The crossing number of a projective graph is quadratic in the face–width

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Abstract

We show that for each integer $g \geq 0$ there is a constant $c_g > 0$ such that every graph that embeds in the projective plane with sufficiently large face–width r has crossing number at least $c_g r^2$ in the orientable surface Σ_g of genus g. As a corollary, we give a polynomial time constant factor approximation algorithm for the crossing number of projective graphs with bounded degree.

1 Introduction

We recall that the face–width of a graph G embedded in a surface Σ is the minimum number of intersections of G with a noncontractible curve in Σ.

Fiedler et al. [7] proved that the orientable genus of a projective graph grows linearly with the face–width. Our aim is to show that for each integer $g \geq 0$, the crossing number cr_g of projective graphs in the closed orientable surface Σ_g of genus g grows quadratically with the face–width.

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Theorem 1.1 *For every integer $g \geq 0$ there are constants $c_g, r_g > 0$, such that if G embeds in the projective plane with face–width at least $r \geq r_g$, then the crossing number $\mathrm{cr}_g(G)$ of G in Σ_g is at least $c_g r^2$.*

We remark that cr_0, the crossing number in the sphere, coincides with the "usual" crossing number in the plane.

Our strategy for proving Theorem 1.1 is to show the existence of sufficiently large grid–like structures, so called diamond grids (Theorem 2.1), in projective graphs, and then prove that diamond grids have large crossing number (Section 3, which concludes with a proof of Theorem 1.1). We remark that our constants are not unreasonable (see Theorem 3.4).

Böröczky, Pach and Tóth showed [2] that for every surface χ there is a constant c_χ such that if a graph with n vertices and maximum degree Δ embeds in χ, then its planar crossing number is at most $c_\chi \Delta\, n$. Djidjev and Vrt'o [5] then significantly improved the constant there for orientable surfaces. The result was also generalized by Wood and Telle to all graph classes with an excluded minor [12, 13] (see also [1]).

Along a similar vein, we also give a straightforward upper bound for the crossing number (in the plane, and thus in any orientable surface) of a projective graph G in terms of its face–width r and its maximum degree Δ, regardless of the number of vertices: $\mathrm{cr}(G) \leq r^2\Delta^2/8$ in Proposition 4.1.

No efficient algorithm is known for approximating the crossing number of arbitrary (not even bounded–degree) graphs within a constant factor. The best result reported in this direction is by Even, Guha, and Schieber [6], who give an $O(\log^3 |V(G)|)$ approximation algorithm for $\mathrm{cr}(G) + |V(G)|$ (not for $\mathrm{cr}(G)$, thus weak in the case of graphs with few crossings) on bounded-degree graphs. As a consequence of the claimed lower and upper bounds we obtain a polynomial time approximation algorithm for the crossing number of projective graphs of bounded degree:

Theorem 1.2 *For every fixed Δ and orientable surface Σ_g, there is a polynomial time approximation algorithm that computes the crossing number cr_g of a projective graph with maximum degree Δ within a constant factor.*

This last statement is proved in Section 4.

2 Finding a large diamond projective grid

Randby [11] gave, for each integer $r > 0$, a full characterization of those projective graphs that are minor–minimal with respect to having face–width r. He showed that all such graphs can be obtained from the "$r \times r$ projective grid" by $Y\Delta$– and ΔY–exchanges. Now although it is not too difficult to show that the $r \times r$ projective grid has crossing number quadratic in r for $r \geq 3$, it is not that straightforward to show that performing $Y\Delta$ and ΔY operations does not decrease the crossing number significantly.

Thus our approach is to find, in projective graphs of given face–width, a related grid–like structure that better suits our purposes. We remark that some other research papers besides Randby [11], e.g. [3], implicitly consider existence of large grid–like subgraphs in densely embedded graphs, but none of which we have found contains an explicit result suited right to our needs. For that reason we think our new Theorem 2.1, with its short and self-contained proof, can be of independent research interest.

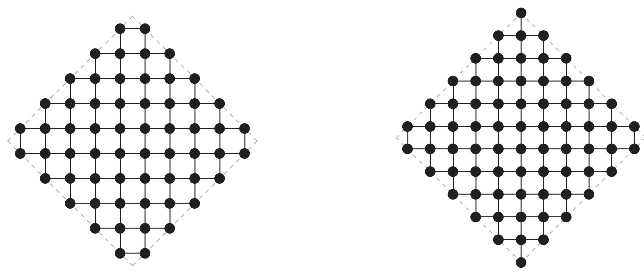

Figure 1: Projective diamond grids of sizes 10 (left) and 11 (right).

The *diamond grid* D_r of size r is a plane graph whose vertices are all integer pairs (i,j) where $|i|+|j| \le r$, such that j is always odd, the parity of i is the opposite of the parity of r, and an edge of D_r joins (i,j) to (i',j') iff $|i-i'|+|j-j'|=2$.

The *projective diamond grid* P_r of size r is obtained from D_r by identifying the opposite pairs of its "boundary" vertices, that is, (i,j) with $(-i,-j)$ whenever $|i|+|j|=r$. On the left (respectively right) hand side of Fig. 1 we illustrate the projective diamond grid of size 10 (respectively, 11).

Theorem 2.1 *Every graph that embeds in the projective plane with face-width r has a minor isomorphic to P_r.*

Proof. Let ϱ denote a closed noncontractible curve intersecting a projective embedding of G in exactly r vertices $v_1, v_2, \ldots, v_r$ in this cyclic order. Cutting the projective plane along ϱ, we get a (planar) disk with boundary ϱ holding two copies u_i, u_i' of each vertex v_i, in cyclic order $u_1, \ldots, u_r, u_1', \ldots, u_r'$. Let G' denote the plane graph derived in this way from G. We claim that G' contains a collection of r pairwise disjoint paths $P_1, \ldots, P_r$, and a collection of $2\lfloor r/2 \rfloor$ pairwise disjoint paths $Q_1, \ldots, Q_{2\lfloor r/2 \rfloor}$, such that:

- each P_i connects u_i to u'_{r+1-i},
- each Q_i connects $u_{\lfloor r/2 \rfloor+1-i}$ to $u_{\lceil r/2 \rceil+i}$ if $i \le \lfloor r/2 \rfloor$, and Q_i connects $u'_{r+\lfloor r/2 \rfloor+1-i}$ to $u'_{i-\lfloor r/2 \rfloor}$ if $\lfloor r/2 \rfloor < i \le 2\lfloor r/2 \rfloor$.

To prove this, first we note that in G' there cannot be a vertex cut of size less than r separating $A = \{u_1, \ldots, u_r\}$ from (disjoint) $B = \{u_1', \ldots, u_r'\}$, since that would contradict that the face–width of G is r. Thus, by Menger's theorem, there exist r pairwise disjoint paths $P_1, \ldots, P_r$ in G' from A to B. Moreover, planarity of G' forces these paths to connect u_1 to u_r', u_2 to u_{r-1}', and so on. For even r, we get r paths $Q_1, \ldots, Q_r$ by the same argument between $C = \{u_1, \ldots, u_{r/2}, u'_{r/2+1}, \ldots, u'_r\}$ and $D = \{u'_1, \ldots, u'_{r/2}, u_{r/2+1}, \ldots, u_r\}$. For odd r, we are seeking only $r-1$ paths $Q_1, \ldots, Q_{r-1}$ from $C \setminus \{u'_{\lceil r/2 \rceil}\}$ to $D \setminus \{u_{\lceil r/2 \rceil}\}$. They are found by an analogous argument in the subgraph $G' - \{u_{\lceil r/2 \rceil}, u'_{\lceil r/2 \rceil}\}$, noticing that the face-width of $G - \{u_{\lceil r/2 \rceil}\}$ is $r-1$.

We now claim that $P_1, \ldots, P_r$, and $Q_1, \ldots, Q_{2\lfloor r/2 \rfloor}$ can be chosen such that, for all i, j, the intersection $P_i \cap Q_j$ is connected (possibly empty).

Among all choices of the two collections of paths we select one for which $|E(P^+) \setminus E(Q^+)|$ is minimized, where $P^+ = P_1 \cup \ldots \cup P_r$ and $Q^+ = Q_1 \cup \ldots \cup Q_{2\lfloor r/2 \rfloor}$. Let $\mathcal{R}_{i-1,i}$ denote the open region between P_{i-1} and P_i. Seeking a contradiction, we take a pair of indices i, j such that i is minimum one for which one of the following is true; (a) for some x, y in the intersection of Q_j with P_i the subpath of Q_j between x, y passes through $\mathcal{R}_{i-1,i}$, (b) for some $x, y \in V(Q_j) \cap V(P_i)$ the subpath of Q_j between x, y enters $\mathcal{R}_{i,i+1}$, or (c) Q_j enters $\mathcal{R}_{i,i+1}$ both before and after intersecting P_i.

If (a) happens, then Q_j cannot intersect P_{i-1} by minimality of i, and so P_i can be re-routed along a section of Q_j in $\mathcal{R}_{i-1,i}$ decreasing $|E(P^+) \setminus E(Q^+)|$, a contradiction. If (b) happens, then no $Q_{j'}$ may intersect the subpath of P_i between x, y unless (a) is true for i, j', or i is not minimal. So Q_j can be re-routed along the section of P_i between x and y decreasing $|E(P^+) \setminus E(Q^+)|$ again. Finally, if (c) happens, then clearly $j \le \lfloor r/2 \rfloor - i$ (or $j > \lfloor r/2 \rfloor + i$, symmetrically). Setting $j' = \lfloor r/2 \rfloor + 1 - i$ (or $j' = \lfloor r/2 \rfloor + i$ in the symmetric case), we see that $Q_{j'}$ sharing one end with P_i has to pass through $\mathcal{R}_{i-1,i}$ by planarity, and so we are back in (a) with i, j'.

Hence, particularly by (a),(b), $P_i \cap Q_j$ is connected for all pairs i, j. By contracting to a vertex the intersection between P_i and Q_j for each i and j where nonempty, we obtain a minor in G' which is a subdivision of a diamond grid of size r, which corresponds back in G to a projective diamond grid minor of size r. □

3 Crossing number of projective diamond grids

A set $\mathcal{C}$ of cycles in a graph is an *I-collection* if each two cycles in $\mathcal{C}$ have connected, nonempty intersection, and no vertex is in more than two cycles of $\mathcal{C}$. The following statement is an easy exercise (see Fig. 2).

Proposition 3.1 *The projective diamond grid P_r of size r contains an I-collection of $r-1$ cycles.*

The first key observation is that each fixed orientable surface cannot host an arbitrarily large embedded I–collection.

Proposition 3.2 *For each nonnegative integer g there is a positive constant M_g such that if an I–collection $\mathcal{C}$ is embedded in Σ_g then $|\mathcal{C}| \le M_g$.*

Proof. Let $\mathcal{C}$ be an I–collection embedded in Σ_g. First we note that the intersection between any two cycles in $\mathcal{C}$ may be contracted to a single vertex, if necessary, and the

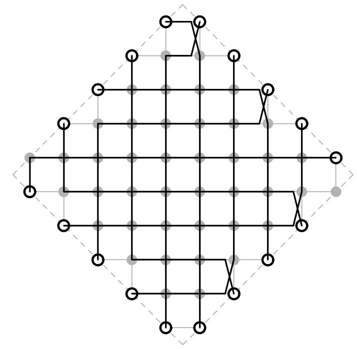

Figure 2: Finding an I-collection of 9 cycles in P_{10}.

result is still an I–collection of the same size. Thus we may assume that the intersection between any two cycles in $\mathcal{C}$ is a single vertex.

Let $\mathcal{C}_0$ denote the subcollection of all contractible cycles of $\mathcal{C}$. It is straightforward to induce from $\mathcal{C}_0$ an embedding of the complete graph on $|\mathcal{C}_0|$ vertices, and so $|\mathcal{C}_0|$ is at most the size of the largest complete graph that embeds in Σ_g, that is, $|\mathcal{C}_0| \leq \frac{1}{2}(7+\sqrt{1+48g})$.

It is an easy observation that no four pairwise homotopic noncontractible curves (in any orientable surface) can pairwise intersect in exactly one point, unless some point belongs to more than two curves. Since $\mathcal{C}$ is an I–collection, it follows that no four curves in $\mathcal{C} \setminus \mathcal{C}_0$ are pairwise homotopic. Thus, after eliminating at most two thirds of the cycles in $\mathcal{C} \setminus \mathcal{C}_0$, we are left with a collection $\mathcal{C}'$ of pairwise nonhomotopic, simple closed curves that pairwise intersect in exactly one point. By [8], there is a constant N_g which depends only on g such that any such $\mathcal{C}'$ has size at most N_g. Thus $|\mathcal{C} \setminus \mathcal{C}_0| \leq 3N_g$, and so $|\mathcal{C}| \leq 3N_g + \frac{1}{2}(7+\sqrt{1+48g})$. □

Secondly, we show that the crossing number of sufficiently large I-collections grows quadratically with their size, which finishes the main proof.

Theorem 3.3 *Let G be a graph that contains an I–collection of size $k > M_g$, where M_g is the constant in Proposition 3.2. Then the crossing number of G in Σ_g is at least $k(k-1)/(M_g(M_g+1))$.*

Proof. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be an I–collection in G, and let $\mathcal{D}$ be a drawing of G in Σ_g. Let M_g be as in Proposition 3.2. Then in any collection $\mathcal{C}' \subseteq \mathcal{C}$ of M_g+1 C_i's there are edges e, f in different C_i's that cross in $\mathcal{D}$. One such a crossing pair e, f gets counted exactly $\binom{k-2}{M_g-1}$-times since we have a free choice of selecting the remaining M_g-1 cycles from $\mathcal{C}$ to form $\mathcal{C}' \supseteq \{e, f\}$ of size M_g+1. Hence the counting argument yields that the total number of crossings in $\mathcal{D}$ is at least $\binom{k}{M_g+1}/\binom{k-2}{M_g-1} = k(k-1)/(M_g(M_g+1))$. □

Proof of Theorem 1.1. By Theorem 2.1, G contains a (projective diamond grid) P_r-minor. It is moreover obvious that if a minor of G contains an I-collection, then an I-collection of the same size is contained also in G itself. Hence it now follows from Proposition 3.1 that G contains an I-collection of $r-1$ cycles, and from Theorem 3.3 that

$\mathrm{cr}_g(G) \geq (r-1)(r-2)/(M_g(M_g+1))$. Thus Theorem 1.1 follows if we set $r_g = M_g + 2$, and $c_g = 1/(M_g+2)^2$ since $M_g + 2 \leq r$. □

It is easy to see that $M_0 = 4$ (planar case) satisfies Proposition 3.1. This gives the following special (planar) version of Theorem 1.1.

Theorem 3.4 *If G embeds in the projective plane with face–width at least $r \geq 6$, then the crossing number $\mathrm{cr}_g(G)$ of G in the plane is at least $\frac{1}{36}r^2$.*

4 Estimating the crossing number of bounded degree projective graphs

The basic idea behind our approximation algorithm is that the crossing number of bounded degree projective graphs is bounded by above and by below by quantities that are within a constant factor of each other. The required lower bound is given in Theorem 1.1.

To obtain the upper bound we perform surgery on the projective plane: cut along an essential curve that intersects the embedded graph as little as possible, then rejoin the pieces and bound the number of crossings thus obtained. This technique is presented in its full generality (applies to all surfaces) by Böröczky, J. Pach, and G. Tóth in [2], in which an even sharper bound of $O(\sum_v \deg^2(v))$ is presented. Using these techniques, we now give a bound that explicitly involves the face–width of the embedding.

Proposition 4.1 *Suppose that G is a graph with maximum degree Δ that embeds in the projective plane with face–width r. Then the crossing number of G in the plane (and thus in any orientable surface) is at most $r^2\Delta^2/8$.*

Proof. Consider ℓ, the *dual edge-width* of G—i.e. the length ℓ of the shortest noncontractible cycle C^* in the topological dual of embedded G in the projective plane. Hence C^* intersects a set F of exactly ℓ edges of G, and if we now perform surgery on the projective plane by cutting along C^*, we get an ordinary plane embedding of $G - F$ in which the ends of edges from F all lie on the outer face. Hence we can easily re-insert the edges of F back by using at most $\binom{\ell}{2} < \ell^2/2$ crossings.

It remains to argue that $\ell \leq r\Delta/2$. Indeed, consider a simple noncontractible curve γ that intersects G in exactly r vertices $u_1, u_2, \ldots, u_r$. Now we may slightly perturb γ to a curve γ' that crosses at most $\deg(u_i)/2$ edges incident with each u_i, and γ' is disjoint from $V(G)$. The faces of G traversed by γ' then define in this order the vertex set of a noncontractible dual cycle C^*, and so $\ell \leq |V(C^*)| \leq r\Delta/2$. □

Proof of Theorem 1.2. The idea of the previous statement readily translates into an approximation algorithm, namely:

- We test whether the input graph G embeds in Σ_g using the $O(n)$-time algorithm by Mohar [10] (if the input G is not given along with a projective embedding, we can easily construct one, also using [10]).

- We construct the topological dual G^* of G in the projective plane.
- Then we compute a shortest noncontractible cycle C^* in G^*. For that one can use an $O(n\sqrt{n})$-time algorithm by Cabello and Mohar [4]. As pointed to us by S. Cabello [private communication], the same goal can be achieved in $O(n \log n)$ time using a suitable preprocessing and then algorithm of Klein [9] (for planar distances).
- Let F be the set of edges of G intersected by the (dual) edges of C^*. Then $G - F$ is actually a plane embedding, and we easily add the edges of F back to $G - F$, making a plane drawing $\mathcal{D}$ with at most $\binom{|F|}{2}$ pairwise crossings.

This whole algorithm can run in time $O(n \log n)$.

Assume now that G does not embed in Σ_g, while G embeds in the projective plane with face–width r. Let r_g be as in Theorem 1.1. If $r < r_g$, then $1 \le \mathrm{cr}_g(G) \le \mathrm{cr}(\mathcal{D}) \le \binom{|F|}{2} < r_g^2\Delta^2/8$ as in Proposition 4.1, and hence the number of crossings in $\mathcal{D}$ is within a constant factor $r_g^2\Delta^2/8$ of $\mathrm{cr}_g(G)$.

If, on the other hand, $r \ge r_g$, then by Theorem 1.1 and Proposition 4.1 we get $c_g r^2 \le \mathrm{cr}_g(G) \le \mathrm{cr}(\mathcal{D}) \le r^2\Delta^2/8$, and so in this case the number of crossings in $\mathcal{D}$ is within a constant factor $\Delta^2/(8c_g)$ of $\mathrm{cr}_g(G)$. □

Remark 4.2 *In the planar case of Theorem 1.2, the described approximation algorithm yields a drawing of G within a factor $4.5\Delta^2$ of $\mathrm{cr}_0(G)$.*

References

[1] D. Bokal, G. Fijavž, and D. Wood, The minor crossing number of graphs with an excluded minor, Electronic J. Combinatorics 15(1):R4, (2008).

[2] K. Böröczky, J. Pach, and G. Tóth, Planar crossing numbers of graphs embeddable in another surface, *Internat. J. of Foundations of Comp. Science*, 17 (2006), 1005–1011.

[3] R. Brunet, B. Mohar, R.B. Richter, *Separating and nonseparating disjoint homotopic cycles in graph embeddings*, J. Combin. Theory Ser. B 66 (1996), 201–231.

[4] S. Cabello and B. Mohar, Finding shortest non-separating and non-contractible cycles for topologically embedded graphs. In: *ESA 2005*, Lecture Notes in Computer Science **3669**, 131–142. Springer, 2005.

[5] H. Djidjev and I. Vrt'o, Planar crossing numbers of genus g graphs. In: *Proc. 33rd ICALP*, Lecture Notes in Computer Science **4051**, 419–430. Springer, 2006.

[6] G. Even, S. Guha, B. Schieber, Improved Approximations of Crossings in Graph Drawings and VLSI Layout Areas, *SIAM J. Comput.* 32(1), 231–252 (2002).

[7] J.R. Fiedler, J.P. Huneke, R.B. Richter, and N. Robertson, Computing the orientable genus of projective graphs. *Journal of Graph Theory* 20 (1995), 297–308.

[8] M. Juvan, A. Malnič, and B. Mohar. Systems of curves on surfaces, *Journal of Combinatorial Theory Series B* 68 (1996), 7–22.

[9] P.N. Klein, Multiple-source shortest paths in planar graphs. In: *SODA 2005, Proceedings*, SIAM (2005), 146–155.

[10] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface. *SIAM J. Discrete Math.* 12 (1999), 6–26.

[11] S. Randby, Minimal embeddings in the projective plane. *J. Graph Theory* 25 (1997), 153–163.

[12] D.R. Wood and J.A. Telle, Planar decompositions and the crossing number of graphs with an excluded minor. *Proc. of 14th International Symposium on Graph Drawing* (GD '06), Lecture Notes in Computer Science **4372**, 150–161. Springer, 2007.

[13] D.R. Wood and J.A. Telle, Planar decompositions and the crossing number of graphs with an excluded minor. *New York J. Mathematics* **13** (2007), 117–146.