# A note on packing graphs without cycles of length up to five

Agnieszka Görlich[*], Andrzej Żak
University of Science and Technology AGH, Al. Mickiewicza 30, 30-059 Kraków, Poland
{forys,zakandrz}@agh.edu.pl

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**Abstract**

The following statement was conjectured by Faudree, Rousseau, Schelp and Schuster:

if a graph $G$ is a non-star graph without cycles of length $m \leqslant 4$ then
$G$ is a subgraph of its complement.

So far the best result concerning this conjecture is that every non-star graph $G$ without cycles of length $m \leqslant 6$ is a subgraph of its complement. In this note we show that $m \leqslant 6$ can be replaced by $m \leqslant 5$.

## 1 Introduction

We deal with finite, simple graphs without loops and multiple edges. We use standard graph theory notation. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of $G$ is denoted by $|G|$ and the size is denoted by $||G||$. We say that $G$ is *packable in its complement* ($G$ is packable, in short) if there is a permutation $\sigma$ on $V(G)$ such that if $xy$ is an edge in $G$ then $\sigma(x)\sigma(y)$ is not an edge in $G$. Thus, $G$ is packable if and only if $G$ is a subgraph of its complement. In [2] the authors stated the following conjecture:

**Conjecture 1** *Every non-star graph $G$ without cycles of length $m \leqslant 4$ is packable.*

In [2] they proved that the above conjecture holds if $||G|| \leqslant \frac{6}{5}|G| - 2$. Woźniak proved that a graph $G$ without cycles of length $m \leqslant 7$ is packable [6]. His result was improved by Brandt [1] who showed that a graph $G$ without cycles of length $m \leqslant 6$ is packable. Another, relatively short proof of Brandt's result was given in [3]. In this note we prove the following statement.

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**Theorem 2** *If a graph $G$ is a non-star graph without cycles of length $m \leqslant 5$ then $G$ is packable.*

The basic ingredient for the proof of our theorem is the lemma presented below. This lemma is both a modification and an extension of Lemma 2 in [4].

**Lemma 3** *Let $G$ be a graph and $k \geqslant 1$, $l \geqslant 1$ be any positive integers. If there is a set $U = \{v_1, ..., v_{k+l}\} \subset V(G)$ of $k + l$ independent vertices of $G$ such that*

1. *$k$ vertices of $U$ have degree at most $l$ and $l$ vertices of $U$ have degree at most $k$;*
2. *vertices of $U$ have mutually disjoint sets of neighbors, i.e. $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$;*
3. *$G - U$ is packable*

*then there exists a packing $\sigma$ of $G$ such that $U$ is an invariant set of $\sigma$, i.e. $\sigma(U) = U$.*

Proof. Let $G' := G - U$ and $\sigma'$ be a packing of $G'$. Below we show that we can find an appropriate packing $\sigma$ of $G$.
For any $v \in V(G')$ we define $\sigma(v) := \sigma'(v)$. Then let us consider a bipartite graph $B$ with partition sets $X := \{v_1, ..., v_{k+l}\} \times \{0\}$ and $Y := \{v_1, ..., v_{k+l}\} \times \{1\}$. For $i, j \in \{1, ..., k+l\}$ the vertices $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in $B$ if and only if $\sigma'(N(v_i)) \cap N(v_j) = \emptyset$. So, if $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in $B$ we can put $\sigma(v_i) = v_j$.
Without loss of generality we can assume that $k \leqslant l$. Note that if $\deg v_i \leqslant l$ in $G$ then $\deg(v_i, 0) \geqslant k$ in $B$. Furthermore, if $\deg v_i \leqslant k$ in $G$ then $\deg(v_i, 0) \geqslant l$ in $B$. Thus $X$ contains $k$ vertices of degree $\geqslant k$ and $l$ vertices of degree $\geqslant l$. In the similar manner we can see that $Y$ contains $k$ vertices of degree $\geqslant k$ and $l$ vertices of degree $\geqslant l$. In particular, every vertex in $Y$ has degree $\geqslant k$. Let $S \subset X$. If $|S| \leqslant k$ then obviously $|N(S)| \geqslant |S|$. Suppose that $k < |S| \leqslant l$. Then there is at least one vertex of degree $l$ in $S$ thus $|N(S)| \geqslant l \geqslant |S|$. Finally, we show that if $|S| > l$, then $N(S) = Y$. Indeed, otherwise let $(v_j, 1) \in Y$ be a vertex which has no neighbor in $S$. Thus $\deg(v_j, 1) \leqslant |X| - |S| < k + l - l = k$, a contradiction. Hence, for any $S \subset X$ we get $|S| \leqslant |N(S)|$. Therefore, by the famous Hall's theorem [5], there is a matching $M$ in $B$. We define $\sigma(v_i) = v_j$ for $i, j \in \{1, ..., k+l\}$ such that $(v_i, 0)$, $(v_j, 1)$ are incident with the same edge in $M$. $\square$

# 2 Proof of Theorem 2

Proof. Assume that $G$ is a counterexample of Theorem 2 with minimal order. Without loss of generality we may assume that $G$ is connected. We choose an edge $xy \in E(G)$ with the maximal sum $\deg x + \deg y$ of degrees of its endvertices among all edges of $G$. Since $G$ is not a star $\deg x \geqslant 2$ and $\deg y \geqslant 2$. Let $U$ be the union of the sets of neighbors of $x$ and $y$ different from $x, y$. Define $k := \deg x - 1$, $l := \deg y - 1$. We may assume that $k \leqslant l$. Consider graph $G' := G - \{x, y\}$. Note that because of the choice of the edge $xy$, $U$ contains $k$ vertices of degree $\leqslant l$ and $l$ vertices of degree $\leqslant k$ in $G'$. Moreover, since $G$

has no cycles of length $\leqslant 5$, the vertices of $U$ are independent in $G'$ and have mutually disjoint sets of neighbors in $G'$. By our assumption $G' - U$ is packable or it is a star.

Assume that $G' - U$ is packable. Thus, by Lemma 3, there is a packing $\sigma'$ of $G'$ such that $\sigma'(U) = U$. This packing can be easily modified in order to obtain a packing of $G$. Namely, note that there are vertices $v, w \in U$ where $v$ is a neighbor of $x$ and $w$ is a neighbor of $y$ such that $\sigma'(v)$ is a neighbor of $x$ and $\sigma'(w)$ is a neighbor of $y$, or $\sigma'(v)$ is a neighbor of $y$ and $\sigma'(w)$ is a neighbor of $x$. In the former case $(x\sigma'(v)y\sigma'(w))\sigma'$ is a packing of $G$ and in the latter case $(x\sigma'(v))(y\sigma'(w))\sigma'$ is a packing of $G$. Thus we get a contradiction.

Assume now that $G' - U$ is a star (with at least one edge). Note that since $G$ has no cycles of lengths up to five, every vertex from $U$ has degree $\leqslant 2$ in $G$. Moreover, $G$ has a vertex which is at distance at least 3 from $y$. Let $z$ denote a vertex which is not in $U$ and is at distance 2 from $x$, or if such a vertex does not exist let $z$ be any vertex which is at distance at least 3 from $y$. Furthermore, let $W$ denote the set of neighbours of $y$. Consider a graph $G'' := G - \{y, z\}$. Thus $W$ consists of $l$ vertices of degree $\leqslant 1$ in $G''$ and one vertex of degree $k \leqslant l$ in $G''$. Note that $G'' - W$ has an isolated vertex, namely a neighbour of $x$. Thus $G'' - W$ is not a star, hence it is packable. Moreover vertices from $W$ are independent and have mutually disjoint sets of neighbours in $G''$. Thus by Lemma 3 there is a packing $\sigma''$ of $G''$ such that $\sigma''(W) = W$. Then $(yz)\sigma''$ is a packing of $G$. Therefore, we get a contradiction again, so the proof is completed. □

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