COUNTING 1324, 4231-AVOIDING PERMUTATIONS

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A complete structural description and enumeration is found for permutations that avoid both 1324 and 4231.

1. INTRODUCTION

A permutation α is said to be a subpermutation of a permutation β if β contains a subsequence whose terms are ordered relatively the same as the terms of α. For example 231 is a subpermutation of 31524 by virtue of the subsequence 352 whereas 231 is not a subpermutation of 51324. Classes of permutations are sets of permutations that are closed downwards under taking subpermutations. They are usually presented as sets $\mathcal{C}$ that avoid a given set B of permutations (i.e. the permutations of $\mathcal{C}$ have no subpermutation in the set B). We express this by the notation $\mathcal{C} = \mathrm{Av}(B)$. Much of the inspiration for elucidating the structure of pattern classes has been driven by the enumeration problem: given $\mathcal{C} = \mathrm{Av}(B)$, how many permutations of each length does $\mathcal{C}$ contain? The answer to such a question can be a formula giving this number c_n in terms of the length n, or it may be a generating function $\sum_{n=1}^{\infty} c_n x^n$ or it may simply be an asymptotic result about the behaviour of c_n as $n \to \infty$.

The subpermutation relation is invariant under 8 symmetries generated by reversal, complementation, and inversion. These symmetries can be used to cut down the number of cases in many investigations since, for every such symmetry $\sigma \to \sigma^{\omega}$, we have

$$\mathcal{C}^{\omega} = \mathrm{Av}(B)^{\omega} = \mathrm{Av}(B^{\omega})$$

For sets B containing only a single permutation β exact enumerations are known for $|\beta| \leqslant 4$ with one notable exception, the case $\beta = 1324$ (or its symmetry 4231) where only lower and upper bounds are known [2, 6]. For sets $B = \{\alpha, \beta\}$ exact enumerations are known when $|\alpha| \leqslant 3$ and $|\beta| \leqslant 4$ but far less is known in the case $|\alpha| = |\beta| = 4$. It is known that there are 56 essentially different pairs (i.e. inequivalent under symmetries) α, β of length 4 and that they give rise to 38 different enumerations [5, 7, 8, 9, 10] (some inequivalent pairs are *Wilf-equivalent* meaning that they nevertheless have the same enumeration). Of these 38 Wilf classes 23 have yet to be enumerated.

Here we consider the class $\mathcal{C} = \mathrm{Av}(1324, 4231)$. This class is of interest because both $\mathrm{Av}(1324)$ and $\mathrm{Av}(4231)$ have unknown enumerations and techniques such as generating trees [13], the insertion coding [4] and the WILFPLUS program [11] seem unable to enumerate it. Our approach in this paper is to use some of the theory of simple permutations developed in [1] which appears to have considerable promise for problems of this type.

In the next section we give some necessary notation and definitions, assemble some technical results, and give a structural decomposition of the class $\mathrm{Av}(1324, 4231)$ together with a recurrence to enumerate its simple permutations. In the final section we put all the ingredients together to give the complete generating function for the class.

2. PRELIMINARY RESULTS

An *interval* of a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is a subsequence $\pi(i)\pi(i+1)\cdots\pi(j)$ whose values form a consecutive set. If a permutation has no intervals except for itself and singletons it is said to be *simple*. For example 68352471 has non-trivial intervals 3524 and 6835247 while 31524 is simple.

Simple permutations are precisely those that do not arise from a non-trivial inflation, in the following sense. Let σ be any permutation of length m and $\alpha_1, \alpha_2, \ldots, \alpha_m$ any sequence of permutations. Then the *inflation* of σ by $\alpha_1, \alpha_2, \ldots, \alpha_m$, which we denote by $\sigma[\alpha_1, \alpha_2, \ldots, \alpha_m]$, is that permutation of length $|\alpha_1| + \cdots + |\alpha_m|$ which decomposes into m segments $\alpha'_1 \alpha'_2 \cdots \alpha'_n$ where each segment α'_i is an interval which is order isomorphic to α_i, and the sequence $a_1 a_2 \cdots a_n$ formed by any (and hence every) choice of a_i from α'_i is order isomorphic to σ. For example the inflation of 3142 by $21, 132, 1, 123$ is

$$3142[21, 132, 1, 123] = 87\ 132\ 9\ 456$$

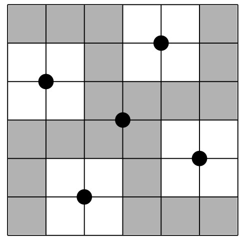

Figure 1: A permutation in $\mathrm{Av}(1324, 4231)$ containing 41352.

The precise connection between simple permutations and inflations is furnished by a result from [1].

Lemma 1. *For every permutation π there is a unique simple permutation σ such that $\pi = \sigma[\alpha_1, \alpha_2, \ldots, \alpha_m]$. Furthermore, except when $\sigma = 12$ or $\sigma = 21$ the intervals of σ that correspond to $\alpha_1, \alpha_2, \ldots, \alpha_m$ are uniquely determined. In the case that $\sigma = 12$ (respectively $\sigma = 21$), the intervals are unique so long as we require the first of the two intervals to be* sum *(respectively,* skew*) indecomposable which means that it cannot be decomposed further as $12[\gamma, \delta]$ (respectively $21[\gamma, \delta]$).*

Our methodology for enumerating $\mathrm{Av}(1324, 4231)$ is first to determine its simple permutations and then, for each one, to determine how many inflations lie in the class. By the previous lemma this will deliver the count of all permutations in the class. It turns out that the simple permutations, apart from two of them, can all be constructed in a recursive way from smaller simple permutations. The two exceptional simple permutations are 25314 and 41352. We deal with these permutations by the following proposition.

Proposition 2. *If the permutation $\pi \in \mathrm{Av}(1324, 4231)$ contains 41352, then it is of the form $41352[\alpha_1, \alpha_2, 1, \alpha_4, \alpha_5]$ where*

1. $\alpha_1 \in \mathrm{Av}(132, 312)$
2. $\alpha_2 \in \mathrm{Av}(132, 231)$
3. $\alpha_4 \in \mathrm{Av}(213, 312)$
4. $\alpha_5 \in \mathrm{Av}(213, 231)$

Conversely, all permutations of this form belong to $\mathrm{Av}(1324, 4231)$ and contain 41352. A similar result holds for the reverse permutation 25314.

Proof. The proposition is readily verified by "diagram-chasing". We begin from a diagram of π which displays 5 points that correspond to the subpermutation 41352, together with the regions that correspond to all the other points of π according to their relation to these five. Then, by exploiting the 1324- and 4231-avoidance, we find that all of the shaded regions in Figure 1 are empty.

The avoidance conditions further imply that the 4 unshaded regions do not overlap either by position or by value. This proves that

$$\pi = 41352[\alpha_1, \alpha_2, 1, \alpha_3, \alpha_4]$$

The conditions on $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ then follow readily from the avoidance conditions. The converse is easily checked. ☐

The subclasses defined by two restrictions of length 3 that feature in this result are well understood. The third one, for example, consists of all permutations that have a division into two segments, the first increasing and the second decreasing. The diagram of such permutations is roughly shaped like a wedge pointing upwards and so we denote this class by $\wedge$. The other classes are symmetries of this class and have wedge-shaped diagrams pointing respectively left, down, up and right and are denoted $<$,$\vee$,$\wedge$ and $>$. We refer to them all as "wedge classes".

The permutations of $\mathrm{Av}(1324, 4231)$ fall into three categories: they contain 41352 or contain 25314, or contain neither of these permutations. Proposition 2 gives a description of the first two of these categories. It follows that $\mathrm{Av}(1324, 4231)$ is the union of three classes:

- $41352[<, \vee, 1, \wedge, >]$ and all its subpermutations,
- $25314[<, \wedge, 1, \vee, >]$ and all its subpermutations, and
- $\mathrm{Av}(1324, 4231, 25314, 41352)$.

Our initial interest in this class came from this observation. First, it is rare to find classes with only a few short basis elements which admit such nice decompositions as unions. Moreover, the third class listed is a subclass of the "convex class" of Albert et al. [3], that is, the class of all permutations which can be drawn in convex position. Furthermore, it was observed that the simple permutations in $\mathrm{Av}(1324, 4231, 25314, 41352)$ can be "drawn on a circle" in the sense of Vatter and Waton [12]. While these observations do not appear in our proof, much of our underlying intuition was drawn from them initially.

Proposition 3. *In any simple permutation in* $\mathrm{Av}(1324, 4231)$ *other than* 25314 *or* 41352*, the maximum and minimum entries occur consecutively.*

Proof. By passing to the reverse permutation if necessary we consider a simple permutation π of length n in $\mathrm{Av}(1324, 4231)$ in which the symbol 1 precedes the symbol n. If the symbols 1 and n do not occur consecutively, then they are horizontally separated by some entry $\pi(i)$. Proposition 2 then implies that either

- there are no entries to the right of n and vertically between 1 and $\pi(i)$, or
- there are no entries to the left of 1 and vertically between $\pi(i)$ and n.

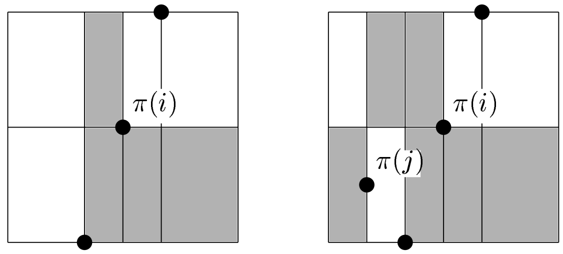

Figure 2: The situation in the proof of Proposition 3.

By considering the reverse-complement of π if necessary, we may assume that the former holds. Now let us choose $\pi(i)$ to be the entry horizontally between 1 and n of minimal value.

We now have the situation depicted on the left of Figure 2 where the shaded regions are empty. Note that the region horizontally between 1 and $\pi(i)$ and vertically above $\pi(i)$ is empty because any entry in this region would give rise to a copy of 1324. Because π is simple, 1 and $\pi(i)$ must be separated, and thus they must be separated by an entry $\pi(j)$ to the left. Selecting the leftmost such entry gives us the situation depicted on the right of Figure 2 (where the region above $\pi(i)$ but between $\pi(j)$ and 1 is empty because of the 1324 avoidance). As shown in the figure, however, there is now no way to separate the interval containing this new entry, 1, and $\pi(i)$, contradicting our assumption that π is simple. ☐

In the next proposition notation such as $\pi \setminus t$ denotes the permutation obtained by renumbering the points of $\pi \setminus \{t\}$ so that it is a permutation. Dually (and used in the proof of Proposition 5) we refer to inserting an entry t into a permutation π (requiring the values greater than or equal to t to be incremented by 1).

Proposition 4. *Every simple permutation* $\pi \in \mathrm{Av}_n(1324, 4231)$ *other than* 1, 12, 21, 25314, *or* 41352 *in which* 1 *occurs before* n *contains one of the following segments*

1. $a1n2$ *with* $a \neq n-1$*; in this case* $\pi \setminus 1$ *is simple,*
2. $(n-1)1nb$ *with* $b \neq 2$*; in this case* $\pi \setminus n$ *is simple, or*
3. $(n-1)1n2$*; in this case* $\pi \setminus \{1, n\}$ *is simple.*

Proof. In the first case $\pi \setminus 1$ is simple because any interval must have been split by position in π by the symbol 1. However, although the interval contains a and n it cannot also contain 2 without being the whole of $\pi \setminus 1$. Hence the interval ends at n. It is not a doubleton since $a \neq n-1$ and therefore it contains a proper interval ending at a; but this would also have been an interval of π. The second case follows by a similar argument, while the third case is even easier. Here $\pi \setminus \{1, n\}$ cannot contain an interval since this would have been split by position in π by $n1$ and so would have contained 2 and $n-1$.

To complete the proof we have to show that there are no more cases. In other words we have to prove that a permutation of the type $\alpha a 1 n b \beta$ with $a \neq n-1$ and $b \neq 2$ is not simple. Note that not both of 2 and $n-1$ can precede a (they would lead to the forbidden sequences $2(n-1)an$ or $(n-1)2a1$). Nor, similarly, can both 2 and $n-1$ follow b.

Next suppose that 2 precedes a and $n-1$ succeeds b. Since 1324 is forbidden and $2ab(n-1)$ is a subsequence we know that $a < b$. The positions of π up to the position containing 2 contain no entries p larger than a (else $p2a1 \sim 4231$) and the positions between the positions of 2 and a also contain no entries larger than a (else $2pan \sim 1324$). On the other hand there can be no entries $p < a$ between the positions of b and $n-1$ (else $1ap(n-1) \sim 1324$) nor between the positions following the position of $n-1$ (else $nb(n-1)p \sim 4231$). It follows that the first a positions form an interval. A similar argument holds when $n-1$ precedes a and 2 succeeds b. □

Proposition 5. *The number s_n of simple permutations of length n in $\mathrm{Av}(1324, 4231)$ satisfies $s_n = 2s_{n-1} + s_{n-2}$ for $n \geqslant 8$.*

Proof. The preceding proposition shows how the simple permutations arise. Conversely it is easy to see that permutations in which 1 or n or both have been inserted into simple permutations according to the previous proposition are necessarily simple. □

Proposition 6. *The simple permutations of $\mathrm{Av}(1324, 4231)$ other than 1, 12, 21, 25314, and 41352 have the generating function $4x^4/(1-2x-x^2)$.*

Further generating functions results we shall need appear in the next proposition.

Proposition 7. *We have the following enumerative results.*

permutations	*generating function*
(a) nonempty permutations in any particular wedge class	$\frac{x}{1-2x}$
(b) sum (or skew) indecomposable permutations in any wedge class	$\frac{x(1-x)}{1-2x}$
(c) non-singleton sum indecomposable permutations in $\mathrm{Av}(132, 4231)$	$\frac{x^2(1-x)}{(1-2x)^2}$
(d) $\mathrm{Av}(213, 4231)$	$\frac{x(1-3x+3x^2)}{(1-x)(1-2x)^2}$

Proof. Consider a wedge class oriented as $>$, i.e., $\mathrm{Av}(231, 213)$. Every non-singleton permutation in this class can be described as either $12[1, \pi]$ or as $21[1, \pi]$ for some $\pi \in \mathrm{Av}(231, 213)$. Thus the generating function, f, for this class satisfies $f = x + 2xf$, while the generating function for the sum indecomposable elements is $x + xf$, verifying both (a) and (b).

For (c), note that the permutations of $\mathrm{Av}(132, 4231)$ in question can be described as $21[\pi, \sigma]$ where π is a skew indecomposable permutation in the wedge class $\mathrm{Av}(132, 312)$ and σ is an arbitrary permutation in the wedge class $\mathrm{Av}(132, 231)$. The generating function for these permutations then follows from (a) and (b).

Finally, every non-singleton permutation in $\mathrm{Av}(213, 4231)$ is of one of two forms, either $12[1, \pi]$ for $\pi \in \mathrm{Av}(213, 4231)$ or $21[\pi, \sigma]$ for a skew indecomposable permutation

π in the wedge class $\mathrm{Av}(213, 312)$ and an arbitrary permutation σ in the wedge class $\mathrm{Av}(213, 231)$. Thus we have from (a) and (b) that

$$f = x + xf + \frac{x}{1-2x} \cdot \frac{x(1-x)}{1-2x},$$

and solving this gives (d). $\square$

3. Main Theorem

Theorem 8. *The generating function (including the empty permutation) for* $\mathrm{Av}(1324, 4231)$ *is*

$$\frac{1 - 12x + 59x^2 - 152x^3 + 218x^4 - 168x^5 + 58x^6 - 6x^7}{(1-x)(1-2x)^4(1-4x+2x^2)}.$$

Proof. With the simple permutations in $\mathrm{Av}(1324, 4231)$ categorized, it remains only to consider their 1324-, 4231-avoiding inflations.

By symmetry, there are precisely as many inflations of 21 as of 12 in $\mathrm{Av}(1324, 4231)$, so we count the latter (the sum decomposable permutations). Suppose that π is a sum decomposable permutation in $\mathrm{Av}(1324, 4231)$. Then $\pi = 12[\alpha_1, \alpha_2]$ for a sum indecomposable α_1. It follows that α_1 must avoid 132 and 4231, while α_2 must avoid 213 and 4231. From Proposition 7 (c) and (d) it follows that the generating functions for inflations of 12 and 21 in our class is

$$2\frac{x^2(1-x)}{(1-2x)^2} \cdot \frac{x(1-3x+3x^2)}{(1-x)(1-2x)^2} = \frac{2x^3(1-3x+3x^2)}{(1-2x)^4}. \tag{1}$$

The inflations of 25314 (and, by symmetry 41352) are also easily counted. Note that the 3 in 25314 may not be inflated at all, while every other entry may be inflated only by a permutation from a particular wedge class; for example, the 2 may be inflated only by permutations from $\mathrm{Av}(132, 312)$. Thus the generating function for inflations of these two simple permutations in our class is, using Proposition 7 (a),

$$2x\left(\frac{x(1-x)}{1-2x}\right)^4 = \frac{2x^5(1-x)^4}{(1-2x)^4}. \tag{2}$$

This leaves us with the remaining simple permutations, which by Proposition 3, all have adjacent minimum and maximum elements. Figure 3 shows the general 'shape' of such simple permutations and, while we shall not appeal to this figure in an essential way, it will be found helpful for the following arguments. We shall show that the first, last, minimum and maximum elements can be inflated by an entire wedge class but that all other points (*interior* points) can only be inflated either by an arbitrary increasing permutation or by an arbitrary decreasing permutation. Consider any interior point which, without loss, we shall take to be to the left of the minimum-maximum pair. This point p say, has at least one predecessor point and so is the middle point of

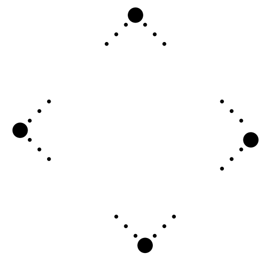

Figure 3: The form of a simple permutation not containing 41352 or 25314.

either a 123 or 321 pattern (but, by the avoidance conditions or Figure 3, not both); so p can be inflated, respectively, by any increasing permutation or by any decreasing permutation. Now consider the first point (similar arguments will apply to the last, minimum and maximum points). It is followed by both larger and smaller points. Because of the 1324 and 4231 avoidance conditions this point can only be inflated by a permutation that avoids 132 and 312. Such permutations lie in the wedge class $<$ and clearly every inflation by such a permutation continues to avoid 1324 and 4231.

Thus, by Proposition 6, the contribution of these permutations to the generating function of $\mathrm{Av}(1324, 4231)$ is

$$\frac{2\left(\dfrac{x(1-x)}{1-2x}\right)^4}{1-\dfrac{2x}{1-x}-\left(\dfrac{x}{1-x}\right)^2} = \frac{2x^4(1-x)^6}{(1-2x)^4(1-4x+2x^2)}. \tag{3}$$

Summing the quantities from (1)–(3) and $1+x$ (which counts the empty and trivial permutations) gives the generating function stated. $\square$

We note finally that the growth rate of $\mathrm{Av}(1324, 4231)$ can be read off from its generating function as the reciprocal of the smallest root of the denominator; it is therefore $2+\sqrt{2}$.

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