On Suborbital Graphs for the Normalizer of $\Gamma_0(N)$

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Abstract

In this study, we deal with the conjecture given in [R. Keskin, Suborbital graph for the normalizer of $\Gamma_0(m)$, European Journal of Combinatorics 27 (2006) 193-206.], that when the normalizer of $\Gamma_0(N)$ acts transitively on $\mathbb{Q}\cup\{\infty\}$, any circuit in the suborbital graph $G(\infty, u/n)$ for the normalizer of $\Gamma_0(N)$, is of the form

$$v \to T(v) \to T^2(v) \to \ \cdots \to T^{k-1}(v) \to v,$$

where $n > 1$, $v \in \mathbb{Q}\cup\{\infty\}$ and T is an elliptic mapping of order k in the normalizer of $\Gamma_0(N)$.

1. Introduction

Let N be a positive integer and let $N(\Gamma_0(N))$ be the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$. The normalizer $N(\Gamma_0(N))$ was studied for the first time by Lehner and Newman in [10]. The correct normalizer was determined by Atkin and Lehner in [3]. A complete description of the elements of $N(\Gamma_0(N))$ is given in [14]. Especially, a necessary and sufficient condition for $N(\Gamma_0(N))$ to act transitively on the set $\hat{\mathbb{Q}} = \mathbb{Q}\cup\{\infty\}$ of the cusps of $N(\Gamma_0(N))$ was given in [2] . If we represent the elements of $N(\Gamma_0(N))$ by the associated matrices, then the normalizer consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$$

where $e \mid (N/h^2)$ such that $(e, (N/h^2)/e) = 1$ and h is the largest divisor of 24 for which $h^2 | N$ with the understanding that the determinant of the matrix is $e > 0$. If $e \mid N$ and

$(e, N/e) = 1$, we represent this as $e \mid\mid N$ and we say that e is an exact divisor of N. Thus we have

$$N(\Gamma_0(N)) = \left\{ A = \begin{pmatrix} a\sqrt{q} & b/h\sqrt{q} \\ cN/h\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det A = 1,\ q \mid\mid (N/h^2);\ a, b, c, d \in \mathbb{Z} \right\}.$$

In [9], it was shown that when $n > 1$ and m is a square-free positive integer, any circuit in the suborbital graph $G(\infty, u/n)$ for $N(\Gamma_0(m))$ is of the form

$$v \to T(v) \to T^2(v) \to T^3(v) \to \ \cdots \to T^{k-1}(v) \to v$$

for a unique elliptic mapping $T \in N(\Gamma_0(m))$ of order k and for some $v \in \hat{\mathbb{Q}}$. After that it was conjectured that the same is true when $N(\Gamma_0(m))$ acts transitively on $\hat{\mathbb{Q}}$(See section 2, for the definition of suborbital graph.). In this paper, we deal with this conjecture. Before discussing this conjecture we also investigate suborbital graphs for some Hecke groups, which are conjugate to $N(\Gamma_0(m))$. Moreover, we give simple and different proofs of some known theorems for the sake of completeness.

2. The Action of $N(\Gamma_0(N))$ on $\hat{\mathbb{Q}}$

Let N be a natural number and let

$$\Gamma_0^+(N) = \left\{ A = \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cN/\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det A = 1,\ 1 \leqslant q,\ q \mid\mid N;\ a, b, c, d \in \mathbb{Z} \right\}.$$

Then $\Gamma_0^+(N)$ is a subgroup of the normalizer of $\Gamma_0(N)$. Moreover, any element of $\Gamma_0^+(N)$ is an Atkin-Lehner involution of $\Gamma_0(N)$. Recall that an Atkin-Lehner involution w_q of $\Gamma_0(N)$ is an element of determinant 1 of the form

$$w_q = \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cN/\sqrt{q} & d\sqrt{q} \end{pmatrix}$$

for some exact divisor q of N. If $h = 1$, then $\Gamma_0^+(N)$ is equal to $N(\Gamma_0(N))$.

Let $m = N/h^2$. Then, it is well-known and easy to see that

$$N(\Gamma_0(N)) = \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix} \Gamma_0^+(m) \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix}^{-1}.$$

We will use this fact in the subsequent theorems.

Theorem 2.1. $\Gamma_0^+(N)$ *acts transitively on the set* $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ *if and only if* N *is a square-free positive integer.*

Proof. Let $\Gamma_0^+(N)$ act transitively on the set $\hat{\mathbb{Q}}$ and assume that N is not a square-free positive integer. Then $N = k^2 m$ for some $k > 1$. Since $\Gamma_0^+(N)$ acts transitively on $\hat{\mathbb{Q}}$, there exists some $T \in \Gamma_0^+(N)$ satisfying $T(\infty) = 1/km$. Since $T \in \Gamma_0^+(N)$, there exists some $q \mid\mid N$ such that

$$T = \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cN/\sqrt{q} & d\sqrt{q} \end{pmatrix} \text{ where } adq - bcN/q = 1.$$

Then $(a\sqrt{q})/(cN/\sqrt{q}) = 1/km$, i.e.,$\dfrac{a}{cN/q} = 1/km$. Since $(a, cN/q) = 1$, $a = \pm 1$ and $cN/q = \pm km$. It follows from $N = k^2 m$, $ck^2 m = \pm kmq$ that $q = \pm kc$ with $(q, c) = 1$. So we have

$$1 = (q, c) = (\pm kc, c) = |c|\,(\pm k, 1) = |c|\,.$$

Thus $c = \pm 1$ and $q = \pm kc = k$. Since q is an exact divisor of N, $(q, N/q) = 1$. Then it follows that $k = (k, km) = (k, N/k) = (q, N/q) = 1$, which contradicts our assumption that $k > 1$. Thus N is a square-free positive integer.

Now suppose that N is a square-free positive integer. Let $k/s \in \hat{\mathbb{Q}}$ with $(k, s) = 1$ and $q = (s, N)$. Then $s = s^* q$ for some integer s^*. Since N is square-free, $(s, N/q) = 1$. Thus we have $(s, kN/q) = 1$. Therefore there exist two integers x and y such that $sx - (N/q)\,ky = 1$. Let

$$T(z) = \frac{x\sqrt{q}z + k\sqrt{q}}{\left(yN/\sqrt{q}\right) z + s^*\sqrt{q}}.$$

Then $T \in \Gamma_0^+(N)$ and $T(0) = k/s^* q = k/s$. Thus the proof follows. ■

Now we can give the following theorem.

Theorem 2.2. *Let $m = N/h^2$. Then $N(\Gamma_0(N))$ acts transitively on the set $\hat{\mathbb{Q}}$ if and only if $\Gamma_0^+(m)$ acts transitively on the set $\hat{\mathbb{Q}}$.*

Proof. Since

$$N(\Gamma_0(N)) = \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix} \Gamma_0^+(m) \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix}^{-1},$$

the proof follows. ■

The following theorem is proved in [2]. We will present a different proof.

Theorem 2.3. *Let N have prime power decomposition $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} \ldots . p_r^{\alpha_r}$. Then $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \leqslant 7, \alpha_2 \leqslant 3, \alpha_i \leqslant 1, i = 3, 4, \ldots, r$.*

Proof. Let $m = N/h^2$ and assume that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$. Then, in view of the above theorem $\Gamma_0^+(m)$ acts transitively on $\hat{\mathbb{Q}}$. Thus m is a square-free positive integer according to Theorem2.1. Let $m = 2^{k_1} 3^{k_2} p_3^{\alpha_3} \ldots . p_r^{\alpha_r}$ with $k_i, \alpha_i \in \{0, 1\}$. Since h is the largest divisor of 24 for which $h^2 \mid N$, then $h = 2^{t_1}3^{t_2}$ for some integers t_1 and t_2 such that $0 \leqslant t_1 \leqslant 3$ and $0 \leqslant t_2 \leqslant 1$. Thus we have

$$N = mh^2 = 2^{k_1+2t_1}3^{k_2+2t_2}p_3^{\alpha_3}....p_r^{\alpha_r} = 2^{\alpha_1}3^{\alpha_2}p_3^{\alpha_3}....p_r^{\alpha_r},$$

where $\alpha_1 = k_1 + 2t_1$, $\alpha_2 = k_2 + 2t_2$. Hence we see that

$$\alpha_1 = k_1 + 2t_1 \leqslant 1 + 2.3 = 7 \text{ and } \alpha_2 = k_2 + 2t_2 \leqslant 1 + 2.1 = 3.$$

Now suppose that $N = 2^{\alpha_1}3^{\alpha_2}p_3^{\alpha_3}....p_r^{\alpha_r}$, where $\alpha_1 \leqslant 7$, $\alpha_2 \leqslant 3$, $\alpha_i \leqslant 1$ for $i = 3, 4, ..., r$. Dividing α_1 and α_2 by 2 we get,

$$\begin{aligned} \alpha_1 &= 2t_1 + r_1,\ 0 \leqslant r_1 \leqslant 1 \\ \alpha_2 &= 2t_2 + r_2,\ 0 \leqslant r_2 \leqslant 1. \end{aligned}$$

Since $\alpha_1 \leqslant 7,\ \alpha_2 \leqslant 3$, we see that $0 \leqslant t_1 \leqslant 3$ and $0 \leqslant t_2 \leqslant 1$. This gives

$$N = 2^{2t_1}3^{2t_2}2^{r_1}3^{r_2}p_3^{\alpha_3}....p_r^{\alpha_r} = \left(2^{t_1}3^{t_2}\right)^2 2^{r_1}3^{r_2}p_3^{\alpha_3}....p_r^{\alpha_r}.$$

Let $h = 2^{t_1}3^{t_2}$ with $0 \leqslant t_1 \leqslant 3$, $0 \leqslant t_2 \leqslant 1$. Then h is the largest divisor of 24 such that h^2 divides N. Let $m = N/h^2 = 2^{r_1}3^{r_2}p_3^{\alpha_3}....p_r^{\alpha_r}$. Then it is clear that m is a square-free positive integer. Thus, by Theorem2.1, $\Gamma_0^+(m)$ acts transitively on $\hat{\mathbb{Q}}$, and it follows that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$ by Theorem2.2. So the proof is completed. ■

3. Suborbital Graphs For $N(\Gamma_0(N))$

Let (G, X) be a transitive permutation group. Then G acts on $X \times X$ by

$$g : (\alpha, \beta) \to (g(\alpha), g(\beta))\ ,\ (g \in G,\ \alpha, \beta \in X).$$

The orbits of this action are called suborbitals of G. The suborbital containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a suborbital graph $G(\alpha, \beta)$ whose vertices are the elements of X, and there is an edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. If there is an edge γ to δ, we will represent this by $\gamma \to \delta$. Briefly, there is an edge $\gamma \to \delta$ in $G(\alpha, \beta)$ iff there exists $T \in G$ such that $T(\alpha) = \gamma$ and $T(\beta) = \delta$. If $\alpha = \beta$, then $O(\alpha, \beta)$ is the diagonal of $X \times X$ and $G(\alpha, \beta)$ is said to be a trivial suborbital graph. We will interested in non-trivial suborbital graph. Since G acts transitively on X, any suborbital graph is equal to $G(\lambda_0, \lambda)$ for a fixed λ_0.

Let $G(\alpha, \beta)$ be a suborbital graph and let $k \geqslant 3$ be a natural number. By a circuit of the length k, we mean different k vertices $v_0, v_1, ..., v_k = v_0$ such that $v_0 \to v_1$ is an edge in the graph $G(\alpha, \beta)$ and for $1 \leqslant r \leqslant k-1$, either $v_r \to v_{r+1}$ or $v_{r+1} \to v_r$ is an edge in the graph $G(\alpha, \beta)$. Let G have an element T of finite order $k \geqslant 3$. It can be seen that if $\alpha \neq T(\alpha)$, then

$$\alpha \to T(\alpha) \to T^2(\alpha) \to \ \cdots \to T^{k-1}(\alpha) \to \alpha$$

is a circuit of the length k in the graph $G(\alpha,\ T(\alpha))$.

We now investigate suborbital graphs for $N(\Gamma_0(N))$. If $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$, then any non-trivial suborbital graph is equal to $G(\infty, u/n)$ for some $u/n \in \mathbb{Q}$. We give the following theorem from [9].

Theorem 3.1. *Let m be a square-free positive integer and let $G(\infty, u/n)$ be suborbital graph for $N(\Gamma_0(m))$. Then any circuit in $G(\infty, u/n)$ is of the form*

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

for a unique elliptic mapping $T \in N(\Gamma_0(m))$ of order k and for some $v \in \hat{\mathbb{Q}}$ where $n > 1$ and $(u, n) = 1$.

Unless $n > 1$, the above theorem may not be correct. Before giving the examples, we give some lemmas and theorems for the graph $G(\infty, 1)$. The following lemma appears in [9] as Corollary 1.

Lemma 1. *Let m be a square-free positive integer and let $G(\infty, 1)$ be suborbital graph for $N(\Gamma_0(m))$. Then, $r/s \to x/y$ is an edge in $G(\infty, 1)$ if and only if $ry - sx = \pm 1$ and $q|s, (m/q)|y$ for some $q|m$.*

Let m be a square-free positive integer and let $G(\infty, 1)$ be suborbital graph for $N(\Gamma_0(m))$. If $r/s \to x/y$ is an edge in $G(\infty, 1)$, then there exists $A \in N(\Gamma_0(m))$ such that $A(\infty) = r/s$ and $A(1) = x/y$. Let

$$T = \begin{pmatrix} -\sqrt{m} & \frac{m+1}{\sqrt{m}} \\ -\sqrt{m} & \sqrt{m} \end{pmatrix}.$$

Then $T(\infty) = 1$ and $T(1) = \infty$. Thus $AT(\infty) = A(1) = x/y$ and $AT(1) = A(\infty) = r/s$, and so $x/y \to r/s$ is an edge in $G(\infty, 1)$. If we represent the edges of $G(\infty, 1)$ as hyperbolic geodesics in the upper-half plane $U = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, then no edges of $G(\infty, 1)$ cross in $\mathcal{U}$(See [9]). Using these facts and the above lemma, we can give the following theorem.

Theorem 3.2. *Let m be a square-free positive integer. A circuit of minimal length in the graph $G(\infty, 1)$ for $N(\Gamma_0(m))$ is of the form*

$$v \to S(v) \to S^2(v) \to S^3(v) \to \cdots \to S^{k-1}(v) \to v$$

for an elliptic mapping $S \in N(\Gamma_0(m))$ and for some $v \in \hat{\mathbb{Q}}$. If $m \geqslant 5$, then $G(\infty, 1)$ does not contain any circuits.

Proof. Let

$$w_0 \to w_1 \to w_2 \to w_3 \to \cdots \to w_{k-2} \to w_{k-1} \to w_0$$

be a circuit of the minimal length in $G(\infty, 1)$. Then

$$A(\infty) = w_0 \quad , \quad A(1) = w_1$$

for some $A \in N(\Gamma_0(m))$. By applying the mapping A^{-1} to vertices of the circuit, we obtain the following circuit,

$$\infty \to 1 \to A^{-1}(w_2) \to \cdots \to A^{-1}(w_{k-2}) \to A^{-1}(w_{k-1}) \to \infty.$$

Since no edges of $G(\infty, 1)$ cross in the upper-half plane, either

$$1 < A^{-1}(w_2) < \cdots < A^{-1}(w_{k-2}) < A^{-1}(w_{k-1})$$

or

$$1 > A^{-1}(w_2) > \cdots > A^{-1}(w_{k-2}) > A^{-1}(w_{k-1}).$$

If $r/s \to x/y$ is an edge in the graph $G(\infty, 1)$, it can be shown that $(2 - r/s) \to (2 - x/y)$ is an edge in the graph $G(\infty, 1)$. To see this, suppose that $r/s \to x/y$ is an edge in $G(\infty, 1)$. Then there exists $A \in N(\Gamma_0(m))$ such that $A(\infty) = r/s$ and $A(1) = x/y$. Let $\Psi(z) = 2 - z$. Then it follows that $\Psi A \Psi \in N(\Gamma_0(m))$, $\Psi A \Psi(\infty) = 2 - r/s$, and $\Psi A \Psi(1) = 2 - x/y$. Thus $(2 - r/s) \to (2 - x/y)$ is an edge in the graph $G(\infty, 1)$. Therefore we may suppose that

$$1 < A^{-1}(w_2) < \cdots < A^{-1}(w_{k-2}) < A^{-1}(w_{k-1}).$$

Let $v_0 = \infty, v_1 = 1, v_{k-1} = 2$, and $v_j = A^{-1}(w_j)$ for $2 \leqslant j \leqslant k-1$. Then

$$v_0 \to v_1 \to v_2 \to \cdots \to v_{k-2} \to v_{k-1} \to v_0$$

is a circuit of the minimal length. Let $v_{k-1} = x/y$. Since $x/y \to \infty = \dfrac{1}{0}$, we see that $x.0 - y.1 = \mp 1$ and therefore $y = 1$. That is, $v_{k-1} = x$. Since $2 \to \infty$ is an edge in $G(\infty, 1)$ and no edges of the circuit cross in the upper-half plane, we see that $v_{k-1} = x = 2$. Since the circuit is of minimal length and $v_j \to v_{j+1}$ is an edge in the circuit, v_{j+1} must be the largest vertex greater than v_j, which is adjacent to v_j. A simple computation shows that $v_2 = (m+1)/m$ and $v_{k-2} = (2m-1)/m$. Let

$$T = \begin{pmatrix} -\sqrt{m} & \frac{2m+1}{\sqrt{m}} \\ -\sqrt{m} & 2\sqrt{m} \end{pmatrix}.$$

Then $T \in N(\Gamma_0(m))$ and

$$T(x/y) = \frac{-m(x/y) + 2m + 1}{-m(x/y) + 2m}.$$

Thus it follows that for $1 < x/y < (2m-1)/m$, we have $1 < x/y < 2$ and $x/y < T(x/y)$. Moreover, $T(\infty) = 1, T(1) = (m+1)/m$, $T((2m-1)/m) = 2$, and $T(2) = \infty$. That is, $T(v_0) = v_1, T(v_1) = v_2$, $T(v_{k-2}) = v_{k-1}$, and $T(v_{k-1}) = \infty = v_0$. By applying the mapping T to the vertices of the circuit

$$v_0 \to v_1 \to v_2 \to \cdots \to v_{k-2} \to v_{k-1} \to v_0,$$

we get the circuit

$$T(v_0) \to T(v_1) \to T(v_2) \to \cdots \to T(v_{k-2}) \to T(v_{k-1}) \to T(v_0).$$

That is, we obtain the circuit

$$v_1 \to v_2 \to T(v_2) \to \cdots \to T(v_{k-2}) \to v_0 \to v_1.$$

Therefore

$$\infty \to 1 \to T(v_1) \to \cdots \to T(v_{k-3}) \to 2 \to \infty$$

is a circuit of length k, whose rational vertices lie between 1 and 2. It follows that

$$\infty \to 1 \to v_2 \to \cdots \to v_{k-2} \to 2 \to \infty$$

and

$$\infty \to 1 \to T(v_1) \to \cdots \to T(v_{k-3}) \to 2 \to \infty$$

are the same circuits. This is illustrated in Figure 1 and Figure 2.

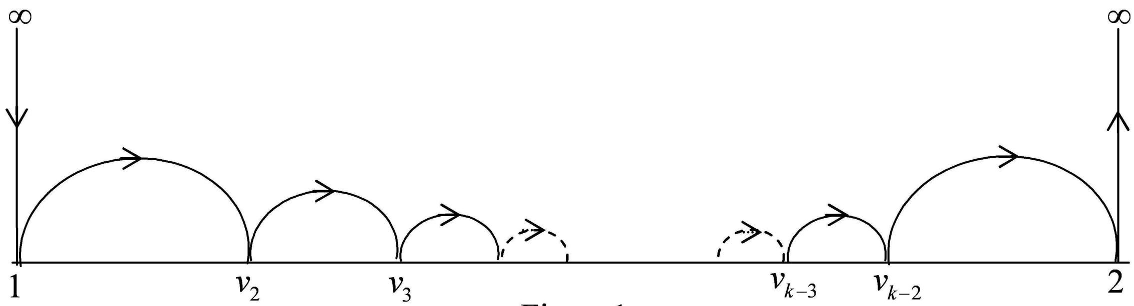

Figure1

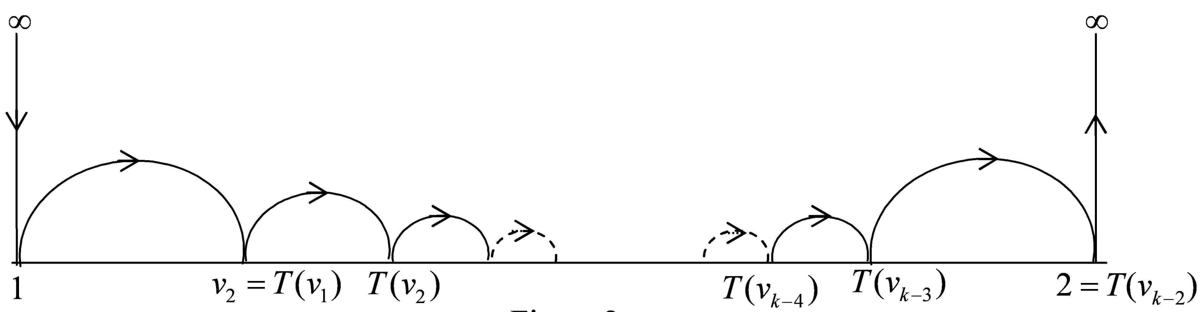

Figure2

Thus $v_3 = T(v_2), v_4 = T(v_3), ..., v_{k-2} = T(v_{k-3})$. Since

$$v_1 = T(v_0), v_2 = T(v_1), ..., v_{k-1} = T(v_{k-2}),$$

we see that $T^k(v_0) = v_0$, $T^k(v_1) = v_1$, and $T^k(v_2) = v_2$. Therefore $T^k = I$ and thus T is an elliptic mapping. Moreover, we get $v_j = T^j(\infty)$. Using $w_j = A(v_j)$, and $\infty = A^{-1}(w_0)$, we see that the circuit

$$w_0 \to w_1 \to w_2 \to \cdots \to w_{k-2} \to w_{k-1} \to w_0$$

is equal to the circuit

$$w_0 \to ATA^{-1}(w_0) \to AT^2A^{-1}(w_0) \to \cdots \to AT^{k-1}A^{-1}(w_0) \to w_0.$$

If we take $S = ATA^{-1}$, then S is an elliptic mapping and thus the proof follows. As T is an elliptic mapping, we see that $m \leqslant 3$. Thus, if $m \geqslant 5$, then the graph $G(\infty, 1)$ contains no circuits. ■

Taking $m = 3$, we get

$$T = \begin{pmatrix} -\sqrt{3} & 7/\sqrt{3} \\ -\sqrt{3} & 2\sqrt{3} \end{pmatrix}$$

and therefore $\infty \to T(\infty) \to T^2(\infty) \to T^3(\infty) \to T^4(\infty) \to T^5(\infty) \to \infty$ is a circuit in $G(\infty, 1)$. That is, we get the circuit

$$\infty \to 1 \to 4/3 \to 3/2 \to 5/3 \to 2 \to \infty.$$

If we apply the mapping $\Psi(z) = 2 - z$, to the vertices of the above circuit, we obtain the circuit $\infty \to 0 \to 1/3 \to 1/2 \to 2/3 \to 1 \to \infty$, which is the same circuit $\infty \to S(\infty) \to S^2(\infty) \to S^3(\infty) \to S^4(\infty) \to S^5(\infty) \to \infty$ for the mapping

$$S = \Psi T \Psi = \begin{pmatrix} \sqrt{3} & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}.$$

Therefore $\infty \to 0 \to 1/3 \to 1/2 \to 2/3 \to 1 \to 4/3 \to 3/2 \to 5/3 \to 2 \to \infty$ is a circuit of length 10. This circuit is illustrated in Figure 3. Thus we obtain many circuits using the same argument.

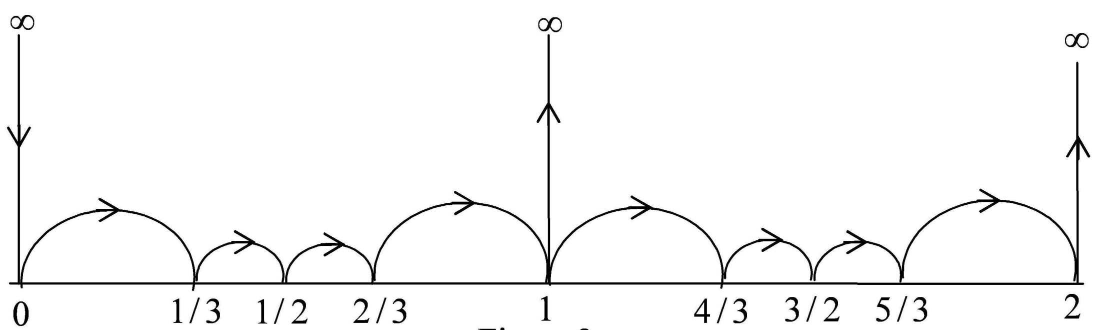

Figure 3

Let r be an odd natural number. By using Lemma1, we see that

$$\infty \to 1 \to 1/2 \to \cdots \to 1/k \to 1/(k+1) \to \cdots \to 1/(r-1) \to 0 \to \infty$$

is a circuit of length $r + 1$ in the graph $G(\infty, 1)$ for $N(\Gamma_0(2))$. In fact, the mapping

$$T = \begin{pmatrix} 1 & -1 \\ r-1 & 2-r \end{pmatrix}$$

is in $N(\Gamma_0(2))$ and $T(\infty) = \frac{1}{r-1}$, $T(1) = 0$. Therefore $\frac{1}{r-1} \to 0$ is an edge in $G(\infty, 1)$. Moreover, if k is an odd natural number, then

$$T = \begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ \sqrt{2}k & \sqrt{2}(\frac{1-k}{2}) \end{pmatrix}$$

is in $N(\Gamma_0(2))$ and $T(\infty) = \frac{1}{k}$, $T(1) = \frac{1}{k+1}$. If k is an even natural number, then

$$S = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

is in $N(\Gamma_0(2))$ and $S(\infty) = \frac{1}{k}$, $S(1) = \frac{1}{k+1}$. This shows that $\frac{1}{k} \to \frac{1}{k+1}$ is an edge in $G(\infty, 1)$. Since, for the mapping

$$A = \begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix},$$

we have $A \in N(\Gamma_0(2))$, $A(\infty) = 0$ and $A(1) = \infty$, we see that $0 \to \infty$ is an edge in $G(\infty, 1)$. Thus

$$\infty \to 1 \to 1/2 \to \cdots \to 1/k \to 1/(k+1) \to \cdots \to 1/(r-1) \to 0 \to \infty$$

is a circuit of the length $r + 1$ in the graph $G(\infty, 1)$.

Lemma 2. *Let $S \in N(\Gamma_0(N))$ and let $N(\Gamma_0(N))$ act transitively on $\hat{\mathbb{Q}}$. If $S(v) = v$ and $S(w) = w$ for different v and w in $\hat{\mathbb{Q}}$, then $S = I$. In particular, if $S(v) = T(v)$ and $S(w) = T(w)$ for $T \in N(\Gamma_0(N))$, then $S = T$.*

Proof. Since $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$, there exists $A \in N(\Gamma_0(N))$ such that $A(\infty) = v$. Hence $(A^{-1}SA)(\infty) = \infty$ and $(A^{-1}SA)(A^{-1}(w)) = A^{-1}(w)$. Since $v \neq w$, we see that $A^{-1}(w) \neq A^{-1}(v) = \infty$. Therefore, $A^{-1}SA$ has two different fixed points. Since $(A^{-1}SA)(\infty) = \infty$ and $A^{-1}SA \in N(\Gamma_0(N))$,

$$A^{-1}SA = \begin{pmatrix} 1 & b/h \\ 0 & 1 \end{pmatrix}$$

for some integer b. If $b \neq 0$, then $A^{-1}SA$ is a parabolic mapping, which has two different fixed points. This is a contradiction. Therefore $b = 0$ and thus $A^{-1}SA = I$, which implies that $S = I$. Now assume that $S(v) = T(v)$ and $S(w) = T(w)$. Then $(S^{-1}T)(v) = v$ and $(S^{-1}T)(w) = w$. Thus the proof follows. ■

Lemma 3. *Let $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$ and let $S, T \in N(\Gamma_0(N))$. If*

$$v \to S(v) \to S^2(v) \to S^3(v) \to \cdots \to S^{k-1}(v) \to v$$

and

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

are the same circuits in $G(\infty, u/n)$, then $T = S$.

Proof. From the hypothesis, we get

$$S(v) = T(v), S^2(v) = T^2(v), ..., S^{k-1}(v) = T^{k-1}(v).$$

Since $S(v) = T(v), S^2(v) = T^2(v)$, it follows that $(S^{-1}T)(v) = v$ and $S(v) = S^{-1}(T^2(v)) = (S^{-1}T)(T(v)) = (S^{-1}T)(S(v))$. Then according to the above lemma, we see that $S^{-1}T = I$, which implies that $S = T$. ■

Let $n > 1$ and $(u, n) = 1$. Then by using Theorem3.1, it can be shown that if

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

is a circuit in the graph $G(\infty, u/n)$ for $N(\Gamma_0(m))$, then T is an elliptic mapping of order k in $N(\Gamma_0(m))$. The following theorem shows that the same is true for the graph $G(\infty, 1)$.

Theorem 3.3. *Let m be a square-free positive integer. If any circuit in the graph $G(\infty, 1)$ for $N(\Gamma_0(m))$ is of the form*

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v,$$

then T is an elliptic mapping of order k.

Proof. Suppose that

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

is a circuit in $G(\infty, 1)$. Then there is a mapping $A \in N(\Gamma_0(m))$ such that $A(\infty) = v$ and $A(1) = T(v)$. If we apply A^{-1} to the vertices of the above circuit, we obtain the circuit

$$A^{-1}(v) \to A^{-1}(T(v)) \to A^{-1}(T^2(v)) \to \cdots \to A^{-1}(T^{k-1}(v)) \to A^{-1}(v).$$

Since $A(\infty) = v$, we get

$$\infty \to \left(A^{-1}TA\right)(\infty) \to \left(A^{-1}T^2A\right)(\infty) \to \cdots \to \left(A^{-1}T^{k-1}A\right)(\infty) \to \infty.$$

Let $B = A^{-1}TA$. Then the above circuit is equal to the circuit

$$\infty \to B(\infty) \to B^2(\infty) \to \cdots \to B^{k-2}(\infty) \to B^{k-1}(\infty) \to \infty.$$

Since no edges of $G(\infty, 1)$ cross in the upper half plane $U = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$, either

$$B(\infty) < B^2(\infty) < \cdots < B^{k-2}(\infty) < B^{k-1}(\infty)$$

or

$$B(\infty) > B^2(\infty) > \cdots > B^{k-2}(\infty) > B^{k-1}(\infty).$$

Assume that

$$B(\infty) < B^2(\infty) < \cdots < B^{k-2}(\infty) < B^{k-1}(\infty).$$

Thus the circuit is as in Figure 4.

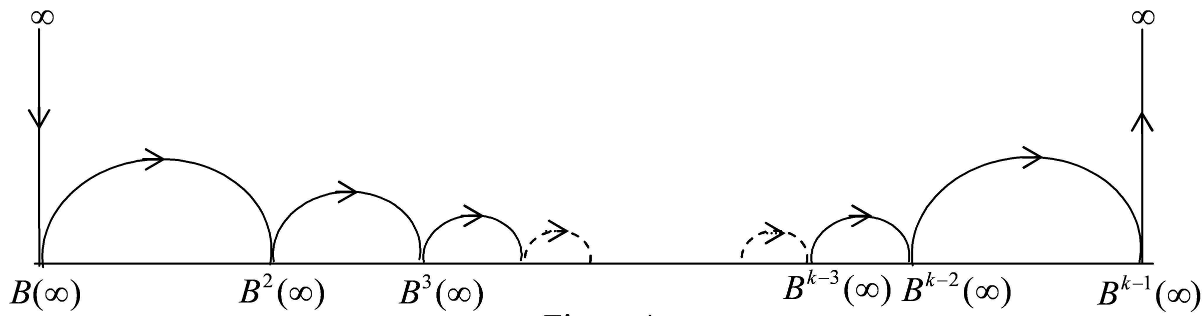

Figure 4

If we apply the mapping B to the vertices of the above circuit, we obtain the circuit

$$B(\infty) \to B^2(\infty) \to B^3(\infty) \to \cdots \to B^{k-1}(\infty) \to B^k(\infty) \to B(\infty).$$

Now we show that $B^k(\infty) = \infty$. Assume that $B^k(\infty) \neq \infty$. It can be easily seen that $B^k(\infty) \neq B^j(\infty)$ for $1 \leqslant j \leqslant k-1$. Then there are only two cases that we have to deal with. The first case; if $B^{k-1}(\infty) < B^k(\infty)$, then the edges $B^k(\infty) \to B(\infty)$ and $B^{k-1}(\infty) \to \infty$ cross in $\mathcal{U}$. The second case; if $B^k(\infty) < B^{k-1}(\infty)$, then the edges $B^{k-1}(\infty) \to B^k(\infty)$ and $\infty \to B(\infty)$ cross in $\mathcal{U}$. So our assumption is impossible. Hence $B^k(\infty) = \infty$, and so $B^k(B(\infty)) = B(B^k(\infty)) = B(\infty)$ and similarly $B^k(B^2(\infty)) = B^2(\infty)$. This shows that $B^k = I$. Thus B is an elliptic mapping of order k. Since B is an elliptic mapping and $B = A^{-1}TA$, we see that T is an elliptic mapping. This completes the proof . ■

The suborbital graph for the Hecke group $H(\sqrt{m})$ on the set of cusps of $H(\sqrt{m})$ was investigated in [8]. $H(\sqrt{m})$ is the Hecke group generated by the mappings

$$z \to z + \sqrt{m} \ \text{ and } \ z \to -1/z \ , \ m = 2, 3.$$

It is well known that $H(\sqrt{m})$ consists of the mappings of the following two types:

$$(i) \ \ T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d} \ , \ a, b, c, d \in \mathbb{Z} \ , ad - bcm = 1$$

$$(ii) \ \ T(z) = \frac{a\sqrt{m}z + b}{cz + d\sqrt{m}} \ , a, b, c, d \in \mathbb{Z}, \ adm - bc = 1.$$

Let

$$M = \begin{pmatrix} m^{-1/4} & 0 \\ 0 & m^{1/4} \end{pmatrix}.$$

Then, it can be shown that

$$H\left(\sqrt{m}\right) = M^{-1} N\left(\Gamma_0(m)\right) M$$

and the set of the cusps of $H(\sqrt{m})$ is

$$\sqrt{m}\hat{\mathbb{Q}} = \left\{ (r/s)\sqrt{m} : r/s \in \hat{\mathbb{Q}} \right\} \cup \{\infty\}$$

The following lemma is proved easily.

Lemma 4. *$H(\sqrt{m})$ acts transitively on $\sqrt{m}\hat{\mathbb{Q}} = \{(r/s)\sqrt{m} : r/s \in \mathbb{Q}\} \cup \{\infty\}$.*

Theorem 3.4. *Let $m = 2, 3$ and $n > 1$. Let $G(\infty, u/n)$ be a suborbital graph for $N(\Gamma_0(m))$ and $G(\infty, (u/n)\sqrt{m})$ be a suborbital graph for $H(\sqrt{m})$. Then the mapping given by $(r/s)\sqrt{m} \to M((r/s)\sqrt{m})$ from $G(\infty, (u/n)\sqrt{m})$ to $G(\infty, u/n)$ is an isomorphism.*

Proof. Let $G(\infty, (u/n)\sqrt{m})$ be the suborbital graph for $H(\sqrt{m})$ and suppose that $(r/s)\sqrt{m} \to (x/y)\sqrt{m}$ is an edge in the graph $G(\infty, (u/n)\sqrt{m})$. Then there exists $T \in H(\sqrt{m})$ such that

$$T(\infty) = (r/s)\sqrt{m},\ T(u/n) = (x/y)\sqrt{m}.$$

Since $H(\sqrt{m}) = M^{-1}N(\Gamma_0(m))M$, there exists $S \in N(\Gamma_0(m))$ such that $T = M^{-1}SM$. Then

$$\begin{aligned} \left(M^{-1}SM\right)(\infty) &= (r/s)\sqrt{m} \\ \left(M^{-1}SM\right)\left((u/n)\sqrt{m}\right) &= (x/y)\sqrt{m}. \end{aligned}$$

Since $M(z) = z/\sqrt{m}$, we see that

$$\begin{aligned} S(\infty) &= S(M(\infty)) = M\left((r/s)\sqrt{m}\right) \\ S(u/n) &= S\left(M\left((u/n)\sqrt{m}\right)\right) = M\left((x/y)\sqrt{m}\right). \end{aligned}$$

This shows that $M((r/s)\sqrt{m}) \to M((x/y)\sqrt{m})$ is an edge in the suborbital graph $G(\infty, u/n)$. Moreover, if $r/s \to x/y$ is an edge in $G(\infty, u/n)$, then $M^{-1}(r/s) \to M^{-1}(x/y)$ is an edge in the graph $G(\infty, (u/n)\sqrt{m})$. ■

Theorem 3.5. *Let $m = 2, 3$ and $(u, n) = 1$ with $n > 1$ and let $G(\infty, (u/n)\sqrt{m})$ be a suborbital graph for $H(\sqrt{m})$. Then any circuit in $G(\infty, (u/n)\sqrt{m})$ is of the form*

$$v_1 \to S(v_1) \to S^2(v_1) \to S^3(v_1) \to \cdots \to S^{k-1}(v_1) \to v_1$$

for a unique elliptic mapping $S \in H(\sqrt{m})$ and for some $v_1 \in \sqrt{m}\hat{\mathbb{Q}}$.

Proof. Let C be a circuit in $G(\infty, (u/n)\sqrt{m})$ and let C^* be the circuit constructed by applying the mapping M to the vertices of the circuit C. Then by the above theorem, C^* is a circuit in the graph $G(\infty, u/n)$ for $N(\Gamma_0(m))$. Since $n > 1$, by Theorem3.1, C^* is of the form

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

for some $v \in \hat{\mathbb{Q}}$ and for a unique elliptic mapping $T \in N(\Gamma_0(m))$ of order k. By applying M^{-1} to the vertices of the circuit C^*, we see that the circuit C is in the form

$$M^{-1}(v) \to M^{-1}(T(v)) \to M^{-1}(T^2(v)) \to \cdots \to M^{-1}(T^{k-1}(v)) \to M^{-1}(v).$$

Since $v = M(M^{-1}(v))$, we see that C is of the form

$$M^{-1}(v) \to M^{-1}(T\left(M\left(M^{-1}(v)\right)\right)) \to \cdots \to M^{-1}(T^{k-1}\left(M\left(M^{-1}(v)\right)\right)) \to M^{-1}(v).$$

Let $v_1 = M^{-1}(v)$ and $S = M^{-1}TM$. Then S is an elliptic mapping of $H(\sqrt{m})$ of order k and thus the circuit C is of the form

$$v_1 \to S(v_1) \to S^2(v_1) \to S^3(v_1) \to \cdots \to S^{k-1}(v_1) \to v_1.$$

■

We now consider suborbital graphs for $N(\Gamma_0(N))$ when $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$. Let $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$ and let $m = N/h^2$. Then m is a square-free positive integer by Theorem2.2. Therefore

$$N\left(\Gamma_0^+(m)\right) = N(\Gamma_0(m))$$

and thus

$$N(\Gamma_0(N)) = \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix} N(\Gamma_0(m)) \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix}^{-1}.$$

If we take $H(z) = hz$, then

$$N(\Gamma_0(N)) = H^{-1}N(\Gamma_0(m))\,H.$$

Therefore we can give the following theorem.

Theorem 3.6. *Suppose that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$. Let $G(\infty, u/n)$ be a suborbital graph for $N(\Gamma_0(N))$ and let $G(\infty, hu/n)$ be a suborbital graph for $N(\Gamma_0(m))$. Then the mapping $r/s \to H(r/s)$, from $G(\infty, u/n)$ to $G(\infty, hu/n)$ is an isomorphism.*

Proof. The proof is exactly the same as in Theorem3.4 and is omitted.

Theorem 3.7. *Suppose that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$ and suppose that $(u, n) = 1$ with $n > 1$. If $(h, n) < n$, then any circuit in the suborbital graph $G(\infty, u/n)$ for $N(\Gamma_0(N))$ is of the form*

$$v_1 \to S(v_1) \to S^2(v_1) \to S^3(v_1) \to \cdots \to S^{k-1}(v_1) \to v_1$$

for a unique elliptic mapping $S \in N(\Gamma_0(N))$ of order k and for some $v_1 \in \hat{\mathbb{Q}}$.

Proof. Let C be a circuit in $G(\infty, u/n)$ and let C^* be the circuit constructed by applying the mapping H to the vertices of the circuit C. Then by the above theorem, C^* is a circuit in the suborbital graph $G(\infty, hu/n)$ for $N(\Gamma_0(m))$. Since the reduced form of hu/n is

$$\frac{hu/(h,n)}{n/(h,n)},$$

we see that $n/(h, n) > 1$. Then, by the Theorem3.1, C^* is of the form

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

for some $v \in \hat{\mathbb{Q}}$ and for a unique elliptic mapping $T \in N(\Gamma_0(m))$ of order k. By applying H^{-1} to the vertices of the circuit C^*, we see that the circuit C is in the form

$$H^{-1}(v) \to H^{-1}(T(v)) \to H^{-1}(T^2(v)) \to \cdots \to H^{-1}(T^{k-1}(v)) \to H^{-1}(v).$$

Since $v = H(H^{-1}(v))$ we see that C is of the form

$$H^{-1}(v) \to H^{-1}(T(H(H^{-1}(v)))) \to \cdots \to H^{-1}(T^{k-1}(H(H^{-1}(v)))) \to H^{-1}(v).$$

Let $v_1 = H^{-1}(v)$ and $S = H^{-1}TH$. Then S is an elliptic mapping of $N(\Gamma_0(N))$ of order k and the circuit C is of the form

$$v_1 \to S(v_1) \to S^2(v_1) \to S^3(v_1) \to \cdots \to S^{k-1}(v_1) \to v_1.$$

■

If $(h,n) = n$, the above theorem may not be correct, since the graph $G(\infty, u/n)$ for $N(\Gamma_0(N))$ is isomorphic to the graph $G(\infty, (h/n)u) = G(\infty, 1)$ for $N(\Gamma_0(m))$. More generally, the following example shows this. Let $N = 32$ and $n = 4$ then $h = 4$. Thus it follows that

$$\infty \to 1/4 \to 1/8 \to \cdots \to 1/4(r-1) \to 0 \to \infty$$

is a circuit of length $r+1$ in the graph $G(\infty, 1/4)$ for $N(\Gamma_0(32))$.

If

$$A = \begin{pmatrix} a\sqrt{q} & b/h\sqrt{q} \\ cN/h\sqrt{q} & d\sqrt{q} \end{pmatrix}$$

is an elliptic mapping in $N(\Gamma_0(N))$, then the order of A depends on q and $a+d$. If $a+d=0$, then the order of A is 2. If $a+d=\pm 1$, then the order of A is equal to $3, 4$ and 6 when q is $1, 2$ and 3 respectively(see [2] for more details.). Using this fact we can give the following theorem.

Theorem 3.8. *Assume that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$. Moreover, assume that T and S are elliptic mappings in $N(\Gamma_0(N))$ of order k and r respectively. If*

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

and

$$w \to S(w) \to S^2(w) \to S^3(w) \to \cdots \to S^{r-1}(w) \to w$$

are two circuits in $G(\infty, u/n)$, then $r = k$. That is, the two circuits have the same length.

Proof. Assume that

$$v \to T(v) \to T^2(v) \to T^3(v) \to \cdots \to T^{k-1}(v) \to v$$

and

$$w \to S(w) \to S^2(w) \to S^3(w) \to \cdots \to S^{r-1}(w) \to w$$

are two circuits in $G(\infty, u/n)$. Then there exist two mappings $A, B \in N\left(\Gamma_0(N)\right)$ such that $A(\infty) = v, A(u/n) = T(v)$ and $B(\infty) = w, B(u/n) = S(w)$. Then $u/n = A^{-1}T(v) = (A^{-1}TA)(\infty)$ and $u/n = B^{-1}S(w) = (B^{-1}SB)(\infty)$. Therefore we get $(A^{-1}TA)(\infty) = (B^{-1}SB)(\infty)$. Let

$$A^{-1}TA = \left(\begin{array}{cc} a\sqrt{q} & b/h\sqrt{q} \\ cN/h\sqrt{q} & d\sqrt{q} \end{array} \right),\ adq - (bcN)/h^2q = 1$$

and

$$B^{-1}SB = \left(\begin{array}{cc} a^*\sqrt{q^*} & b^*/h\sqrt{q^*} \\ c^*N/h\sqrt{q^*} & d^*\sqrt{q^*} \end{array} \right),\ a^*d^*q^* - (b^*c^*N)/h^2q^* = 1$$

Then, since $(A^{-1}TA)(\infty) = (B^{-1}SB)(\infty)$, we obtain

$$\frac{a}{(cN/h)/q} = \frac{a^*}{(c^*N/h)/q^*}$$

and therefore

$$\frac{a}{cN/h^2q} = \frac{a^*}{c^*N/h^2q^*}.$$

A simple calculation shows that $q = q^*$. Then it follows that $A^{-1}TA$ and $B^{-1}SB$ have the same order and therefore T and S have the same order. That is, the two circuits have the same length. ■

A circuit of length $3, 4$, and 6 is called a triangle, a rectangle, and a hexagon respectively. Since any elliptic mapping of $N\left(\Gamma_0(N)\right)$ is of order $2, 3, 4$, or 6, it follows that when $(h, n) < n$, $G(\infty, u/n)$ may contain only triangle, rectangle, or hexagon by Theorem3.8.

Theorem 3.9. *Let m be a square-free positive integer and $(u, n) = 1$. Then $G(\infty, u/n)$ contains a triangle if and only if $m \mid n$, $u^2 \mp u + 1 \equiv 0 \pmod{n}$, a rectangle if and only if $2|m, m|2n$, and $2u^2 \mp 2u + 1 \equiv 0 \pmod{n}$, and a hexagon if and only if $3|m$, $m|3n$ and $3u^2 \mp 3u + 1 \equiv 0 \pmod{n}$.*

Proof. If $n = 1$, then the proof is easy. Assume that $G(\infty, u/n)$ contains a circuit when $n > 1$. Then, by Theorem3.1, this circuit must be of the form

$$v \to T(v) \to T^2(v) \to T^3(v) \to \ \cdots \to T^{k-1}(v) \to v$$

for a unique elliptic mapping T of order k. Since $v \to T(v)$ is an edge in $G(\infty, u/n)$, there exists $A \in N\left(\Gamma_0(m)\right)$ such that $A(\infty) = v$ and $A(u/n) = T(v)$. Then $(A^{-1}TA)(\infty) = u/n$. Let

$$A^{-1}TA = \left(\begin{array}{cc} a\sqrt{q} & b/\sqrt{q} \\ cm/\sqrt{q} & d\sqrt{q} \end{array} \right),\ q|m, adq - (bcm)/q = 1.$$

Then it follows that $u/n = a/(cm/q)$. $A^{-1}TA$ is an elliptic mapping, since T is an elliptic mapping. Therefore $a + d = \mp 1$, since the order of T is not 2. From the equalities $u/n = a/(cm/q)$ and $a+d = \mp 1$, we see that $qu^2 \mp qu + 1 \equiv 0 \pmod{n}$, $m \mid qn$. If the circuit is a triangle, a rectangle, and a hexagon, then $q = 1, 2$, and 3 respectively. The proof then follows. For the other part of the theorem, assume that $qu^2 \mp qu + 1 \equiv 0 \,(\mathrm{mod}\, n)$, $m \mid qn$. Then the mapping

$$T(z) = \frac{-u\sqrt{q}z + (qu^2 \mp qu + 1)/n\sqrt{q}}{(-nq/\sqrt{q})z + (u \mp 1)\sqrt{q}}$$

is in $N(\Gamma_0(m))$ and $T(\infty) = u/n$. The order of T is $3, 4$ and 6, when $q = 1, 2$, and 3 respectively. Moreover, if we represent the order of T by k, we get the circuit

$$\infty \to T(\infty) \to T^2(\infty) \to \cdots \to T^{k-1}(\infty) \to \infty$$

in $G(\infty, u/n)$, as required. ■

Corollary 1. *Let m be a square-free positive integer and let $n > 1$ with $(u, n) = 1$. If the graph $G(\infty, u/n)$ for $N(\Gamma_0(m))$ contains a triangle, then for any prime divisor p of n greater than 3, we have $p \equiv 1 \,(\mathrm{mod}\, 3)$. If $G(\infty, u/n)$ contains a rectangle, then n is an odd natural number and $n \equiv 1 \,(\mathrm{mod}\, 4)$. If $G(\infty, u/n)$ contains a hexagon, then for any odd prime divisor p of n we have $p \equiv 1 \,(\mathrm{mod}\, 3)$.*

Proof. Assume that $G(\infty, u/n)$ contains a triangle. Then $u^2 \mp u + 1 \equiv 0 \,(\mathrm{mod}\, n)$. It follows that $(2u \mp 1)^2 + 3 \equiv 0 \,(\mathrm{mod}\, n)$. Thus if $p|n$ and $p > 3$, then $(2u \mp 1)^2 + 3 \equiv 0 \,(\mathrm{mod}\, p)$. It follows that $p \equiv 1 \,(\mathrm{mod}\, 3)$. If $G(\infty, u/n)$ contains a rectangle, then $2u^2 \mp 2u + 1 \equiv 0 \,(\mathrm{mod}\, n)$. This shows that n is an odd natural number and $(2u + 1)^2 + 1 \equiv 0 \,(\mathrm{mod}\, n)$. Then for any prime divisor of n, we have $(2u \mp 1)^2 + 1 \equiv 0 \,(\mathrm{mod}\, p)$ and therefore $p \equiv 1 \,(\mathrm{mod}\, 4)$. Therefore $n \equiv 1 \,(\mathrm{mod}\, 4)$. If $G(\infty, u/n)$ contains a hexagon, then $3u^2 \mp 3u + 1 \equiv 0 \,(\mathrm{mod}\, n)$. This shows that $3 \nmid n$ and $36u^2 \mp 36u + 12 \equiv 0 \,(\mathrm{mod}\, n)$. That is, $(6u \mp 3)^2 + 3 \equiv 0 \,(\mathrm{mod}\, n)$. Let p be an odd prime divisor of n. Then $(6u \mp 3)^2 + 3 \equiv 0 \,(\mathrm{mod}\, p)$. Thus it follows that $p \equiv 1 \,(\mathrm{mod}\, 3)$. ■

Let $N(\Gamma_0(N))$ act transitively on $\hat{\mathbb{Q}}$ and assume that $n > 1$ and $(h, n) < n$. Then, by using Theorem3.7, a necessary and sufficient condition for the graph $G(\infty, u/n)$ to contain a triangle, a rectangle or a hexagon may be given. Because the graph $G(\infty, u/n)$ and the graph $G(\infty, hu/n)$ for $N(\Gamma_0(m))$ is isomorphic, where $m = N/h^2$.

Corollary 2. *Assume that $N(\Gamma_0(N))$ acts transitively on $\hat{\mathbb{Q}}$. Then any circuit of the minimal length in $G(\infty, u/n)$ is of the form*

$$v \to T(v) \to T^2(v) \to \cdots \to T^{k-1}(v) \to v$$

for a unique elliptic mapping $T \in N(\Gamma_0(N))$ of order k and for some $v \in \hat{\mathbb{Q}}$. Moreover, the graph $G(\infty, u/n)$ contains a circuit if and only if there exists an elliptic mapping $T \in N(\Gamma_0(N))$ of order greater than 2 such that $T(\infty) = u/n$.

Proof. By Theorem3.2, Theorem3.6, and Theorem3.7, it is seen that any circuit of the minimal length in $G(\infty, u/n)$ is of the form

$$v \to T(v) \to T^2(v) \to \cdots \to T^{k-1}(v) \to v$$

for a unique elliptic mapping $T \in N(\Gamma_0(N))$ and for some $v \in \hat{\mathbb{Q}}$. Assume that $T(\infty) = u/n$ for some elliptic mapping of order greater than 2, then

$$\infty \to T(\infty) \to T^2(\infty) \to \cdots \to T^{k-1}(\infty) \to \infty$$

is a cicuit in $G(\infty, u/n)$, where k is the order of T. Now suppose that $G(\infty, u/n)$ contains a circuit, then $G(\infty, u/n)$ contains a circuit of the minimal length and thus this circuit must be of the form

$$v \to T(v) \to T^2(v) \to \cdots \to T^{k-1}(v) \to v$$

for an elliptic mapping of $N(\Gamma_0(N))$. Since $v \to T(v)$ is an edge in $G(\infty, u/n)$, there exists $A \in N(\Gamma_0(N))$ such that $A(\infty) = v$ and $A(u/n) = T(v)$. Then it follows that ATA^{-1} is an elliptic mapping and $ATA^{-1}(\infty) = u/n$. ■

If m is a square-free positive integer and $G(\infty, 1)$ is the graph for $N(\Gamma_0(m))$, then it can be seen easily that the length of any circuit is an even number. Therefore we can give the following corollary.

Corollary 3. *Let $N(\Gamma_0(N))$ act transitively on $\hat{\mathbb{Q}}$. Then the length of any circuit in $G(\infty, u/n)$ is either 3 or an even natural number.*

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