Heterochromatic matchings in edge-colored graphs

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Submitted: Dec 2, 2007; Accepted: Oct 28, 2008; Published: Nov 14, 2008
Mathematics Subject Classifications: 05C38, 05C15

Abstract

Let G be an (edge-)colored graph. A heterochromatic matching of G is a matching in which no two edges have the same color. For a vertex v, let $d^c(v)$ be the color degree of v. We show that if $d^c(v) \geq k$ for every vertex v of G, then G has a heterochromatic matching of size $\lceil \frac{5k-3}{12} \rceil$. For a colored bipartite graph with bipartition (X, Y), we prove that if it satisfies a Hall-like condition, then it has a heterochromatic matching of cardinality $\lceil \frac{|X|}{2} \rceil$, and we show that this bound is best possible.

1 Introduction and notation

We consider simple undirected graphs. Let $G = (V, E)$ be a graph. An *edge coloring* of G is a function $C : E \to \{0, 1, 2, \cdots\}$. If G is assigned such a coloring C, then we say that G is an *edge colored graph*, or simply *colored graph*. Denote by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$.

We study heterochromatic matchings, the case H is a matching. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [13]), the maximum heterochromatic matching problem is NP-complete, even for bipartite graphs (see [9]).

The heterochromatic subgraphs have received increasing attention in the last decade as mentioned below. Albert, Frieze and Reed [2] proved that the colored complete graph K_n has a heterochromatic Hamiltonian cycle if n is sufficiently large and no color appears more than $\lceil cn \rceil$ times, where $c < 1/32$. Suzuki [17] gave a sufficient and necessary condition for the existence of a heterochromatic spanning tree in a colored connected graph. For more references, see [3, 6, 7, 8, 10].

Theorem 1.1 [16] *Every $n \times n$ Latin square has a partial transversal of length at least $n - 5.53(\log n)^2$, namely every properly edge-colored complete bipartite graph $K_{n,n}$ with n colors has a heterochromatic matching with at least $n - 5.53(\log n)^2$ edges.*

For colored complete graphs, Kaneko and Suzuki gave the following result.

Theorem 1.2 [12] *For $n \geq 3$, each proper edge coloring of K_{2n} has a heterochromatic perfect matching.*

Let G be a colored graph. For a vertex set S, a *color neighborhood* of S is defined as a set $T \subseteq N(S)$ such that there are $|T|$ edges between S and T that are incident at distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S with maximum size. In particular, if $S = \{v\}$, then let $d^c(v) = |N^c(v)|$ and call it the *color degree* of v. Given a set S and a color neighborhood T of S, denote by $C(S,T)$ a set of $|T|$ distinct colors on some such set of $|T|$ edges between S and distinct vertices of T.

In [15], we obtained the following result concerning heterochromatic matchings in colored bipartite graphs meeting a color degree condition.

Theorem 1.3 [15] *For a colored bipartite graph G, if $d^c(v) \geq k \geq 3$ for each vertex $v \in V(G)$, then G has a heterochromatic matching of cardinality $\lceil \frac{2k}{3} \rceil$.*

In this paper, we study heterochromatic matchings in general graphs and obtain the following result.

Theorem 1.4 *Let G be a colored graph. If $d^c(v) \geq k$ for each vertex $v \in V(G)$, then G has a heterochromatic matching of cardinality $\lceil \frac{5k-3}{12} \rceil$.*

We propose the following strengthening of Theorem 1.4.

Conjecture 1.1 *Let G be a colored graph. Suppose that $d^c(v) \geq k \geq 4$ for each vertex v of G, then there exists a heterochromatic matching with $\lceil \frac{k}{2} \rceil$ edges.*

The complete graph K_{k+1} with a proper edge coloring satisfies $d^c(v) = k$ for each vertex v, and K_{k+1} contains no heterochromatic matching of cardinality more than $\lceil \frac{k}{2} \rceil$. Thus if the above conjecture holds, it would be best possible.

In [14], large heterochromatic matchings under some color neighborhood conditions in colored bipartite graphs were studied and the following result was obtained.

Theorem 1.5 **[14]** *Let G be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality $\lceil \frac{3n}{8} \rceil$.*

In the case of 3-partite 3-uniform hypergraphs, Aharoni [1] verified a conjecture of Ryser. Using this result, we improve the bound in Theorem 1.5 as follows.

Theorem 1.6 *Let G be a colored bipartite graph with bipartition (X, Y). If $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching of cardinality $\lceil \frac{|X|}{2} \rceil$.*

Moreover, we show that the bound in Theorem 1.6 is sharp.

2 Proof of Theorem 1.4

Before the proof of Theorem 1.4, we give some notations and a proposition. For a heterochromatic matching M of G, let V_M denote $V(M)$. For a vertex $v \in V(G - V_M)$, let $b_M(v)$ denote $C(M) \cap C(\{vx : \ x \in V(G - V_M)\})$. For a subset V_1 of $V(G - V_M)$, let $b_M(V_1)$ denote $\{b_M(v) : \ v \in V_1\}$. For simplicity, let $b_M = b_M(V(G - V_M))$.

Relative a heterochromatic matching M, an *alternating* 3-path AP_M in G is a path $x'yxy'$ such that $C(xy') = C(x'y) \notin C(M)$, in which $xy \in E(M)$ and $x', y' \in V(G - V_M)$. Given two alternating 3-paths $AP_M^1 = x_1'y_1x_1y_1'$ and $AP_M^2 = x_2'y_2x_2y_2'$, AP_M^1 is *different* from AP_M^2, by the phrase we mean that $C(x_1'y_1) \neq C(x_2'y_2)$ and $x_1y_1 \neq x_2y_2$.

Easily, we can get the following proposition by Theorem 1.2.

Proposition 2.1 *For $m \geq 5$, each proper edge coloring of K_m has a heterochromatic matching of cardinality $\lceil \frac{m-1}{2} \rceil$.*

Proof of Theorem 1.4

For $k \leq 3$, Theorem 1.4 holds clearly. So we assume that $k \geq 4$. Suppose the conclusion is false, then we choose a heterochromatic matching M such that
(R_1) $|M| = t$ is maximum;
(R_2) subject to (R_1), $|b_M|$ is maximum.

Let $C(M) = \{c_1, c_2, \cdots, c_t\}$. Since for each vertex v, $d^c(v) \geq k \geq 4$ and $t \leq \lceil \frac{5k-3}{12} \rceil - 1$, it holds that $|V(G - V_M)| \geq 2$. Choose $v_x, v_y \in V(G - V_M)$. Let $N^c(v_x), N^c(v_y)$ be maximum color neighborhoods of v_x, v_y, respectively. Let $N^c(v_x) = S_1 \cup S_2$ $(S_1 \cap S_2 = \emptyset)$, where $C(v_x, S_1) \cap C(M) = \emptyset$ and $C(v_x, S_2) \subseteq C(M)$. Further let $N^c(v_y) = S_3 \cup S_4$ $(S_3 \cap S_4 = \emptyset)$, in which $C(v_y, S_3) \cap C(M) = \emptyset$ and $C(v_y, S_4) \subseteq C(M)$. Clearly $|S_2|, |S_4| \leq t$.

Claim 2.1 $S_1, S_3 \subseteq V_M$.

Proof. Otherwise, there exists a vertex $v \in V(G - V_M)$ such that $C(v_xv)$(or $C(v_yv)$) $\notin C(M)$, then $M \cup \{v_xv\}$(or $\{v_yv\}$) is a heterochromatic matching of cardinality $t + 1$, a contradiction. □

Claim 2.2 *There exists an AP_M in G.*

Proof. Since $|N^c(v_x)| = |S_1| + |S_2| \geq k$, it follows that $|S_1| \geq k - |S_2| \geq k - t$. Similarly $|S_3| \geq k - |S_4| \geq k - t$. Hence $|S_1| + |S_3| \geq 2(k-t) = 2k - 2t > 2t = |V_M|$. Then there exists an edge $xy \in M$ such that x is adjacent with v_y and y is adjacent with v_x, moreover $C(xv_y), C(v_x y) \notin C(M)$. If $C(xv_y) \neq C(v_x y)$, letting $M' = M \cup \{xv_y, v_x y\} - \{xy\}$, we see that M' is a heterochromatic matching and $|M'| = t + 1$, a contradiction. Thus $C(xv_y) = C(v_x y)$, and it follows that $v_x y x v_y$ is an AP_M. $\square$

Let l be the maximum number of the vertex-disjoint AP_Ms in G satisfying that every pair of AP_Ms are different. Clearly $1 \leq l \leq t$. For $1 \leq i \leq l$, assume that AP_M^i has edges $\{x_i' y_i, x_i y_i, x_i y_i'\}$, where $x_i y_i \in E(M)$, $x_i', y_i' \in V(G - V_M)$ and $C(x_i y_i') = C(x_i' y_i) = c_i'$.

Let V_L denote $\{x_1', x_2', \cdots, x_l'\} \cup \{y_1', y_2', \cdots, y_l'\}$ and let V_{M_l} denote $\{x_1, x_2, \cdots, x_l\} \cup \{y_1, y_2, \cdots, y_l\}$, where $\{x_1 y_1, x_2 y_2, \cdots, x_l y_l\} = E(M_l) \subseteq E(M)$. We abbreviate $C_l = C(M_l) = \{c_1, c_2, \cdots, c_l\}$ and $C_L = \{c_1', c_2', \cdots, c_l'\}$. Clearly $C(M) - C(M_l) = C(M - M_l)$. Let $S_0 = V - V_M - V_L$, and we have the following claim.

Claim 2.3 $|S_0| \geq 2$.

Proof. Otherwise, we have that $|S_0| \leq 1$. If $|S_0| = 1$, then assume that $S_0 = \{u\}$. Since for each vertex v of G, $d^c(v) \geq k$, then $2(t+l) + 1 \geq k + 1$. If $2(t+l) + 1 = k + 1$, then G is a colored complete graph such that $|V(G)| = k + 1$ and $d^c(v) = k \geq 4$ for each vertex v of G. That is, G is an proper-edge-colored complete graph of order at least 5. Thus, by Proposition 2.1, G has a heterochromatic matching of size $\lceil \frac{k}{2} \rceil \geq \lceil \frac{5k-3}{12} \rceil > t$, a contradiction. So we conclude that $2(t+l) \geq k+1$, then $l \geq \frac{k+1}{2} - t$. Now consider the vertices x_1', y_1' and we have the following facts.

Fact 2.1 *Suppose y_1' has a neighbor $v \in V(M_l) \backslash \{x_1\}$ and $C(vy_1') \notin C(M - M_l)$, where without loss of generality, let $v = x_i$ $(2 \leq i \leq l)$. Then*
(1) $C(x_i y_1') = c_i'$.
(2) $|b_M(x_i')| \geq 1$.
(3) $C_l \cap b_M(x_i') = \emptyset$.

Proof. Suppose, to the contrary, $C(x_i y_1') \neq c_i'$, then let

$$M' = \begin{cases} M \cup \{x_i y_1', x_i' y_i\} - \{x_i y_i\} & C(x_i y_1') \notin C_l \text{ or } C(x_i y_1') = c_i; \\ M \cup \{x_i y_1', x_i' y_i, x_j' y_j\} - \{x_i y_i, x_j y_j\} & C(x_i y_1') = c_j, 1 \leq j \leq l,\ j \neq i. \end{cases}$$

Then M' is a heterochromatic matching of cardinality $t + 1$, which is a contradiction. Thus it holds that $C(x_i y_1') = c_i'$.

If there is an edge $e \in E(G - V_M)$ such that $C(e) = c_i$, then $e = x_i' y_i'$. Otherwise, assume that e is not incident with x_i', then $M' = M \cup \{x_i' y_i, e\} - \{x_i y_i\}$ is a heterochromatic matching such that $|M'| > t$, a contradiction. If $|b_M(x_i')| = 0$, letting $M' = M \cup \{x_i' y_i\} - \{x_i y_i\}$, we see that M' is a heterochromatic matching such that $|M'| = t$ and $|b_{M'}| \geq |b_M| + |b_{M'}(x_i)| \geq |b_M| + 1$, a contradiction with the choice of M. Thus $|b_M(x_i')| \geq 1$.

If $C_l \cap b_M(x'_i) \neq \emptyset$, we assume that $c_j \in b_M(x'_i)$, $1 \leq j \leq l$. There exists an edge $x'_i z \in E(G - V_M)$ such that $C(x'_i z) = c_j$. Then let

$$M' = \begin{cases} M \cup \{x'_i z, x_i y'_i\} - \{x_i y_i\} & j = i, z = y'_1; \\ M \cup \{x'_i z, x_i y'_1\} - \{x_i y_i\} & j = i, z \neq y'_1; \\ M \cup \{x'_i z, x'_j y_j\} - \{x_j y_j\} & j \neq i, z = y'_j; \\ M \cup \{x'_i z, x_j y'_j\} - \{x_j y_j\} & j \neq i, z \neq y'_j. \end{cases}$$

Clearly, M' is a heterochromatic matching and $|M'| > t$, a contradiction. □

Similarly to Fact 2.1, we can prove the following fact, for simplicity, we omit the proof.

Fact 2.1′. *Suppose x'_1 has a neighbor $v \in V(M_l) \backslash \{y_1\}$ and $C(x'_1 v) \notin C(M - M_l)$, where without loss of generality, let $v = y_i$ $(2 \leq i \leq l)$. Then*
(1) $C(x'_1 y_i) = c'_i$.
(2) $|b_M(y'_i)| \geq 1$.
(3) $C_l \cap b_M(y'_i) = \emptyset$.

Let $N^c(y'_1)$ be a maximum color neighborhood of y'_1 such that $x_1 \in N^c(y'_1)$. Assume that $N^c(y'_1) = P_1 \cup P_2$ $(P_1 \cap P_2 = \emptyset)$, where $C(y'_1, P_1) \cap (C(M - M_l) \cup \{c_1\}) = \emptyset$ and $C(y'_1, P_2) \subseteq C(M - M_l) \cup \{c_1\}$. Further let $P_1^1 = P_1 \cap (V(M_l) \backslash \{y_1\})$, $|P_1^1| = p_1$ and $P_1^2 = P_1 \backslash P_1^1$. Clearly $|P_2| \leq t - l + 1$.

Let $N^c(x'_1)$ be a maximum color neighborhood of x'_1 such that $y_1 \in N^c(x'_1)$. Assume that $N^c(x'_1) = P_3 \cup P_4$ $(P_3 \cap P_4 = \emptyset)$, where $C(x'_1, P_3) \cap (C(M - M_l) \cup \{c_1\}) = \emptyset$ and $C(x'_1, P_4) \subseteq C(M - M_l) \cup \{c_1\}$. Further let $P_3^1 = P_3 \cap (V(M_l) \backslash \{x_1\})$, $|P_3^1| = p_3$ and $P_3^2 = P_3 \backslash P_3^1$. Clearly $|P_4| \leq t - l + 1$.

By symmetry and without loss of generality, we assume that $P_1^1 = \{x_{k_1}(x_{k_1} = x_1), x_{k_2} \cdots, x_{k_{p_1}}\}$ and let $P_1^{1'}$ denote $\{x'_{k_2}, \cdots x'_{k_{p_1}}\}$. Similarly we assume that $P_3^1 = \{y_{j_1}(y_{j_1} = y_1), y_{j_2}, \cdots, y_{j_{p_3}}\}$ and let $P_3^{1'}$ denote $\{y'_{j_2}, \cdots y'_{j_{p_3}}\}$. Firstly, we assume that $P_1^{1'}, P_3^{1'} \neq \emptyset$.

Fact 2.2 $|b_M(P_1^{1'})| \geq p_1 - 1$ *and* $|b_M(P_3^{1'})| \geq p_3 - 1$.

Proof. If $|b_M(P_1^{1'})| < p_1 - 1$ then $M' = M \cup \{x'_{k_2} y_{k_2}, \cdots, x'_{k_{p_1}} y_{k_{p_1}}\} - \{x_{k_2} y_{k_2}, \cdots, x_{k_{p_1}} y_{k_{p_1}}\}$ is a heterochromatic matching such that $|M'| = t$ and $|b_{M'}| \geq |b_M| + p_1 - 1 - |b_M(P_1^{1'})| > |b_M|$, a contradiction. Thus $|b_M(P_1^{1'})| \geq p_1 - 1$. Similarly, we can prove that $|b_M(P_3^{1'})| \geq p_3 - 1$. □

Without loss of generality, we assume that $b_M(P_1^{1'}) = \{c_{l+1}, c_{l+2}, \cdots, c_{l+p_2}\}$. Let $V(M_{p_2}) = \{x_{l+1}, x_{l+2}, \cdots, x_{l+p_2}\} \cup \{y_{l+1}, y_{l+2}, \cdots, y_{l+p_2}\}$. Similarly, we assume that $b_M(P_3^{1'}) = \{c_{i_1}, c_{i_2}, \cdots, c_{i_{p_4}}\}$. Let $V(M_{p_4})$ denote $\{x_{i_1}, x_{i_2}, \cdots, x_{i_{p_4}}\} \cup \{y_{i_1}, y_{i_2}, \cdots, y_{i_{p_4}}\}$.

Fact 2.3 *Suppose x'_1 and y'_1 have a common neighbor $v \in V(M_{p_4}) \cup V(M_{p_2})$, then $C(vx'_1) \in C(M - M_l) \cup \{c_1\}$ or $C(vy'_1) \in C(M - M_l) \cup \{c_1\}$.*

Proof. By contradiction. Otherwise, $C(vx'_1), C(vy'_1) \notin C(M - M_l) \cup \{c_1\}$. Without loss of generality, assume that $v = x_{i_1} \in V(M_{p_4})$ and since $c_{i_1} \in b_M(P_3^{1'})$, moreover we can assume that $c_{i_1} \in b_M(y'_{j_2})$. By the definition of the $b_M(y'_{j_2})$, we conclude that there is an edge $y'_{j_2}z \in E(G - V_M)$ such that $C(y'_{j_2}z) = c_{i_1}$. We distinguish the following cases.

Case 1. $z = x'_1$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}y'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}y'_1) \notin C_l; \\ M \cup \{x_{i_1}y'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}y'_1) = c_{j_2}; \\ M \cup \{x_{i_1}y'_1, x'_j y_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_j y_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l \text{ and } j \ne j_2. \end{cases}$$

Case 2. $z = y'_1$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}x'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}x'_1) \notin C_l; \\ M \cup \{x_{i_1}x'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}x'_1) = c_{j_2}; \\ M \cup \{x_{i_1}x'_1, x'_j y_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_j y_j\} & C(x_{i_1}x'_1) = c_j, 2 \le j \le l \text{ and } j \ne j_2. \end{cases}$$

Case 3. $z \notin \{x'_1, y'_1\}$. Then let

$$M' = \begin{cases} M \cup \{x_{i_1}y'_1, y'_{j_2}z\} - \{x_{i_1}y_{i_1}\} & C(x_{i_1}y'_1) \notin C_l; \\ M \cup \{x_{i_1}y'_1, y'_{j_2}z, x'_{j_2}y_{j_2}\} - \{x_{i_1}y_{i_1}, x_{j_2}y_{j_2}\} & C(x_{i_1}y'_1) = c_{j_2}; \\ M \cup \{x_{i_1}y'_1, x_j y'_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_j y_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l, j \ne j_2, z \ne y'_j; \\ M \cup \{x_{i_1}y'_1, x'_j y_j, y'_{j_2}z\} - \{x_{i_1}y_{i_1}, x_j y_j\} & C(x_{i_1}y'_1) = c_j, 2 \le j \le l, j \ne j_2, z = y'_j. \end{cases}$$

In any case, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. □

For simplicity, let V' denote $V(M - M_l)$.

Fact 2.4 *$P_1^2 \subseteq V' \cup \{y_1\}$ and $P_3^2 \subseteq V' \cup \{x_1\}$.*

Proof. Suppose, to the contrary, there is a vertex $z \in P_1^2$ and $z \notin V(M - M_l) \cup \{y_1\}$. Since $P_1^2 = P_1 \backslash P_1^1$, then $z \notin V(M_l)$ and $C(y'_1z) \notin C(M - M_l) \cup \{c_1\}$. We distinguish the following two cases.

Case 1. $z \in V(G - V_M - V_L)$. In fact, if $V(G - V_M - V_L) \ne \emptyset$, then $z = u$. Then let

$$M' = \begin{cases} M \cup \{y'_1z\} & C(y'_1z) \notin C_l; \\ M \cup \{y'_1z, x'_j y_j\} - \{x_j y_j\} & C(y'_1z) = c_j, 2 \le j \le l. \end{cases}$$

Case 2. $z \in V_L$. Assume that $z = x'_i (1 \le i \le l)$, then let

$$M' = \begin{cases} M \cup \{y'_1z\} & C(y'_1z) \notin C_l; \\ M \cup \{y'_1z, x_j y'_j\} - \{x_j y_j\} & C(y'_1z) = c_j, 2 \le j \le l. \end{cases}$$

In both cases, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. Thus it holds that $P_1^2 \subseteq V' \cup \{y_1\}$. Similarly, we have that $P_3^2 \subseteq V' \cup \{x_1\}$. □

By Facts 2.3 and 2.4, we conclude that

$$\begin{aligned} |P_1^2 \cap V'| + |P_3^2 \cap V'| &\le 2|V'| - |M_{p_2}| - |M_{p_4}| \\ &\le 4(t - l) - p_2 - p_4 \end{aligned}$$

On the other hand, $|P_1^2 \cap V'| \geq k - |P_2| - |P_1^1| - |P_1^2 \cap \{y_1\}| \geq k - (t - l + 1) - p_1 - 1$ and $|P_3^2| \geq k - |P_4| - |P_3^1 \cap \{x_1\}| \geq k - t + l - p_3 - 2$. Since $t \leq \lceil \frac{5k-3}{12} \rceil - 1$, $l \geq \frac{k+1}{2} - t$ and, by Fact 2.2, $p_2 \geq p_1 - 1$, $p_4 \geq p_3 - 1$, it follows that

$$\begin{aligned}
|P_1^2 \cap V'| &+ |P_3^2 \cap V'| - [4(t-l) - p_2 - p_4] \\
&\geq 2k - 2t + 2l - p_1 - p_3 - 4 - 4t + 4l + p_1 + p_3 - 2 \\
&\geq 2k - 6t + 6l - 6 \\
&\geq 5k - 12t - 3 \\
&> 0.
\end{aligned}$$

Note that if $P_1^{1'} = \emptyset$ or $P_3^{1'} = \emptyset$, the above two inequalities also hold, which is a contradiction.

So we have that $|S_0| \geq 2$, which completes the proof of Claim 2.3. $\square$

Now let $w_1, w_2 \in S_0$. Choose a maximum color neighborhood $N^c(w_1)$ of w_1. Assume that $N^c(w_1) = T_1 \cup T_2$ ($T_1 \cap T_2 = \emptyset$), where $C(w_1, T_1) \cap C(M - M_l) = \emptyset$ and $C(w_1, T_2) \subseteq C(M - M_l)$. Further let $T_1^1 = T_1 \cap V(M_l)$, $|T_1^1| = t_1$ and $T_1^2 = T_1 \backslash T_1^2$. Clearly $|T_2| \leq t - l$.

Similarly, choose a maximum color neighborhood $N^c(w_2)$ of w_2. And let $N^c(w_2) = T_3 \cup T_4$ ($T_3 \cap T_4 = \emptyset$), where $C(w_2, T_3) \cap C(M - M_l) = \emptyset$ and $C(w_2, T_4) \subseteq C(M - M_l)$. Further let $T_3^1 = T_3 \cap V(M_l)$, $|T_3^1| = t_2$ and $T_3^2 = T_3 \backslash T_3^1$. Clearly $|T_4| \leq t - l$.

Claim 2.4 *Suppose $w (w \in \{w_1, w_2\})$ has a neighbor $v \in V(M_l)$ and $C(wv) \notin C(M - M_l)$, where without loss of generality, let $v = x_i$ ($1 \leq i \leq l$). Then $C(wx_i) = c_i'$.*

Proof. Otherwise, if $C(wx_i) \notin C(M - M_l)$ and $C(wx_i) \neq c_i'$. Then let

$$M' = \begin{cases} M \cup \{wx_i, x_i'y_i\} - \{x_iy_i\} & C(wx_i) \notin C_l \text{ or } C(wx_i) = c_i; \\ M \cup \{wx_i, x_i'y_i, x_j'y_j\} - \{x_iy_i, x_jy_j\} & C(wx_i) = c_j, 1 \leq j \leq l,\ j \neq i. \end{cases}$$

Then M' is a heterochromatic matching of cardinality $t + 1$, which is a contradiction. $\square$

Claim 2.5 $T_1^2 \subseteq V(M - M_l)$ *and* $T_3^2 \subseteq V(M - M_l)$.

Proof. By symmetry, we only prove that $T_1^2 \subseteq V(M - M_l)$. Otherwise, there is an edge w_1z such that $C(w_1z) \notin C(M - M_l)$, in which $z \in T_1^2$ and $z \notin V(M - M_l)$. Since $T_1^2 = T_1 \backslash T_1^1$, $z \notin V(M_l)$. We distinguish the following two cases.

Case 1. $z \in V(G - V_M - V_L)$. Then let

$$M' = \begin{cases} M \cup \{w_1z\} & C(w_1z) \notin C_l; \\ M \cup \{w_1z, x_j'y_j\} - \{x_jy_j\} & C(w_1z) = c_j, 1 \leq j \leq l. \end{cases}$$

Case 2. $z \in V_L$. Without loss of generality, assume that $z = x_1'$, then let

$$M' = \begin{cases} M \cup \{w_1z\} & C(w_1z) \notin C_l; \\ M \cup \{w_1z, x_jy_j'\} - \{x_jy_j\} & C(w_1z) = c_j, 1 \leq j \leq l. \end{cases}$$

In both cases, M' is a heterochromatic matching and $|M'| > t$, which is a contradiction. $\square$

Since $|N^c(w_1)| = |T_1| + |T_2| \geq k$, it follows that $|T_1^2| \geq k - |T_2| - |T_1^1| \geq k - (t-l) - t_1$. Similarly it holds that $|T_3^2| \geq k - (t-l) - t_2$. Then

$$\begin{aligned} |T_1^2| + |T_3^2| - |V'| &\geq 2k - 2t + 2l - t_1 - t_2 - 2(t-l) \\ &\geq 2k - 4t + 4l - t_1 - t_2 \\ &\geq 2l + 1. \end{aligned}$$

So there exists an edge $x_0y_0 \in E(M - M_l)$, where $x_0 \in T_1^2$, $y_0 \in T_3^2$ and $C(w_1x_0) \notin C_l \cup C_L$. Note that $C(w_1x_0), C(w_2y_0) \notin C(M - M_l)$.

If $C(w_2y_0) \in C_l$, suppose $C(w_2y_0) = c_i$, $1 \leq i \leq l$. Let $M' = M \cup \{w_1x_0, w_2y_0, x_iy_i'\} - \{x_iy_i, x_0y_0\}$, then M' is a heterochromatic matching and $|M'| > t$, a contradiction.

If $C(w_2y_0) \notin C_l$ and $C(w_2y_0) \neq C(w_1x_0)$, then let $M' = M \cup \{w_1x_0, w_2y_0\} - \{x_0y_0\}$. Thus M' is a heterochromatic matching and $|M'| > t$, a contradiction.

If $C(w_2y_0) = C(w_1x_0)$, then we obtain an $AP_M = w_2y_0x_0w_1$, where $C(w_2y_0) = C(w_1x_0) \notin C(M) \cup C_L$, $x_0y_0 \in E(M - M_l)$ and $w_1, w_2 \in V(G - V_M)$. So there exists $(l+1)$ vertex-disjoint AP_Ms, in which every pair of AP_Ms are different, which is a contradiction.

The proof of Theorem 1.4 is complete. $\square$

3 Proof of Theorem 1.6

Firstly, we give some preliminaries. A *hypergraph* is a set of subsets, called *hyperedges*, of some ground set, whose elements are called *vertices*. A hypergraph H is called *r-uniform* (or an *r-graph*) if all its hyperedges are of the same size, r. An r-uniform hypergraph is called *r-partite* if its vertex set $V(H)$ can be partitioned into sets $V_1, \cdots, V_r$ in such a way that each hyperedge meets each V_i in precisely one vertex.

A *matching* in a hypergraph is a set of disjoint hyperedges. The *matching number*, $\nu(H)$, of a hypergraph H is the maximal size of a matching in H.

A *cover* of a hypergraph H is a subset of $V(H)$ meeting all hyperedges of H. The *covering number*, $\tau(H)$, of H is the minimal size of a cover of H. Obviously, $\tau \geq \nu$ for all hypergraphs. In a r-uniform hypergraph $\tau \leq r\nu$, since the union of the hyperedges of a maximal matching forms a cover.

Ryser gave a conjecture as follows.

Conjecture 3.1 *In a r-partite r-uniform hypergraph (where $r > 1$), $\tau \leq (r-1)\nu$.*

This conjecture appeared in the Ph.D thesis of Henderson, a student of Ryser. For small values of r, only the case $r = 3$ was studied for general ν. The bounds for this case were improved successively: $\tau \leq \frac{25}{9}\nu$ [11], $\tau \leq \frac{8}{3}\nu$ [18], $\tau \leq \frac{5}{2}\nu$ [19]. Finally, it was proved by Aharoni [1].

Theorem 3.1 [1] *In a tripartite 3-graph, $\tau \leq 2\nu$.*

Proof of Theorem 1.6

Construct a 3-partite 3-uniform hypergraph H as follows. Let $V_1 = X, V_2 = Y$ and $V_3 = C(G)$. A hyperedge $e = \{x, y, c\} \in E(H)$ if and only if in graph G, $x \in X, y \in Y$ and $C(xy) = c$. Clearly, a matching of a hypergraph H is a heterochromatic matching of G. Let M be a maximum heterochromatic matching. Then $|M| = \nu(H)$.

We conclude that $\tau(H) \geq |X|$. Otherwise, assume that $D = D_1 \cup D_2 \cup D_3$ is a cover of H with $|D| \leq |X| - 1$, in which $D_1 \in V_1, D_2 \in V_2$ and $D_3 \in V_3$. Now consider $F = X \backslash D_1$ in graph G, then there exists a maximum color neighborhood $N^c(F)$ such that $|N^c(F)| \geq |F| = |X| - |D_1|$. Thus in the hypergraph H, there exists a hyperedge set E_1 with $|E_1| \geq |F|$ such that
(i) for each hyperedge $e = \{x, y, c\} \in E_1$, it holds that $x \in F$;
(ii) for two hyperedges $e = \{x, y, c\}$, $e' = \{x', y', c'\}$, it holds that $y \neq y'$ and $c \neq c'$.

By (i), D_1 does not meet any hyperedge of E_1. And $D = D_1 \cup D_2 \cup D_3$ is a cover of H, so $D_2 \cup D_3$ meets each hyperedge of E_1. Thus by (ii) and since $D_2 \cap D_3 = \emptyset$, we conclude that $|D_2| + |D_3| \geq |E_1| \geq |F| = |X| - |D_1|$. Therefore, $|D_1| + |D_2| + |D_3| \geq |X|$, a contradiction. So $\tau(H) \geq |X|$ and by Theorem 3.1, $|X| = \tau(H) \leq 2\nu(H) = 2|M|$. That is $|M| \geq \frac{|X|}{2}$, which completes the proof. $\square$

Let $G = sC_4$, a graph with s components, each a C_4. Let C be a proper edge coloring of G with $2s$ colors so that each color appears exactly twice, both times in the same C_4. Any bipartition (X, Y) for G meets the condition in Theorem 1.6. Yet the largest heterochromatic matching has cardinality $s = \frac{|X|}{2}$. Thus this example shows that the bound in Theorem 1.6 is best possible.

Acknowledgements

The authors are indebted to Ron Aharoni for his helpful discussion. We deeply thank the referee for the constructive comments. This research is supported by the National Natural Science Foundation (10871119) of China, the French-Chinese foundation for sciences and their applications and the China Scholarship Council.

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