $\mathrm{d}(x_2, z_1) = \mathrm{d}(x_1, x_2)$. Since the line $x_2 y_2$ is not contained in $\langle x_2, y_1 \rangle$, we have $\mathrm{d}(y_2, y_1) = \mathrm{d}(y_2, x_2) + \mathrm{d}(x_2, y_1) = \mathrm{d}(x_1, x_2)$. Together with $\mathrm{d}(y_2, x_1) = \mathrm{d}(x_1, x_2) - 1$, this implies that $\mathrm{d}(y_2, z_1) = \mathrm{d}(x_1, x_2)$. Since the line $x_2 z_2$ is not contained in $\langle x_2, y_1 \rangle$, we have $\mathrm{d}(z_2, y_1) = \mathrm{d}(z_2, x_2) + \mathrm{d}(x_2, y_1) = \mathrm{d}(x_1, x_2)$. Since $\mathrm{d}(x_1, y_2) = \mathrm{d}(x_1, x_2) - 1$, we have $\mathrm{d}(z_2, x_1) = \mathrm{d}(x_1, x_2)$. Finally, since $\mathrm{d}(z_2, x_1) = \mathrm{d}(z_2, y_1) = \mathrm{d}(x_1, x_2)$, we have $\mathrm{d}(z_2, z_1) = \mathrm{d}(x_1, x_2) - 1$. We can conclude that for every point u_i of M_i, $i \in \{1, 2\}$, there exists a unique point of M_{3-i} at distance $\mathrm{d}(x_1, x_2) - 1$ from u_i.

Notice that $L_1 \neq M_1$ and $L_2 \neq M_2$. So, $\langle L_1, M_1 \rangle$ and $\langle L_2, M_2 \rangle$ are quads. We will now define a special line L_i' of $\langle L_i, M_i \rangle$ through y_i disjoint from L_i ($i \in \{1, 2\}$). Since L_1 and L_2 are special lines, we can distinguish two cases by Lemma 4.1.

(i) Suppose $\langle L_i, M_i \rangle$ is a grid-quad of Type I. Then let L_i' denote the unique line of $\langle L_i, M_i \rangle$ through y_i disjoint from L_i. Then L_i' is special.

(ii) Suppose $\langle L_i, M_i \rangle$ is a $Q^-(5,2)$-quad. Then there are precisely two special lines of $\langle L_i, M_i \rangle$ through y_i. Let L_i' denote any special line of $\langle L_i, M_i \rangle$ through y_i not meeting L_i.

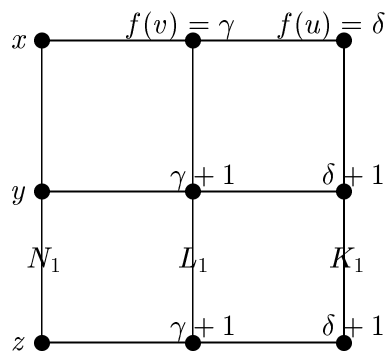
The lines L_i and L_i' are contained in a unique (3×3)-subgrid of $\langle L_i, M_i \rangle$. We denote by L_i'' the unique line of this subgrid which is disjoint from L_i and L_i'. Then also L_i'' is special and $z_i \in L_i''$. Since $L_i \notin S$, at most one of L_i, L_i', L_i'' belongs to S. Since $|M_1| = |M_2| = 3$, there exist points $u_1 \in M_1$ and $u_2 \in M_2$ such that (i) $\mathrm{d}(u_1, u_2) = \mathrm{d}(x_1, x_2) - 1 = \mathrm{d}(L_1, L_2) - 1$; (ii) for every $i \in \{1, 2\}$, the unique line $U_i \in \{L_i, L_i', L_i''\}$ containing u_i does not belong to S.

By the induction hypothesis, U_1 and U_2 are connected by a path which entirely consists of points of $\mathcal{S}_n$ which are contained in the complement of S. Hence, also L_1 and L_2 are connected by a path of $\mathcal{S}_n$ which entirely consists of points of $\mathcal{S}_n$ which are contained in the complement of S. ∎

Lemma 4.5 *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{G}_n$, $n \geqslant 4$, such that every line meeting M_1 and M_2 is special. Let $\mathcal{S}_{n-1}(M_1)$ denote the geometry isomorphic to $\mathcal{S}_{n-1}$ defined on the set of special lines of M_1 (recall $\widetilde{M_1} \cong \mathbb{G}_{n-1}$). Let f be a semi-valuation of $\mathbb{G}_n$ and let S denote the set of special lines L of M_1 such that the unique points of the lines L and $\pi_{M_2}(L)$ with smallest f-values are collinear. Then S is a subspace of $\mathcal{S}_{n-1}(M_1)$.*

Proof. Let $\{K_1, L_1, N_1\}$ be an arbitrary line of $\mathcal{S}_{n-1}(M_1)$ such that $K_1, L_1 \in S$. We need to prove that $N_1 \in S$. Put $K_2 = \pi_{M_2}(K_1)$, $L_2 = \pi_{M_2}(L_1)$ and $N_2 = \pi_{M_2}(N_1)$.

Case I. Suppose the unique point u of K_1 with smallest f-value is collinear with the unique point v of L_1 with smallest f-value. In the following picture we sketch this situation and indicate the values of the points of K_1 and L_1.

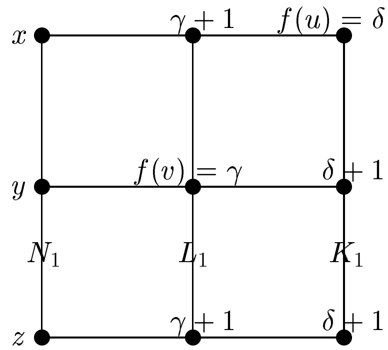


If $\gamma = \delta$, then using the fact that every line meeting K_1 and L_1 contains a unique point with smallest value, we obtain that $f(x) = \gamma - 1$, $f(y) = \gamma$ and $f(z) = \gamma$. So, x is the unique point of N_1 with smallest f-value.

If $\gamma \neq \delta$, then using the fact that every line meeting K_1 and L_1 contains a unique point with smallest value, we obtain $f(x) = \max\{\gamma, \delta\}$ and $f(y) = f(z) = \max\{\gamma + 1, \delta + 1\} = f(x) + 1$. So, again x is the unique point of N_1 with smallest f-value.

Now, since $K_1, L_1 \in S$, $\pi_{M_2}(u)$ is the unique point of K_2 with smallest f-value and $\pi_{M_2}(v)$ is the unique point of L_2 with smallest f-value. Since u and v are collinear, also $\pi_{M_2}(u)$ and $\pi_{M_2}(v)$ are collinear. Repeating the above reasoning for the lines K_2, L_2, N_2 instead of K_1, L_1, N_1, we find that $\pi_{M_2}(x)$ is the unique point of N_2 with smallest f-value. Since x is collinear with $\pi_{M_2}(x)$, we have $N_1 \in S$ as we needed to prove.

Case II. The unique point u of K_1 with smallest f-value is not collinear with the unique point v of L_1 with smallest f-value. This situation is sketched in the following picture, where the values of the points of K_1 and L_1 are mentioned.



Since the f-values of two collinear points differ by at most 1, we have $|(\gamma+1)-\delta| \leqslant 1$ and $|\gamma-(\delta+1)| \leqslant 1$. It follows that $\gamma=\delta$. Since every line meeting K_1 and L_1 contains a unique point with smallest f-value, we have $f(x)=\gamma+1$, $f(y)=\gamma+1$ and $f(z)=\gamma$. So, z is the unique point of N_1 with smallest f-value.

Now, since $K_1, L_1 \in S$, $\pi_{M_2}(u)$ is the unique point of K_2 with smallest f-value and $\pi_{M_2}(v)$ is the unique point of L_2 with smallest f-value. Since u and v are not collinear, also $\pi_{M_2}(u)$ and $\pi_{M_2}(v)$ are not collinear. Repeating the above reasoning for the lines K_2, L_2, N_2 instead of K_1, L_1, N_1, we find that $\pi_{M_2}(z)$ is the unique point of N_2 with smallest f-value. Since z is collinear with $\pi_{M_2}(z)$, we have $N_1 \in S$ as we needed to prove. ■

Lemma 4.6 (1) *Let M_1 and M_2 be two disjoint big maxes of $\mathbb{G}_n$, $n \geqslant 3$, such that every line meeting M_1 and M_2 is special. Put $M_3 := \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. Then every quad meeting M_1, M_2 (and M_3) is either a grid-quad of Type I or a $Q^-(5,2)$-quad.*

(2) *Every point x of $\mathbb{G}_n$ not contained in $M_1 \cup M_2 \cup M_3$ is contained in a unique quad Q_x which intersect M_1, M_2 (and M_3) in lines. This quad Q_x is a $Q^-(5,2)$-quad.*

(3) *Let L be a line if M_1. Then $\langle L, \pi_{M_2}(L)\rangle$ is a grid-quad of Type I if L is an ordinary line and a $Q^-(5,2)$-quad if L is a special line.*

Proof. (1) Suppose Q is a quad meeting M_1 in a line L_1 and M_2 in a line L_2. Let $x \in L_1$. Since Q contains the points x and $\pi_{M_2}(x) \in L_2$, it contains the special line $x\pi_{M_2}(x)$. Hence, Q is either a grid-quad of Type I or a $Q^-(5,2)$-quad by Lemma 4.1.

(2) Suppose x is a point of $\mathbb{G}_n$ not contained in $M_1 \cup M_2 \cup M_3$. If Q is a quad through x meeting M_1 and M_2 in lines, then Q necessarily contains the points $\pi_{M_1}(x)$ and $\pi_{M_2}(x)$. If $x\pi_{M_1}(x) = x\pi_{M_2}(x)$, then $\{x, \pi_{M_1}(x), \pi_{M_2}(x)\}$ is a line meeting M_1 and M_2, a contradiction, since $x \notin M_3$. Hence, $x\pi_{M_1}(x) \neq x\pi_{M_2}(x)$ and Q necessarily coincides with the quad $Q_x := \langle x\pi_{M_1}(x), x\pi_{M_2}(x)\rangle$. Since Q_x meets M_1 and M_2 in lines it is either a grid-quad or a $Q^-(5,2)$-quad by part (1). Since $Q_x \cap (M_1 \cup M_2 \cup M_3)$ is a subgrid of Q_x and $x \notin M_1 \cup M_2 \cup M_3$, Q_x necessarily is a $Q^-(5,2)$-quad.

(3) Let $x \in L$ and let L' denote the unique line through x meeting M_2. Then $\langle L, \pi_{M_2}(L)\rangle = \langle L, L'\rangle$. If L is special, then $\langle L, L'\rangle$ is a $Q^-(5,2)$-quad since L and L' are two distinct special lines through x.

Conversely, suppose that $\langle L, L' \rangle$ is a $Q^-(5,2)$-quad. There are precisely n special lines through x, two of these special lines are contained in $\langle L, L' \rangle$ and $n-1$ of these special lines are contained in M_1 (recall $\widetilde{M_1} \cong \mathbb{G}_{n-1}$). It follows that $L = \langle L, L' \rangle \cap M_1$ is a special line. ■

Lemma 4.7 *Let M_1 and M_2 be two disjoint (big) maxes of $DH(2n-1,4)$, $n \geqslant 2$, and put $M_3 := \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. Then every point x of $DH(2n-1,4)$ not contained in $M_1 \cup M_2 \cup M_3$ is contained in a unique quad Q_x which intersects M_1, M_2 (and M_3) in lines.*

Proof. Similarly as in the proof of Lemma 4.6(2), we have that $x\pi_{M_1}(x) \neq x\pi_{M_2}(x)$ and that Q_x is the unique quad of $DH(2n-1,4)$ containing the lines $x\pi_{M_1}(x)$ and $x\pi_{M_2}(x)$. ■

As already mentioned in Section 1, the following lemma was proved in [19, Proposition 7.7] in the case $n = 3$ and in [21, Proposition 6.13] in the case $n = 4$.

Lemma 4.8 *Regard $\mathbb{G}_n$, $n \in \{3,4\}$, as a subgeometry of $DH(2n-1,4)$ which is isometrically embedded into $DH(2n-1,4)$. Then every valuation of $\mathbb{G}_n$ is induced by a unique (classical) valuation of $DH(2n-1,4)$.*

Lemma 4.9 *Let M_1 and M_2 be two disjoint big maxes of the near polygon $\mathbb{G}_3$ such that every line meeting M_1 and M_2 is special. Let f be a valuation of $\mathbb{G}_3$ having the property that there exists a line K of M_1 such that the unique point of K with smallest f-value is not collinear with the unique point of $\pi_{M_2}(K)$ with smallest f-value. Then there exists a special line L of M_1 such that the unique point of L with smallest f-value is not collinear with the unique point of $\pi_{M_2}(L)$ with smallest f-value.*

Proof. We regard $\mathbb{G}_3$ as a subgeometry of $DH(5,4)$ which is isometrically embedded into $DH(5,4)$. Then by Lemma 4.8, there exists a unique point x of $DH(5,4)$ such that f is induced by the classical valuation of $DH(5,4)$ with center x. Put $M_3 := \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. We have $\widetilde{M_1} \cong \widetilde{M_2} \cong \widetilde{M_3} \cong Q^-(5,2)$. So, M_1, M_2 and M_3 are quads of both $\mathbb{G}_3$ and $DH(5,4)$.

We prove that $x \notin M_1 \cup M_2 \cup M_3$. Suppose to the contrary that $x \in M_i$ for a certain $i \in \{1,2,3\}$. Let u_j, $j \in \{1,2,3\}$, denote the unique point of $\pi_{M_j}(K)$ nearest to x. Then u_j is the unique point of $\pi_{M_j}(K)$ with smallest f-value. Since $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_{M_i}(y)) + \mathrm{d}(\pi_{M_i}(y), y)$ for every $j \in \{1,2,3\}$ and every point $y \in \pi_{M_j}(K)$, we have $u_j = \pi_{M_j}(u_i)$. So, $\{u_1, u_2, u_3\}$ is a line meeting M_1, M_2 and M_3. This contradicts the fact that the unique point of K with smallest f-value is not collinear with the unique point of $\pi_{M_2}(K)$ with smallest f-value.

So, $x \notin M_1 \cup M_2 \cup M_3$. By Lemma 4.7, there exists a unique quad Q_x of $DH(5,4)$ through x intersecting M_1, M_2 and M_3 in lines. By Lemma 4.1(2), there exists a special line L in M_1 disjoint from the line $Q_x \cap M_1$. Let x_i, $i \in \{1,2\}$, denote the unique point of M_i collinear with x and let y_i denote the unique point of $\pi_{M_i}(L)$ collinear with x_i. Since $x \notin M_1 \cup M_2 \cup M_3$, x_1 and x_2 are not collinear. Hence, also y_1 and y_2 are not collinear.

Now, for every $i \in \{1,2\}$ and every point z of $\pi_{M_i}(L)$, we have $\mathrm{d}(x,z) = \mathrm{d}(x,x_i) + \mathrm{d}(x_i,z)$. So, y_i, $i \in \{1,2\}$, is the unique point of $\pi_{M_i}(L)$ nearest to x, or equivalently, the unique point of $\pi_{M_i}(L)$ with smallest f-value.

Summarizing, we have that the unique point of the special line L with smallest f-value is not collinear with the unique point of $\pi_{M_2}(L)$ with smallest f-value. ■

Lemma 4.10 *Let f be a semi-valuation of the near polygon $\mathbb{G}_n$, $n \geqslant 2$, and let Q be a $Q^-(5,2)$-quad of $\mathbb{G}_n$. Then Q contains a unique point x^* with smallest f-value and $f(x) = f(x^*) + d(x^*, x)$ for every point x of Q.*

Proof. It is easy to show (see e.g. De Bruyn [14, Lemma 2.2]) that every semi-valuation of a thick generalized quadrangle is equivalent to either a classical valuation or an ovoidal valuation. Since the generalized quadrangle $Q^-(5,2)$ has no ovoids (see e.g. Payne and Thas [24, 3.4.1]), f is equivalent with a classical valuation of $Q^-(5,2)$. The lemma follows. ■

If K and L are two lines of a near polygon, then by Theorem 1.3 of [11] precisely one of the following two cases occurs: (a) there exists a unique point $k^* \in K$ and a unique point $l^* \in L$ such that $\mathrm{d}(k,l) = \mathrm{d}(k,k^*) + \mathrm{d}(k^*,l^*) + \mathrm{d}(l^*,l)$ for every point $k \in K$ and every point $l \in L$; (b) for every point k in K, there exists a unique point $l \in L$ such that $\mathrm{d}(k,l) = \mathrm{d}(K,L)$. If case (b) occurs, then K and L are parallel.

Lemma 4.11 *Let M_1 and M_2 be two disjoint maxes of the dual polar space $DH(2n-1,4)$, $n \geqslant 3$, and let $M_3 = \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. Let Q and R be two quads of $DH(2n-1,4)$ which intersect M_1 and M_2 in lines. If x is a point of R such that the unique points of $Q \cap M_1$ and $Q \cap M_2$ nearest to x are not collinear, then*

(1) $K := Q \cap M_1$ and $L := R \cap M_1$ are parallel lines;

(2) Q and R are parallel quads;

(3) $x \in R \setminus (M_1 \cup M_2 \cup M_3)$.

Proof. (1) Suppose to the contrary that K and L are not parallel and let $k^* \in K$ and $l^* \in L$ denote the unique points such that $\mathrm{d}(k,l) = \mathrm{d}(k,k^*) + \mathrm{d}(k^*,l^*) + \mathrm{d}(l^*,l)$ for all $k \in K$ and all $l \in L$. Since the map $M_1 \to M_2; x \mapsto \pi_{M_2}(x)$ is an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$, the lines $\pi_{M_2}(K)$ and $\pi_{M_2}(L)$ are not parallel and $\mathrm{d}(k,l) = \mathrm{d}(k, \pi_{M_2}(k^*)) + \mathrm{d}(\pi_{M_2}(k^*), \pi_{M_2}(l^*)) + \mathrm{d}(\pi_{M_2}(l^*), l)$ for all $k \in \pi_{M_2}(K)$ and $l \in \pi_{M_2}(L)$. For every $i \in \{1,2\}$ and every $y \in Q \cap M_i = \pi_{M_i}(K)$, we have $\mathrm{d}(x,y) = \mathrm{d}(x, \pi_{M_i}(x)) + \mathrm{d}(\pi_{M_i}(x), y) = \mathrm{d}(x, \pi_{M_i}(x)) + \mathrm{d}(\pi_{M_i}(x), \pi_{M_i}(l^*)) + \mathrm{d}(\pi_{M_i}(l^*), \pi_{M_i}(k^*)) + \mathrm{d}(\pi_{M_i}(k^*), y)$. So, k^* is the unique point of $K = Q \cap M_1$ nearest to x and $\pi_{M_2}(k^*)$ is the unique point of $\pi_{M_2}(K) = Q \cap M_2$ nearest to x. This contradicts the fact that the unique points of $Q \cap M_1$ and $Q \cap M_2$ nearest to x are not collinear.

(2) By part (1), K and L are parallel. Put $\delta := \mathrm{d}(K,L)$. For every point u of R, there exists a unique point $\pi_Q(u) \in Q$ nearest to u and $\mathrm{d}(u,v) = \mathrm{d}(u, \pi_Q(u)) + \mathrm{d}(\pi_Q(u), v)$ for every $v \in Q$. We prove that $\pi_Q(u)$ has distance δ from u. It suffices to prove the following things:

(a) If $u \in R \cap (M_1 \cup M_2 \cup M_3)$, then $\{d(u,v) \mid v \in Q \cap (M_1 \cup M_2 \cup M_3)\} = \{\delta, \delta+1, \delta+2\}$.

(b) If $u \in R \setminus (M_1 \cup M_2 \cup M_3)$, then $\{d(u,v) \mid v \in Q \cap (M_1 \cup M_2 \cup M_3)\} = \{\delta+1, \delta+2\}$. Moreover, there is more than one $v \in Q \cap (M_1 \cup M_2 \cup M_3)$ for which $d(u,v) = \delta + 1$.

(a) Suppose $u \in R \cap M_i$ for some $i \in \{1,2,3\}$. Let u' denote the unique point of $Q \cap M_i$ nearest to u. Then $d(u,u') = \delta$ and $d(u,v) = \delta + 1$ for every $v \in (Q \cap M_i) \setminus \{u'\}$. Now, let $j \in \{1,2,3\} \setminus \{i\}$. Then $d(u, \pi_{M_j}(u')) = d(u,u') + d(u', \pi_{M_j}(u')) = \delta + 1$. If $v \in (Q \cap M_i) \setminus \{u'\}$, then $d(u, \pi_{M_j}(v)) = d(u,v) + d(v, \pi_{M_j}(v)) = \delta + 2$. This proves (a).

(b) Suppose $u \in R \setminus (M_1 \cup M_2 \cup M_3)$. Let u_i, $i \in \{1,2,3\}$, denote the unique point of $M_i \cap R$ collinear with u and let u_i' denote the unique point of $M_i \cap Q$ nearest to u. Then $d(u,u_i') = d(u,u_i) + d(u_i,u_i') = \delta + 1$ and for every $v \in (Q \cap M_i) \setminus \{u_i'\}$, we have $d(u,v) = d(u,u_i) + d(u_i,v) = \delta + 2$. This proves (b).

Similarly, for every point u of Q, there exists a unique point $\pi_R(u) \in R$ nearest to u and $d(u,v) = d(u, \pi_R(u)) + d(\pi_R(u), v)$ for every $v \in R$. With a similar reasoning as above, one can show that $\pi_R(u)$ has distance δ from u. It follows that Q and R are parallel quads.

(3) Suppose to the contrary that $x \in R \cap M_i$ for some $i \in \{1,2,3\}$. Let x_j, $j \in \{1,2,3\}$, denote the unique point of $Q \cap M_j$ nearest to x. For every $y \in Q \cap M_i$ and $j \in \{1,2,3\}$, we have $d(x, \pi_{M_j}(y)) = d(x,y) + d(y, \pi_{M_j}(y))$. So, $x_j = \pi_{M_j}(x_i)$ for every $j \in \{1,2,3\}$. This would imply that x_1 and x_2 are collinear, a contradiction. It follows that $x \in R \setminus (M_1 \cup M_2 \cup M_3)$. ∎

Lemma 4.12 *Regard $\mathbb{G}_4$ as a subgeometry of $DH(7,4)$ which is isometrically embedded into $DH(7,4)$. Let f be a semi-valuation of $\mathbb{G}_4$. Let M_1 and M_2 be two disjoint big maxes of $\mathbb{G}_4$ such that every line meeting M_1 and M_2 is special and let Q be a $Q^-(5,2)$-quad of $\mathbb{G}_4$ which intersect M_1 and M_2 in lines such that the unique points of $Q \cap M_1$ and $Q \cap M_2$ with smallest f-values are not collinear. Then f is uniquely determined by the values that it takes on the set $M_1 \cup M_2 \cup Q$.*

Proof. By Proposition 2.16, the semi-valuation f of $\mathbb{G}_4$ is equivalent with a unique valuation f' of $\mathbb{G}_4$. By Lemma 4.8, the valuation f' of $\mathbb{G}_4$ is induced by a unique classical valuation f'' of $DH(7,4)$. It suffices to prove that the center x of f'' is uniquely determined by the values that f takes on the set $M_1 \cup M_2 \cup Q$.

Put $M_3 = \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$ and let $\overline{M_i}$, $i \in \{1,2,3\}$, denote the unique max of $DH(7,4)$ containing M_i. By Lemma 4.7, there exists a quad Q_x of $DH(7,4)$ through x intersecting $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$ in lines. (Clearly, this is also valid if x would be contained in $\overline{M_1} \cup \overline{M_2} \cup \overline{M_3}$.) Now, the unique points of $Q \cap M_1$ and $Q \cap M_2$ with smallest f-value are not collinear, or equivalently, the unique points of $Q \cap M_1$ and $Q \cap M_2$ nearest to x are not collinear. By Lemma 4.11(2), Q and Q_x are parallel quads and $x \in Q_x \setminus (\overline{M_1} \cup \overline{M_2} \cup \overline{M_3})$. Let x_i, $i \in \{1,2,3\}$, denote the unique point of $\overline{M_i}$ collinear with x. Then x_1, x_2 and x_3 are mutually noncollinear. Since $d(x,y) = d(x,x_i) + d(x_i,y) = 1 + d(x_i,y)$ for every $i \in \{1,2,3\}$ and every $y \in \overline{M_i}$, the valuation of $\widetilde{M_i}$ induced by f is also induced by the valuation of $\widetilde{\overline{M_i}}$ with center x_i. We also know that the valuation of Q induced by f is classical (recall Lemma 4.10) and that the center of this classical valuation is the unique point of Q nearest to x.

The above discussion allows us to construct x from the values that f takes on the set $M_1 \cup M_2 \cup Q$. Let f_i, $i \in \{1,2\}$, denote the valuation of $\widetilde{M_i}$ induced by f. Then by Lemma 4.8, f_i is induced by a unique classical valuation of $\widetilde{\overline{M_i}}$. We denote by x_i^* the center of this classical valuation of $\widetilde{\overline{M_i}}$. By the above, x_1^* and x_2^* lie at distance 2 from each other. So, they determine a unique quad Q^* which is parallel with Q. If y^* denotes the unique point of Q with smallest f-value, then x necessarily is the unique point of Q^* nearest to y^*. ■

Lemma 4.13 *Regard $\mathbb{G}_n$, $n \geqslant 4$, as a subgeometry of $DH(2n-1,4)$ which is isometrically embedded into $DH(2n-1,4)$. Let f be a semi-valuation of $\mathbb{G}_n$. Let M_1 and M_2 be two disjoint big maxes of $\mathbb{G}_n$ such that every line meeting M_1 and M_2 is special, and let Q be a $Q^-(5,2)$-quad of $\mathbb{G}_n$ which intersects M_1 and M_2 in lines such that the unique points of $Q \cap M_1$ and $Q \cap M_2$ with smallest f-values are not collinear. Then f is uniquely determined by the values that it takes on the set $M_1 \cup M_2 \cup Q$.*

Proof. Notice first that contrary to the situation in the proof of Lemma 4.12, we do not know (yet) whether the valuation of $\mathbb{G}_n$ which is equivalent with f is induced by a classical valuation of $DH(2n-1,4)$. Put $M_3 = \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$ and let $\overline{M_i}$, $i \in \{1,2,3\}$, denote the unique max of $DH(2n-1,4)$ containing M_i. Let x be an arbitrary point of $\mathbb{G}_n$ not contained in $M_1 \cup M_2$.

Suppose first that $x \in M_3$. Then there exists a unique line L through x meeting M_1 in a point x_1 and M_2 in a point x_2. If $f(x_1) = f(x_2)$, then $f(x) = f(x_1) - 1 = f(x_2) - 1$. If $f(x_1) \neq f(x_2)$, then $f(x) = \max\{f(x_1), f(x_2)\}$. So, $f(x)$ is uniquely determined by the values that f takes on the set $M_1 \cup M_2 \cup Q$.

By the previous paragraph, we may suppose that $x \notin M_1 \cup M_2 \cup M_3$. Then by Lemma 4.6(2) there exists a unique $Q^-(5,2)$-quad Q_x through x which intersect M_1, M_2 and M_3 in (special) lines.

Suppose first that the unique point u_1 of $M_1 \cap Q_x$ with smallest f-value is collinear with the unique point u_2 of $M_2 \cap Q_x$ with smallest f-value. Let u denote the point of the line u_1u_2 with smallest f-value. Now, f takes three values on the subgrid $Q_x \cap (M_1 \cup M_2 \cup M_3)$ of Q_x, namely $f(u)$, $f(u)+1$ and $f(u)+2$. It follows that the valuation of Q_x induced by f is classical with center u (recall also Lemma 4.10). So, the f-values of the points of Q_x (in particular, of x) are uniquely determined by the values that f takes on the set $M_1 \cup M_2 \cup Q$.

Suppose next that the unique point of $M_1 \cap Q_x$ with smallest f-value is not collinear with the unique point of $M_2 \cap Q_x$ with smallest f-value. Let $\mathcal{S}_{n-1}(M_1)$ denote the geometry isomorphic to $\mathcal{S}_{n-1}$ defined on the set of special lines of $\widetilde{M_1}$. Let S denote the set of special lines L of $\widetilde{M_1}$ such that the unique points of L and $\pi_{M_2}(L)$ with smallest f-values are collinear. Then S is a subspace of $\mathcal{S}_{n-1}(M_1)$ by Lemma 4.5. It is a proper subspace since $Q \cap M_1 \notin S$. So, the complement of S is connected by Lemma 4.4. It follows that there exists a sequence $Q = Q_1, Q_2, \dots, Q_k = Q_x$ of $k \geqslant 1$ $Q^-(5,2)$-quads which intersect

M_1 and M_2 in lines and which satisfy: (1) for every $i \in \{1, \dots, k\}$, $Q_i \cap M_1$ is a special line not belonging to S; (2) for every $i \in \{1, \dots, k-1\}$, $Q_i \cap M_1$ and $Q_{i+1} \cap M_1$ are collinear points of $\mathcal{S}_{n-1}(M_1)$. It suffices to prove that for every $i \in \{1, \dots, k-1\}$, the values $f(x)$, $x \in Q_{i+1}$, are uniquely determined by the values that f takes on the set $M_1 \cup M_2 \cup Q_i$. By Lemma 4.1 there are two possibilities for $\langle Q_i \cap M_1, Q_{i+1} \cap M_1 \rangle$. Either $\langle Q_i \cap M_1, Q_{i+1} \cap M_1 \rangle$ is a special grid-quad of Type I or a $Q^-(5,2)$-quad. In any case, $\langle Q_i \cap M_1, Q_{i+1} \cap M_1 \rangle$ is contained in a $\mathbb{G}_3$-hex $F \subseteq M_1$ by Lemma 4.2(1)+(2). The convex sub-octagon $\langle F, \pi_{M_2}(F) \rangle$ contains $Q_i \cup Q_{i+1}$ and is isomorphic to $\mathbb{G}_4$ by Lemma 4.2(3). By Lemma 4.12, the values $f(x)$, $x \in Q_{i+1}$, are uniquely determined by the values that f takes on the set $F \cup \pi_{M_2}(F) \cup Q_i$ and hence (a fortiori) also by the values that f takes on the set $M_1 \cup M_2 \cup Q_i$. This was precisely what we needed to show. ■

5 Proof of Theorem 1.1

We regard $\mathbb{G}_n$, $n \geqslant 2$, as a subgeometry of $DH(2n-1,4)$ which is isometrically embedded into $DH(2n-1,4)$. Recall that by De Bruyn [16], there exists up to isomorphism a unique isometric embedding of $\mathbb{G}_n$ into $DH(2n-1,4)$.

Let f be a valuation of the near polygon $\mathbb{G}_n$. We will prove by induction on n that f is induced by a unique (classical) valuation of $DH(2n-1,4)$. This trivially holds if $n=2$. By Lemma 4.8, this claim also holds if $n \in \{3,4\}$. So, in the sequel we will suppose that $n \geqslant 5$. Let M_1 and M_2 be two disjoint big maxes of $\mathbb{G}_n$ such that every line meeting M_1 and M_2 is special. Recall that $\widetilde{M_1} \cong \widetilde{M_2} \cong \mathbb{G}_{n-1}$. Put $M_3 := \mathcal{R}_{M_1}(M_2) = \mathcal{R}_{M_2}(M_1)$. Let $\overline{M_i}$, $i \in \{1,2,3\}$, denote the unique max of $DH(2n-1,4)$ containing M_i. Then $\overline{M_3} := \mathcal{R}_{\overline{M_1}}(\overline{M_2}) = \mathcal{R}_{\overline{M_2}}(\overline{M_1})$. By Proposition 2.9, there exists for any two distinct $i,j \in \{1,2,3\}$ a natural isomorphism between $\widetilde{\overline{M_i}}$ and $\widetilde{\overline{M_j}}$. This isomorphism induces an isomorphism between $\widetilde{M_i}$ and $\widetilde{M_j}$. Let f_i, $i \in \{1,2,3\}$, denote the valuation of $\widetilde{M_i}$ induced by f. For every point x of M_1, we define $f_1'(x) := f_2(\pi_{M_2}(x))$ and $f_1''(x) := f_3(\pi_{M_3}(x))$. By Propositions 2.10 and 2.13, f_1, f_1' and f_1'' are two by two neighboring valuations of $\widetilde{M_1}$ and $f_1'' = f_1 * f_1'$. We distinguish two cases.

Case I: f_1 and f_1' are equal.

In this case, $f_1 = f_1' = f_1''$. Let x^* denote a point of $M_1 \cup M_2 \cup M_3$ such that $f(x) \geqslant f(x^*)$ for every point $x \in M_1 \cup M_2 \cup M_3$. Let $i^* \in \{1,2,3\}$ such that $x^* \in M_{i^*}$. Considering the unique line through x^* meeting M_1, M_2 and M_3, we see that $f(\pi_{M_i}(x^*)) = f(x^*)+1$ for every $i \in \{1,2,3\} \setminus \{i^*\}$. Since $f_1 = f_1' = f_1''$, we necessarily have that $f(\pi_{M_i}(x)) = f(x)+1$ for every point x of M_{i^*} and every $i \in \{1,2,3\} \setminus \{i^*\}$. For every $i \in \{1,2,3\}$ there exists by the induction hypothesis a unique point $x_i^* \in \overline{M_i}$ such that the valuation f_i of $\widetilde{M_i}$ is induced by the classical valuation of $\widetilde{\overline{M_i}}$ with center x_i^*. Taking into account the natural isomorphisms between the near polygons $\widetilde{\overline{M_1}}$, $\widetilde{\overline{M_2}}$ and $\widetilde{\overline{M_3}}$, we see that $\{x_1^*, x_2^*, x_3^*\}$ must be a line of $DH(2n-1,4)$ meeting $\overline{M_1}$, $\overline{M_2}$ and $\overline{M_3}$. Put $y^* := x_{i^*}^*$ and let f^* be the valuation of $\mathbb{G}_n$ induced by the classical valuation of $DH(2n-1,4)$ with center y^*. Since $\mathrm{d}(y^*,x) = \mathrm{d}(y^*,\pi_{M_i}(y^*)) + \mathrm{d}(\pi_{M_i}(y^*),x) = \mathrm{d}(y^*,x_i^*) + \mathrm{d}(x_i^*,x)$ for every $i \in \{1,2,3\}$ and every point $x \in \overline{M_i}$, the valuation of $\widetilde{M_i}$ induced by f^* is equal to the valuation of $\widetilde{M_i}$ induced by the classical valuation of $\widetilde{\overline{M_i}}$ with center x_i^*, i.e. is equal to f_i.

Claim. *We prove that if f' is a valuation of $\mathbb{G}_n$ and $\epsilon \in \mathbb{Z}$ such that $f'(x) = f(x) + \epsilon$ for every point $x \in M_1 \cup M_2$, then $\epsilon = 0$ and $f' = f$.*

PROOF. (i) Let x_3 be an arbitrary point of M_3 and let L be the unique line through x_3 intersecting M_1 in a point x_1 and M_2 in a point x_2. If $f(x_1) = f(x_2)$, then $f(x_3) = f(x_1) - 1 = f(x_2) - 1$. If $f(x_1) \neq f(x_2)$, then $f(x_3) = \max\{f(x_1), f(x_2)\}$. Similarly, if $f'(x_1) = f'(x_2)$, then $f'(x_3) = f'(x_1) - 1 = f'(x_2) - 1$ and if $f'(x_1) \neq f'(x_2)$, then $f'(x_3) = \max\{f'(x_1), f'(x_2)\}$. Since $f'(x_1) = f(x_1) + \epsilon$ and $f'(x_2) = f(x_2) + \epsilon$, we have $f'(x_3) = f(x_3) + \epsilon$.

(ii) Let x be an arbitrary point of $\mathbb{G}_n$ not contained in $M_1 \cup M_2 \cup M_3$. Then by Lemma 4.6(2), there exists a unique $Q^-(5,2)$-quad Q_x through x which intersects M_1, M_2 and M_3 in lines. So, $G := Q_x \cap (M_1 \cup M_2 \cup M_3)$ is a (3×3)-grid. Since $f_1 = f_1' = f_1''$, the grid G is easily seen to contain a unique point u with smallest f-value. Moreover, $f(v) = f(u) + \mathrm{d}(u,v)$ for every point $v \in G$. By (i) we then also know that $f'(v) = f'(u) + \mathrm{d}(u,v)$ for every point $v \in G$. Hence, the valuations of Q_x induced by f and f' coincide with the classical valuation of Q_x with center u. This implies that $f'(y) = f(y) + \epsilon$ for every $y \in Q_x$. In particular, $f'(x) = f(x) + \epsilon$.

By (i) and (ii), f and f' differ by a constant ϵ. Since f and f' have minimal value 0, we have $\epsilon = 0$ and $f = f'$. (qed)

Since f_1 is the valuation of $\widetilde{M_1}$ induced by f and also the valuation of $\widetilde{M_1}$ induced by f^*, there exists an $\epsilon \in \mathbb{Z}$ such that $f^*(x) = f(x) + \epsilon$ for every $x \in M_1$.

If $y^* = x_1^*$, then $i^* = 1$ and $f(\pi_{M_2}(x)) = f(x) + 1$ for every $x \in M_1$. Since $\mathrm{d}(y^*,\pi_{M_2}(x)) = \mathrm{d}(y^*,x) + 1$, we also have $f^*(\pi_{M_2}(x)) = f^*(x) + 1$ for every $x \in M_1$.

If $y^* = x_2^*$, then $i^* = 2$ and $f(\pi_{M_2}(x)) = f(x) - 1$ for every $x \in M_1$. Since $\mathrm{d}(y^*,\pi_{M_2}(x)) = \mathrm{d}(y^*,x) - 1$, we also have $f^*(\pi_{M_2}(x)) = f^*(x) - 1$ for every $x \in M_1$.

If $y^* = x_3^*$, then $i^* = 3$ and $f(\pi_{M_2}(x)) = f(x)$ for every $x \in M_1$. Since $\mathrm{d}(y^*,\pi_{M_2}(x)) = \mathrm{d}(y^*,x)$, we also have $f^*(\pi_{M_2}(x)) = f^*(x)$ for every $x \in M_1$.

It follows that $f^*(x) = f(x) + \epsilon$ for every $x \in M_1 \cup M_2$. By the above Claim we then have that $f^* = f$. So, f is induced by a classical valuation of $DH(2n-1,4)$. Corollary 3.5 then implies that f is induced by a unique classical valuation of $DH(2n-1,4)$.

Case II: f_1 and f_1' are not equal.

By the induction hypothesis, there exists for every $i \in \{1,2,3\}$ a unique point $x_i \in \overline{M_i}$ such that the valuation f_i of $\widetilde{M_i}$ is induced by the classical valuation of $\widetilde{\overline{M_i}}$ with center x_i. Since the map $\overline{M_j} \to \overline{M_1}; x \mapsto \pi_{\overline{M_1}}(x)$, $j \in \{2,3\}$, is an isomorphism between $\widetilde{\overline{M_j}}$ and $\widetilde{\overline{M_1}}$, the valuation f_1' of $\widetilde{M_1}$ is induced by the classical valuation of $\widetilde{\overline{M_1}}$ with center $\pi_{\overline{M_1}}(x_2)$ and the valuation f_1'' of $\widetilde{M_1}$ is induced by the classical valuation of $\widetilde{\overline{M_1}}$ with center $\pi_{\overline{M_1}}(x_3)$. By Lemma 3.6, $L := \{x_1, \pi_{\overline{M_1}}(x_2), \pi_{\overline{M_1}}(x_3)\}$ is a line of $\overline{M_1}$. Now, let R denote the unique quad of $DH(2n-1,4)$ containing L and $\pi_{\overline{M_2}}(L)$. Then $G_R := R \cap (\overline{M_1} \cup \overline{M_2} \cup \overline{M_3})$ is a (3×3)-subgrid of R and $\{x_1, x_2, x_3\}$ is an ovoid of G_R.

Since $f_1 \neq f_1'$, there exists by Corollary 2.3(2) a line K of M_1 such that the unique point of K with smallest f_1-value is distinct from the unique point of K with smallest f_1'-value, or equivalently, such that the unique point u_1 of K with smallest f-value is not collinear with the unique point u_2 of $\pi_{M_2}(K)$ with smallest f-value. Here, u_i, $i \in \{1,2\}$, is the unique point of $\pi_{M_i}(K)$ nearest to x_i. Let u_3 denote the unique point of $\pi_{M_3}(K)$ nearest to x_3.

Now, consider an arbitrary $Q^-(5,2)$-quad T of M_1 through the line K. By Lemma 4.2(3), $\langle T, \pi_{M_2}(T)\rangle$ is a $\mathbb{G}_3$-hex. Applying Lemma 4.9 to the near polygon $\widetilde{\langle T, \pi_{M_2}(T)\rangle} \cong \mathbb{G}_3$, the big maxes T and $\pi_{M_2}(T)$ of $\widetilde{\langle T, \pi_{M_2}(T)\rangle}$ and the valuation of $\widetilde{\langle T, \pi_{M_2}(T)\rangle}$ induced by f, we see that we may without loss of generality suppose that the line K which we introduced in the previous paragraph is a special line of $\mathbb{G}_n$.

Let Q be the quad $\langle K, \pi_{M_2}(K)\rangle$. Since K is a special line, Q is a $Q^-(5,2)$-quad of both $\mathbb{G}_n$ and $DH(2n-1,4)$ (recall Lemma 4.6(3)). Let y be one of the three points of $R \setminus G_R$ such that $\Gamma_1(y) \cap G_R = \{x_1, x_2, x_3\}$. Then $\mathrm{d}(y,x) = \mathrm{d}(y,x_1) + \mathrm{d}(x_1,x) = 1 + \mathrm{d}(x_1,x)$ for every point x of $\overline{M_1}$. It follows that u_1 is the unique point of K nearest to y. In a similar way, one proves that u_2 is the unique point of $\pi_{M_2}(K)$ nearest to y. Since u_1 and u_2 are not collinear, Lemma 4.11 tells us that K and L are parallel lines and that Q and R are parallel quads.

We claim that u_i, $i \in \{1,2,3\}$, is the unique point of Q nearest to x_i. Suppose that this would not be the case. Then $\pi_Q(x_i) \notin M_i$. But then the unique point of $Q \cap M_i$ collinear with $\pi_Q(x_i)$ would lie closer to x_i than $\pi_Q(x_i)$ itself, clearly a contradiction.

Since $u_1 \neq u_2$, f can take two distinct values on the (3×3)-subgrid $G_Q := Q \cap (M_1 \cup M_2 \cup M_3)$ of Q. The points of G_Q with smallest f-value form the ovoid $\{u_1, u_2, u_3\}$ of G_Q. So, the unique point u^* of Q with smallest f-value (recall Lemma 4.10) is collinear with u_1, u_2 and u_3. Now, let x^* denote the unique point of R nearest to u^*. Since x_i, $i \in \{1,2,3\}$, is the unique point of R nearest to u_i, the point x^* is one the the three points of $R \setminus G_R$ collinear with x_1, x_2 and x_3. Now, let f^* denote the valuation of $\mathbb{G}_n$ induced by the classical valuation of $DH(2n-1,4)$ with center x^* and let $\epsilon \in \mathbb{Z}$ be such that $f^*(u^*) + \epsilon = f(u^*)$. We prove that $f^*(x) + \epsilon = f(x)$ for every point x of $M_1 \cup M_2 \cup Q$.

Since u^* is the unique point of Q nearest to x^*, we have $f^*(x) + \epsilon = f^*(u^*) + \epsilon + \mathrm{d}(u^*,x) = f(u^*) + \mathrm{d}(u^*,x) = f(x)$ for every point x of Q.

Let $i \in \{1,2\}$. Since $\mathrm{d}(x^*,y) = \mathrm{d}(x^*,x_i) + \mathrm{d}(x_i,y)$ for every $y \in \overline{M_i}$, the valuation of $\widetilde{M_i}$ induced by f^* coincides with the valuation of $\widetilde{M_i}$ induced by the classical valuation of $\widetilde{\overline{M_i}}$ with center x_i, i.e. with the valuation f_i of $\widetilde{M_i}$ induced by f. It follows that $f(x) - f^*(x)$ is independent from the point $x \in M_i$. By the previous paragraph, $f(x) - f^*(x) = f(u_i) - f^*(u_i) = \epsilon$.

By the two previous paragraphs, $f^*(x) + \epsilon = f(x)$ for every point x of $M_1 \cup M_2 \cup Q$. Lemma 4.13 then implies that $f^*(x) + \epsilon = f(x)$ for every point x of $\mathbb{G}_n$. Since the minimal values attained by f and f^* are equal to 0, we have $\epsilon = 0$ and $f = f^*$. So, f is induced by the classical valuation of $DH(2n-1,4)$ with center x^*. By Corollary 3.5, f is induced by a unique classical valuation of $DH(2n-1,4)$.

6 Proof of Theorem 1.2

We devote this short section to the proof of Theorem 1.2.

We regard $\mathbb{G}_n$, $n \geqslant 2$, as a subgeometry of $DH(2n-1,4)$ which is isometrically embedded into $DH(2n-1,4)$. Let f_1 and f_2 be two distinct valuations of $\mathbb{G}_n$. Then by Theorem 1.1 there exists a unique point x_i, $i \in \{1,2\}$, of $DH(2n-1,4)$ such that the valuation f_i of $\mathbb{G}_n$ is induced by the classical valuation f_i' of $DH(2n-1,4)$ with center x_i.

Suppose x_1 and x_2 are collinear. Then f_1' and f_2' are two neighboring valuations of $DH(2n-1,4)$ by Corollary 2.12(2). Proposition 2.8 then implies that f_1 and f_2 are neighboring valuations of $\mathbb{G}_n$.

Conversely, if f_1 and f_2 are neighboring valuations of $\mathbb{G}_n$, then by Lemma 3.6, x_1 and x_2 are collinear.

References

[1] R. J. Blok and A. E. Brouwer. The geometry far from a residue. *Groups and geometries* (Siena, 1996), 29–38, Trends Math., Birkhäuser, Basel, 1998.

[2] A. E. Brouwer. The uniqueness of the near hexagon on 729 points. *Combinatorica* 2 (1982), 333–340.

[3] A. E. Brouwer. The uniqueness of the near hexagon on 759 points. *Finite geometries*, pp. 47–60, Lecture Notes in Pure and Appl. Math. 82, Dekker, New York, 1983.

[4] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata* 49 (1994), 349–368.

[5] A. E. Brouwer and E. W. Lambeck. An inequality on the parameters of distance regular graphs and the uniqueness of a graph related to M_{23}. *Ann. Discrete Math.* 34 (1987), 101–106.

[6] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[7] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[8] B. N. Cooperstein. On the generation of dual polar spaces of unitary type over finite fields. *European J. Combin.* 18 (1997), 849–856.

[9] B. De Bruyn. On near-polygons and the Coxeter cap in PG(5, 3). *J. Geom.* 68 (2000), 23–33.

[10] B. De Bruyn. New near polygons from Hermitian varieties. *Bull. Belg. Math. Soc. Simon Stevin* 10 (2003), 561–577.

[11] B. De Bruyn. *Near polygons.* Frontiers in Mathematics, Birkhäuser, Basel, 2006.

[12] B. De Bruyn. The universal embedding of the near polygon $\mathbb{G}_n$. *Electron. J. Combin.* 14 (2007), Research paper 39, 12pp.

[13] B. De Bruyn. On the Grassmann-embeddings of the hermitian dual polar spaces. *Linear Multilinear Algebra* 56 (2008), 665–677.

[14] B. De Bruyn. An alternative definition of the notion valuation in the theory of near polygons. *Electron. J. Combin.* 16 (2009), Research paper 16, 14pp.

[15] B. De Bruyn. On the Grassmann modules for the unitary groups. *Linear and Multilinear Algebra*, to appear.

[16] B. De Bruyn. Isometric embeddings between dense near polygons. Preprint.

[17] B. De Bruyn and A. Pasini. Generating symplectic and Hermitian dual polar spaces over arbitrary fields nonisomorphic to $\mathbb{F}_2$. *Electron. J. Combin.* 14 (2007), Research Paper 54, 17 pp.

[18] B. De Bruyn and P. Vandecasteele. Valuations of near polygons. *Glasg. Math. J.* 47 (2005), 347–361.

[19] B. De Bruyn and P. Vandecasteele. The distance-2-sets of the slim dense near hexagons. *Ann. Comb.* 10 (2006), 193–210.

[20] B. De Bruyn and P. Vandecasteele. The valuations of the near hexagons related to the Witt designs $S(5,6,12)$ and $S(5,8,24)$. *J. Combin. Des.* 14 (2006), 214–228.

[21] B. De Bruyn and P. Vandecasteele. The valuations of the near octagon $\mathbb{G}_4$. *Discrete Math.*, to appear.

[22] A. Kasikova and E. E. Shult. Absolute embeddings of point-line geometries. *J. Algebra* 238 (2001), 265–291.

[23] P. Li. On the universal embedding of the $U_{2n}(2)$ dual polar space. *J. Combin. Theory Ser. A* 98 (2002), 235–252.

[24] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Research Notes in Mathematics 110. Pitman, Boston, 1984.

[25] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179–185.

[26] E. E. Shult. On Veldkamp lines. *Bull. Belg. Math. Soc. Simon Stevin* 4 (1997), 299–316.

[27] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.

[28] J. Tits. *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Mathematics 386. Springer-Verlag, Berlin-New York, 1974.