More Constructions for Turán's (3, 4)-Conjecture

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Abstract

For Turán's (3, 4)-conjecture, in the case of $n = 3k+1$ vertices, $\frac{1}{2}6^{k-1}$ non-isomorphic hypergraphs are constructed that attain the conjecture. In the case of $n = 3k+2$ vertices, 6^{k-1} non-isomorphic hypergraphs are constructed that attain the conjecture.

1 Introduction

Turán [8] posed the following problem about edges of hypergraphs. Suppose that an m-uniform hypergraph has exactly n vertices. Given $r > m$, if every possible subset of r vertices contains some m that do not form an edge, how many edges can the hypergraph have, as a function of n, m, and r? Let $t_m(n, r)$ be the greatest number of edges that the hypergraph can have. Turán [7] solved the case where $m = 2$.

The next simplest case is when $m = 3$ and $r = 4$. Turán had a conjecture for this case, which we call Turán (3, 4)-conjecture.

Conjecture 1.1 *Let H be a 3-uniform hypergraph in which every set of four vertices contains at most three edges. Let the number of vertices of H be n. Then the number of edges of H is at most*

$$\begin{cases} \frac{5}{2}k^3 - \frac{3}{2}k^2 & \text{if} \quad n = 3k \\ \frac{5}{2}k^3 + k^2 - \frac{1}{2}k & \text{if} \quad n = 3k+1 \\ \frac{5}{2}k^3 + \frac{7}{2}k^2 + k & \text{if} \quad n = 3k+2. \end{cases}$$

If all possible sets of m vertices formed an edge, there would be $\binom{n}{m}$ edges. Hence, $t_m(n, r)/\binom{n}{m}$ is the fraction of the potential edges. If a hypergraph on $n+1$ vertices attains $t_m(n+1, r)$, then removing one vertex (and all edges containing it) leaves a hypergraph with n vertices and at most $t_m(n,r)$ edges. Average over all the ways to remove a vertex and we get

$$\frac{t_m(n+1,r)}{\binom{n+1}{m}} \leq \frac{t_m(n,r)}{\binom{n}{m}}.$$

Much work has since been done on the following problem.

Problem 1.2 *Fix $r > m$. Let H be an m-uniform hypergraph on n vertices. Suppose further that every possible subset of r vertices contains some m that do not form an edge. Let $t_m(n,r)$ denote the greatest number of edges that H could possibly have. Compute*

$$\lim_{n\to\infty} \frac{t_m(n,r)}{\binom{n}{m}}.$$

As we have seen, the limit is of a (weakly) decreasing sequence of positive numbers, so it must exist. Turán's theorem [7] established that if $m = 2$, the answer is $\frac{r-2}{r-1}$, but this is the only case where the answer is known. Conjecture 1.1 would imply an answer of $\frac{5}{9}$ to the $m = 3, r = 4$ case of Problem 1.2.

Turán established $\frac{5}{9}$ as a lower bound by giving the following construction that attains the bound of his conjecture. Divide the n vertices into three parts as evenly as possible, and arrange the parts cyclically so that each has one to its "left" and one to its "right". The edges of the hypergraph are those for which one vertex is from each part, or two are from one part and one from the part to its left.

For $n \geq 7$, this is not the only construction that attains the conjecture, however. Brown [1] showed that there are at least $k-1$ non-isomorphic constructions that attain the bound if $n = 3k$. Kostochka [5] generalized Brown's constructions to give 2^{k-2} non-isomorphic constructions if $n = 3k$. These constructions are easiest to describe in terms of which edges are not in the hypergraph, and Conjecture 1.1 can be reformulated as a lower bound on the number of missing edges, given by $\binom{n}{3}$ minus the formulas in the conjecture.

Kostochka further observed that by removing one or two vertices from his hypergraphs, one could obtain many constructions that attain the bound of Conjecture 1.1 if n is not a multiple of 3. Removing some vertices can give on the many hypergraphs, but many of them are isomorphic to each other or do not attain the bound. This paper improves on that result by showing that there are on the order of 6^k non-isomorphic hypergraphs that attain the bound of Conjecture 1.1 if $n = 3k+1$ or $n = 3k+2$.

Theorem 1.3 *Let $k \geq 2$. If $n = 3k+1$, then there are at least $\frac{1}{2}(6)^{k-1}$ hypergraphs that attain the bound of Conjecture 1.1, no two of which are isomorphic. If $n = 3k+2$, then there are at least 6^{k-1} hypergraphs that attain the bound of Conjecture 1.1, no two of which are isomorphic.*

Some upper bounds for the $m = 3, r = 4$ case of Problem 1.2 are also known. In particular, if we can compute $t_3(n,4)$ for any particular value of n, then we get $t_3(n,4)/\binom{n}{3}$ as an upper bound on the limit. Some better upper bounds were given by de Caen [3] of

$\approx .6213$, Giraud (unpublished, see [4]) of $(\sqrt{21}-1)/6 \approx .5971$, and Chung and Lu [2] of $(3+\sqrt{17})/12 \approx .5936$. Conjecture 1.1 has been verified for the cases $n \leq 13$ by Spencer [6].

The layout of this paper is as follows. In Section 2, we give our construction. We show that all of the hypergraphs we construct attain the bound of Conjecture 1.1. Finally, we count how many hypergraphs we have. Section 3 shows that no two of the hypergraphs of Construction 2.1 are isomorphic to each other. In Section 4 we discuss whether there are hypergraphs other than those of Construction 2.1 that attain the bound of Conjecture 1.1.

2 The construction

In this section, we give a way to construct many 3-uniform hypergraphs. We then show that these hypergraphs all attain the bound of Conjecture 1.1. To avoid trivial exceptions, assume that there are $n \geq 5$ vertices.

Construction 2.1 Divide n vertices into 3 columns and $\lceil \frac{n}{3} \rceil$ rows, such that each choice of a column and row has at most one vertex. All empty spots must be in the top row. Arrange the columns cyclically so that each has one to its "right" and one to its "left"; if you start at one column and go to its right three times, you end up back at the original column.

Color all vertices either red or blue, except that vertices in the bottom row should be left uncolored. If there are $n = 3k$ vertices, then each row must have all three of its vertices the same color. Color the top row red.

If there are $n = 3k + 1$ vertices, then color the top vertex in each column red. Additionally, for all choices of $j \leq k$, if we restrict to the top j rows, the number of red vertices in a column must not be more than in the column to its left, except that the column with the top vertex may have one more red vertex than the column to its left.

If there are $n = 3k + 2$ vertices, color all vertices in the top row red. Furthermore, for all choices of $j \leq k$, if we restrict to the top j rows, the number of red vertices in a column must not be fewer than in the column to its left, except that the column without a vertex in the top row may have one red vertex fewer than the column to its left.

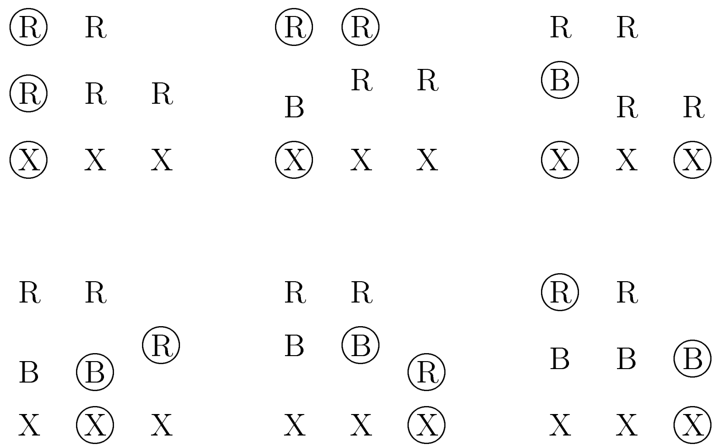
If a row contains both red and blue vertices, make the vertices of one color higher than the vertices of the other color in that row.

Construct a 3-uniform hypergraph with the n vertices as its vertex set. The edges in the hypergraph are all possible sets of three vertices except for

1. three vertices in the same column, with the top two the same color;
2. two vertices in one column, with the higher of the two red, and one vertex in the column to its right;
3. two vertices in one column, with the higher of the two blue, and one vertex in the column to its left; and

4. two vertices in one column and one vertex in a different column, with the highest vertex in the left column blue, the highest in the right column red, and the two lowest vertices in the same column.

If $n = 8$, the following diagram shows all six ways to arrange and color the vertices. An R is a red vertex, a B is a blue vertex, and an X is an uncolored vertex. The circled vertices are arbitrarily chosen sets of three vertices in a hypergraph that do not form an edge.



We can check cases to show that, for any four possible vertices, some three of them do not form an edge.

The intuitive idea of the coloring conditions is that the red vertices and the blue vertices must each be distributed among the columns as evenly as possible throughout the hypergraph.

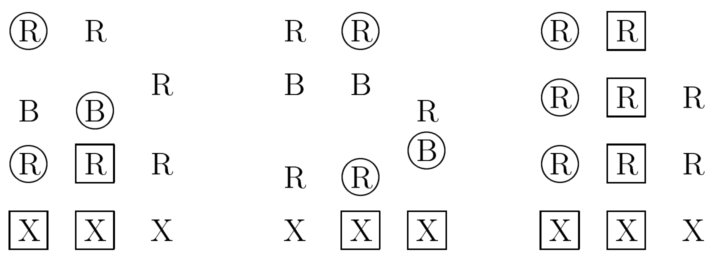
An equivalent explanation of the coloring condition if $n = 3k + 1$ is that you can hit all of the red vertices by starting at the top vertex and jumping to the next each time by moving one column to the right and possibly down some number of rows, but not up. If $n = 3k + 2$, you move left one column each time instead of right, and start at the right vertex of the two in the top row. There is, of course, an equivalent formulation of this in terms of blue vertices.

If we color all vertices red, Construction 2.1 reduces to Turán's in [8]. In the case of $n = 3k$, if we fix j and color the top j rows red, and the rest of the colored vertices blue, this gives us Brown's construction in [1]. The general case of $n = 3k$ is equivalent to Kostochka's construction in [5].

It follows from the structure of the proof of Theorem 2.4 that if the conditions on each color of vertices being distributed evenly among the columns were not met, then a hypergraph would not attain the bound of the conjecture. In contrast, the reason we do not color vertices in the bottom row and require some vertices at the top to be red is to avoid giving several hypergraphs that are isomorphic to each other. In the remainder of this section, we wish to show that all of the hypergraphs here attain the bound of Conjecture 1.1.

Definition 2.2 A *color set* of vertices in Construction 2.1 consists of all red vertices from one column and all blue vertices from the column to its right. The *size* of a color set is its number of vertices. We say that a vertex is *below a color set* if the vertex is in one of the columns used to define the color set and the vertex is lower than all vertices of the color set in the same column, even if it is higher than a vertex of the color set in the other column.

The following diagram shows the vertex sets of a few hypergraphs from Construction 2.1. In each, one color set is circled, and the vertices below that color set have boxes around them.



Note that all of the colored vertices in a hypergraph are partitioned into three color sets. The next lemma states that if we remove some rows from the bottom of Construction 2.1, the remaining vertices are divided into color sets as evenly as possible.

Throughout this paper, when we discuss removing vertices from a hypergraph, we mean taking a section hypergraph. That is, remove some vertices and all edges that contained at least one of the removed vertices, while leaving the rest of the edges intact. Similarly, when we talk of removing rows, we mean removing all vertices in the removed rows.

Lemma 2.3 *Let $k = \left\lfloor \frac{n}{3} \right\rfloor$. If the bottom j rows of a hypergraph from Construction 2.1 are removed for some $1 \leq j \leq k$, then the number of vertices remaining in each color set is either $k - j$ or $k - j + 1$.*

Proof: If $n = 3k$, then a color set has exactly one vertex in each row. As such, if we remove the bottom j rows, each color set has $k - j$ vertices.

Otherwise, let the number of red vertices be $3p + q$ with $0 \leq q \leq 2$. Let column A be the one with the top vertex if $n = 3k + 1$ and the left column with a vertex in the top row if $n = 3k + 2$, column B be the column to the right of A, and column C be the column to the right of B. One can count the number of red and blue vertices remaining in each column to get the following table.

vertices	q	A red	B blue	B red	C blue	C red	A blue
$3k+1$	0	p	$k-j-p$	p	$k-j-p$	p	$k-j-p+1$
$3k+1$	1	$p+1$	$k-j-p$	p	$k-j-p$	p	$k-j-p$
$3k+1$	2	$p+1$	$k-j-p-1$	$p+1$	$k-j-p$	p	$k-j-p$
$3k+2$	0	p	$k-j-p+1$	p	$k-j-p$	p	$k-j-p+1$
$3k+2$	1	p	$k-j-p$	$p+1$	$k-j-p$	p	$k-j-p+1$
$3k+2$	2	$p+1$	$k-j-p$	$p+1$	$k-j-p$	p	$k-j-p$

Note that in all cases, the size of each color set is either $k-j$ or $k-j+1$. □

Theorem 2.4 *All of the hypergraphs of Construction 2.1 attain the bound of Conjecture 1.1.*

Proof: Let $k = \lfloor \frac{n}{3} \rfloor$. Fix j with $1 \leq j \leq k$ and remove the bottom $j-1$ rows. Decolor the new bottom row. Suppose that one column now has a red vertices and b blue vertices, the column to its right has c red vertices and d blue vertices, and the column to its right has e red vertices and f blue vertices.

We wish to count the number of missing edges with at least one vertex in the new bottom row. For missing edges of the first type, we get $\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+\binom{d}{2}+\binom{e}{2}+\binom{f}{2}$. The number of missing edges of the second type for which the column with two vertices has an vertex in the new bottom row is $a(c+d+1)+c(e+f+1)+e(a+b+1)$. The number of analogous missing edges of the third type is $b(e+f+1)+d(a+b+1)+f(c+d+1)$. The number of missing edges of the second or third type for which the only vertex in the new bottom row is in the column with only one vertex is $\binom{a+b}{2}+\binom{c+d}{2}+\binom{e+f}{2}$. The number of missing edges of the fourth type is $bc+de+af$.

If we add all of these up and rearrange, we get

$$\frac{1}{2}\big((a+d)^2+(b+e)^2+(c+f)^2+(a+b+c+d+e+f)^2\big).$$

Note that the last term does not depend on the coloring, so we only need to minimize the rest of the sum. The way to minimize the sum of the squares of three integers with a fixed sum $(n-3j-3)$ is to make the three integers as close to each other as possible.

Note that removing the bottom j rows and then decoloring the new bottom row has the same effect on the size of color sets as simply removing the bottom $j+1$ rows. As such, by Lemma 2.3, each color set of the new construction has either $k-j$ or $k-j-1$ vertices; that is, the sizes of the color sets are as close as possible. Since $a+d$, $b+e$, and $c+f$ are the numbers of vertices in the three color sets of the new construction, all such constructions attainable from Construction 2.1 have exactly the same number of missing edges. Therefore, all hypergraphs of Construction 2.1 have exactly the same number of missing edges whose lowest vertex is in the j-th row from the bottom. Sum over j to get that all of these hypergraphs have the same number of edges. Since the one given by

Turán is among them and it attains the bound of Conjecture 1.1, all of our hypergraphs attain this bound. □

If $n = 3k$, Construction 2.1 gives precisely Kostochka's construction from [5]. Otherwise, we wish to count how many non-isomorphic hypergraphs this construction gives. There is only one way to color the top (partially filled) row, and only one way to (not) color the bottom row.

For most intermediate rows, we can pick zero, one, two, or three red vertices. If we pick one or two, the row will have both colors, so we must specify which color of vertices is higher, and there are two ways to pick which color is higher. As such, there are six ways to fill in a row.

The exception to this is the second row in the $n = 3k + 1$ case. Two vertices are the highest in their respective columns, and hence red. The remaining vertex can be red or blue, and if blue, higher or lower than the red vertices in its row. This creates three ways to fill in the row. As such, we have $\frac{1}{2}(6)^{k-1}$ hypergraphs if $n = 3k + 1$ and 6^{k-1} hypergraphs if $n = 3k + 2$.

3 Checking for isomorphisms

In the previous section, we constructed many hypergraphs that attain the bound of Conjecture 1.1. A priori, these could include many ways of presenting the same hypergraph, so that there would be far fewer distinct hypergraphs than given. In this section, we show that no two hypergraphs of Construction 2.1 are isomorphic.

The main idea of the proof is that we start by defining some combinatorial invariants that are clearly preserved by isomorphisms. If some invariant differs for two hypergraphs, then they cannot be isomorphic to each other. We then show how to easily compute the invariants in most cases from the structure of Construction 2.1. This leads to a way to essentially pick out the bottom vertex of each column as given in the construction. We can then show that no two hypergraphs are isomorphic by removing the bottom row of vertices and proceeding by induction.

Our proof is similar in flavor to Kostochka's proof that no two of his hypergraphs are isomorphic in [5]. Our proof is somewhat more complicated, as Kostochka was able to exploit all three columns being identical, which is untrue if 3 does not divide n. Throughout this section, we assume that the hypergraphs have $n = 3k + 1$ or $n = 3k + 2$ vertices, though the lemmas would generally hold (sometimes with slight modifications) for $n = 3k$ vertices. First, we need the combinatorial invariants.

Definition 3.1 An *empty cluster* is a set of more than one third of the vertices of a hypergraph such that no three of the vertices form an edge and any proper superset of it contains the vertex set of an edge. We can specify that it contains j vertices by calling it an *empty cluster of size j*.

An *empty core* is an intersection of one or more empty clusters of the same size such that the intersection contains at least two vertices and any other empty cluster of the same size or larger contains at most one vertex of the empty core.

An *empty union* is a union of all empty clusters that contain all the vertices of a particular empty core and are of the same size as the empty clusters used to define the empty core.

A *column leg* is an intersection of two empty unions. For each vertex of a column leg, one can count the number of edges containing that vertex and no other vertex of the column leg. The vertices of a column leg contained in the most such edges form a *column foot*.

The terminology of the last paragraph may seem peculiar. It is chosen to be descriptive in that a leg contains a foot, and both are a set of vertices at the bottom of a column, as shown in Lemmas 3.17 and 3.19.

These were defined as they were because it is clear from the definitions that they are preserved by isomorphisms. That is, an isomorphism between two hypergraphs must send empty cores to empty cores, and so forth. The preceding invariants give us ways to show that two hypergraphs are not isomorphic, for example, if the sizes of their column legs do not match.

Definition 3.2 An *indistinguishable pair* of vertices is a pair of distinct vertices a and b such that for all vertices c and d distinct from a and b, the vertices acd form an edge exactly if bcd do. Equivalently, the vertex map swapping a and b induces an isomorphism of the hypergraph. An *indistinguishable set* of vertices is a set of vertices that are pairwise indistinguishable.

We break up the proof that the hypergraphs are not isomorphic into a series of lemmas. The next two lemmas show that most empty clusters are closely related to color sets.

Lemma 3.3 *Any set of vertices contained in two columns such that all but the lowest in the left column are red and all but the lowest in the right column are blue has no edges among any three of the vertices.*

Proof: If three vertices are in the same column, they form a missing edge of type 1. If two vertices are in the left column and one is in the right, the higher of the two in the left is red, so they form a missing edge of type 2. If two vertices are in the right column and one is in the left, the higher of the two in the right column is blue, so they form a missing edge of type 3. □

Lemma 3.4 *Let S be an empty cluster of vertices. All vertices of S in a column except the lowest are the same color. If a column has more than one vertex of S, we can refer to the color of all vertices except the lowest as the color of the column of S. S has vertices in exactly two columns.*

If the left column is blue or the right column red, the other column has exactly one vertex, it is the opposite color, it is the highest vertex of S, and S has exactly $k+1$ vertices. We call such a set a backwards empty cluster.

Proof: If there are two vertices other than the lowest in the same column of S that are different colors, then these two vertices and the lowest in the column form an edge.

If S has vertices in all three columns, then S contains an edge consisting of one vertex from each column. If all vertices of S are in one column, we can add a vertex to S in the column to the right if S is red or to the left if S is blue. Hence, S has vertices in exactly two columns.

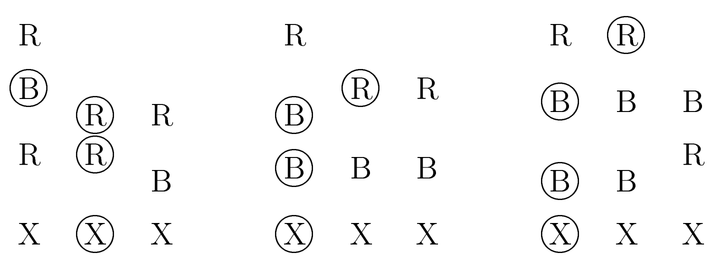
Suppose that the left column is blue. For its top two vertices not to form an edge with a vertex from the right column, the vertex from the right column must be red and higher than any vertex in the left column. The top row contains only red vertices, so the left column contributes at most k vertices to S, and so S has at most $k+1$ vertices.

Similarly, suppose that the right column is red. Its top two vertices and a vertex from the left column form an edge unless the vertex from the left column is blue and higher than any vertex in the right column. The blue vertex cannot be in the top row, so the red column cannot have a vertex of S in the top row.

Thus, S has at most $k+1$ vertices in both cases. S must have at least $k+1$ vertices to be an empty cluster, so S has exactly $k+1$ vertices.

Suppose that both columns have more than one vertex. If the left column is blue then the right column is red and vice versa. In either case, all vertices of each column are required to be higher than all vertices of the other column, which is impossible. $\square$

The preceding lemma gives an easy way to pick out any backwards empty clusters from the structure of Construction 2.1. This next diagram gives a few examples. In each case, the circled vertices form a backwards empty cluster.



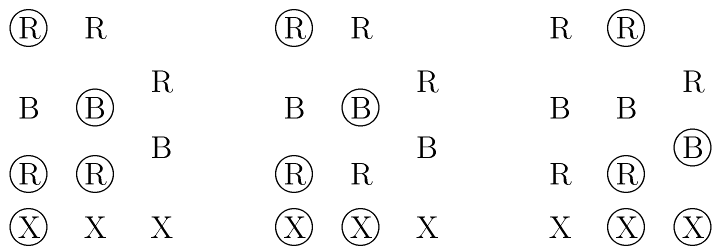
Now we can show how to easily compute several of the invariants from the structure of Construction 2.1.

Lemma 3.5 *There are no empty clusters of size $k+3$ or larger. All empty clusters of size $k+2$ consist of a color set of size k and one additional vertex in each column (of the color set) that is below the color set. Conversely, any set of vertices that fits this description is an empty cluster of size $k+2$.*

Proof: By Lemma 3.4, for an empty cluster to have more than $k+1$ vertices, either one column can have only one vertex or else its left column must be red and its right column blue. In the former case, the other column must have $k+1$ vertices, of which the top k are all the same color. In both this and the latter case, all but two vertices of the empty cluster must be in a single color set. By Lemma 2.3, a color set has size either $k-1$ or k. As such, the largest empty clusters possible are of size $k+2$ and are obtained by adding one additional vertex in each column to a color set of size k, as in the statement of the lemma.

For the converse, by Lemma 3.3, there are no edges among the $k+2$ vertices. As shown in the previous paragraph, the $k+2$ vertices cannot be a proper subset of a larger empty cluster. $\square$

We now have an easy characterization of all empty clusters of size $k+2$. The next diagram shows the three such empty clusters of a particular hypergraph. Note that the two on the left correspond to the same color set, while the one on the right comes from a different color set. The third color set has only $k-1$ vertices, and does not lead to an empty cluster of size $k+2$.



As an empty cluster must have at least $k+1$ vertices by definition, all empty clusters are of size $k+1$ or $k+2$. By Lemma 2.3, each color set has size $k-1$ or k. Hence, we can give them convenient labels.

Definition 3.6 A *small empty cluster* is an empty cluster of size $k+1$. A *large empty cluster* is an empty cluster of size $k+2$. A *small empty core* is an empty core defined by an intersection of small empty clusters. A *large empty core* is an empty core defined by an intersection of large empty clusters. A *small color set* is a color set of size $k-1$. A *large color set* is a color set of size k.

The case $k=2$ is an exception to some of the later lemmas in this section. Therefore, we deal with it separately. Now that we have an easy way to find the large empty clusters, carrying out the computations of the next lemma is straightforward to do by hand, so some details are omitted.

Lemma 3.7 *If $k \leq 2$, then no two constructions are isomorphic.*

Proof: If $k \leq 1$, then there is at most one construction with any particular number of vertices. If $k=2$ and $n=3k+1=7$, there are three possible hypergraphs. One can count the number of vertices contained in exactly 11 edges in each of these constructions and get totals of zero, one, and two.

If $n=3k+2=8$, there are six possible hypergraphs. Among these six hypergraphs, one has five large empty clusters (second row all red), one has four large empty clusters (second row all blue), and the rest have three large empty clusters. One can compute that the last four hypergraphs each have exactly two vertices contained in exactly one large empty cluster; in each case, these are the two lowest vertices of the column with the unique vertex of its color in the second row. If we count the number of edges containing these two vertices for the various hypergraphs, we get totals of 15 and 14 (one red vertex in second row, higher than the blue vertices), 14 and 14 (one red vertex in second row, lower than the blue vertices), 14 and 13 (one blue vertex in second row, higher than the red vertices), and 13 and 13 (one blue vertex in second row, lower than the red vertices). □

Lemma 3.7 leaves only the cases where $k \geq 3$. As such, for the rest of this section, we assume that $k \geq 3$.

Lemma 3.8 *The small empty clusters are precisely the following constructions.*

1. *Pick a small color set and add one additional vertex in each column (of the color set) that is below the color set.*

2. *Pick a large color set, discard its lowest vertex in a column, add a vertex in that column that is above the discarded vertex but below the next lowest vertex of the color set in that column, and add a vertex in the other column that is below the color set.*

3. *Pick a backwards empty cluster.*

Proof: The third type is trivially an empty cluster. The others have no edges by Lemma 3.3. They cannot have vertices added to get a larger empty cluster by Lemma 3.5, so they are small empty clusters.

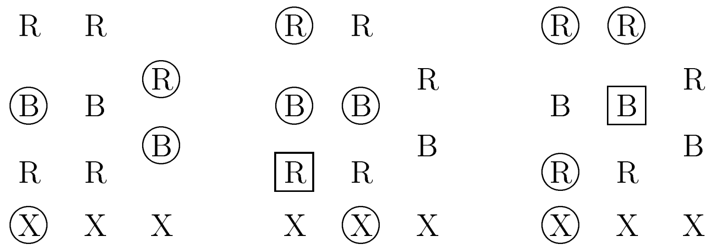
It thus suffices to show that nothing else can be a small empty cluster. By Lemma 3.4, the empty cluster involves exactly two columns. If the left column is blue or the right column is red, then it is a backwards empty cluster. Otherwise, each column can have at most one vertex of the empty cluster outside of the color set associated with the two columns of the empty cluster.

If the color set is small, then all of its vertices must be in the empty cluster in order to get $k + 1$ vertices total. Adding another vertex at the bottom of each column gives precisely the first construction.

Otherwise, the two columns have a large color set. If we include all k vertices of the color set in an empty cluster, then by Lemma 3.4, any additional vertices must be below the color set. Adding one vertex does not prevent adding one in the other column, so this is not a small empty cluster.

To get $k + 1$ vertices for an empty cluster, at least $k - 1$ of them must come from the color set. As such, we can exclude only one of the k vertices. We then add two additional vertices below the lowest in their columns that remain to complete the prospective empty cluster. The new vertex in the column with the excluded vertex must be above the excluded vertex to prevent adding back the removed vertex, which gives precisely the second construction. □

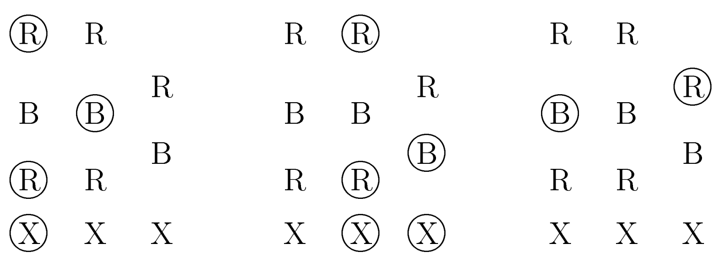
The next diagram gives a few examples of small empty clusters, using the same hypergraph as the previous diagram. The one on the left is the first type of small empty cluster, and the two on the right are the second type. The two on the right have a box around the excluded vertex of the large color set used.



The next definition is used to avoid some awkward wording in the lemmas that follow it.

Definition 3.9 An *expanded color set* is a color set together with the bottom vertex of any column for which the second lowest vertex is in the color set.

Note that all of the vertices of a hypergraph of Construction 2.1 are partitioned into three expanded color sets. The next diagram shows the three expanded color sets for the hypergraph used in two previous diagrams.



Lemma 3.10 *An empty cluster has exactly one expanded color set with which it shares more than one vertex.*

Proof: By Lemma 3.4, each column of a normal empty cluster has at most one vertex outside of the expanded color set defined from the two columns that the empty cluster uses. Each other expanded color set intersects at most one column of the empty cluster, so it contains at most one vertex of the empty cluster. Likewise, all but one vertex of a backwards empty cluster is in the same expanded color set.

Conversely, since $k \geq 3$, an empty cluster has at least four vertices. By the pigeonhole principle, it has at least two vertices in common with at least one expanded color set. $\square$

Lemma 3.11 *An empty core cannot be a proper subset of another empty core.*

Proof: Suppose that there is an empty core that is a proper subset of another empty core. If the empty cores are both large or both small, then the one that is a superset of the other is not really an empty core because it can be intersected with the empty clusters of the other to leave fewer vertices, but still at least two. If one empty core is large and the other is small, then the small empty core is not an empty core because intersecting it with a large empty cluster that contains the large empty core leaves at least the intersection of the two empty cores, which is an empty core, and thus contains at least two vertices. □

Lemma 3.12 *An expanded color set is an empty core.*

Proof: If the color set is large, then by Lemma 3.5, all large empty clusters that use the color set use all vertices of it, as well as an additional vertex below the color set in each column. The intersection of all large empty clusters that use this color set gives the expanded color set, as all are forced to use a vertex below the color set exactly if it is the only such vertex in its column. Any other large empty cluster uses at most one vertex of the expanded color set, so it is an empty core.

If the color set is small, then by Lemmas 3.8 and 3.5, all normal empty clusters in the pair of columns are small and use all of the vertices of the color set. The intersection of all of these small empty clusters is precisely the expanded color set, as again, all are forced to use a vertex below the color set exactly if it is the only such vertex in its column. It suffices to show that every other empty cluster either is small and contains the entire expanded color set or else contains at most one vertex of it. Lemma 3.10 shows the latter condition if we can show that the empty cluster contains at least two vertices of some other expanded color set.

Suppose first that the other empty cluster is a normal empty cluster using a different pair of columns. By Lemmas 3.8 and 3.5, every normal empty cluster uses at least $k-1$ vertices of the color set with its same pair of columns. Since $k \geq 3$, this is at least two vertices.

Otherwise, the other empty cluster is backwards. By Lemma 3.4, it has $k-1$ of its vertices in the a single color set. If it is a different color set from that of our prospective empty core, then since $k \geq 3$, it has at least two vertices in a different expanded color set. If it is the same color set, then the backwards empty core contains all $k-1$ vertices of the color set, as well as the bottom vertex of their column, so it contains the entire expanded color set. □

Lemma 3.13 *The empty cores are precisely the expanded color sets.*

Proof: Lemma 3.12 shows that the expanded color sets are empty cores. It suffices to show that nothing else can be an empty core.

Case 1: The empty clusters used to define an additional empty core do not all use the same pair of columns.

They must have exactly one column in common.

Case 1 a: At least one of the empty clusters is backwards.

The intersection is a subset of the vertices of one column of a backwards empty cluster. An empty core needs more than one vertex, and a subset of the column with multiple vertices either gives us an empty core we already had or else violates Lemma 3.11.

Case 1 b: None of the empty clusters are backwards.

By Lemma 3.4, for one of the empty clusters, all but the lowest vertex in the column is red, and for another, all but the lowest vertex in the column is blue. Any vertex in the intersection is either red, blue, or uncolored. In each case, its color forces it to be the lowest vertex in its column for at least one of the empty clusters. Every vertex of the intersection is the lowest of the intersection in the column, and all are in one column, so the intersection has at most one vertex, and thus cannot contain an empty core.

Case 2: All empty clusters used to define the empty core use the same pair of columns.

Case 2 a: The empty core consists of a single empty cluster.

By Lemma 3.10, the empty cluster has multiple vertices in common with exactly one expanded color set. If the empty cluster contains the entire expanded color set, then either it is an empty core we already had, or else it is a proper superset of one we already had, in violation of Lemma 3.11.

If the empty cluster does not contain the entire expanded color set, then intersecting it with the expanded color set would leave fewer vertices, so the expanded color set could not have been an empty core unless the empty cluster is small and the color set is large. In this case, the small empty cluster has at least two vertices in common with a large empty cluster used to define the large empty core, so it is not an empty core.

Case 2 b: The empty core is defined by an intersection of multiple normal empty clusters.

By Lemmas 3.5 and 3.8, all normal empty clusters contain an entire expanded color set, except for small empty clusters whose pair of columns corresponds to a large color set. An intersection of normal empty clusters all containing the same expanded color set will contain that entire expanded color set. Either we get an empty core we already had, or else a proper superset of one, which would contradict Lemma 3.11.

Suppose that a small empty cluster of type 2 (as defined in Lemma 3.8) contains a vertex outside of its expanded color set. If it is below the color set, then all vertices of the color set in its column must be in the empty cluster, so the excluded vertex of the color set is the lowest one of the other column. If it is not below the color set, then the excluded vertex is the lowest one of the color set in the same column as the chosen vertex outside of the expanded color set. Either way, the vertex outside of the expanded color set uniquely determines the vertex of the color set excluded from the empty cluster.

If a vertex outside of the expanded color set is in the prospective empty core, then it is in each of the normal empty clusters used to define the empty core. All of these must exclude exactly the same vertex of the color set, so all but one vertex of the color set is in the empty core. Since $k \geq 3$ and the color set is large (and hence has k vertices), the empty core contains at least two vertices of the color set. Intersecting the empty core with a large empty cluster that uses the color set thus leaves at least two vertices, so the intersection of small empty clusters cannot produce a small empty core.

Case 2 c: The empty core is defined by an intersection of multiple backwards empty clusters.

A backwards empty cluster has in one of its columns a vertex in every row except for the top one. If the backwards empty clusters have the same column with all but one of their vertices, then to be distinct, the vertex in the other column must vary. The intersection of the backwards empty clusters then consists of the k vertices in a single column, for which all but the bottom are the same color. These are all contained in the same expanded color set, so by Lemma 3.11, it is not a new empty core.

Otherwise, the backwards empty clusters put their respective columns of k vertices in different columns. Any vertex in the intersection must be the unique vertex of a backwards empty cluster in one of its columns, and hence the highest of the empty cluster. If every vertex in a set is the highest in the set, then the set has at most one vertex, so it is not an empty core.

Case 2 d: The empty core is defined by an intersection of at least one normal empty cluster and at least one backwards empty cluster.

By Lemma 3.4, the backwards empty cluster has no red vertices in the left column and no blue vertices in the right column. By the same lemma, if a vertex of a normal empty cluster is in the left column and not red or is in the right column and not blue, then it is the lowest vertex of the empty cluster in its column. As such, every vertex in the intersection must be the lowest in its column for the normal empty cluster.

If the intersection does not have a vertex from both columns, then it has at most one vertex, and is not an empty core. If it does a vertex in each column, any other vertices of the normal empty core must be above all vertices of the backwards empty cluster in the same column. Since the backwards empty cluster has a vertex in each column no lower than the second row, any additional vertices of the normal empty cluster must be in the top row and in the color set. Since the top row is always red, this allows at most one additional vertex, so the normal empty cluster has at most three vertices. Since $k \geq 3$, an empty cluster must have at least 4 vertices, a contradiction. $\square$

Some subsequent lemmas hold for nearly all of our constructions, with only a couple of exceptions that must be handled separately.

Definition 3.14 *The exceptional constructions are the cases where*

1. $n = 3k + 1$ *and the unique blue vertex is in the second row of the column with the highest vertex and is higher than the red vertices in its row, or*

2. $n = 3k + 2$ *and all vertices outside of the top and bottom rows are blue.*

The next diagram shows the exceptional constructions in the case where $k = 3$.

R				R	R	
B				B	B	B
	R	R				
R	R	R		B	B	B
X	X	X		X	X	X

This next lemma explains why the exceptional constructions don't satisfy the conditions of some other lemmas.

Lemma 3.15 *The exceptional constructions are precisely the hypergraphs that use a backwards empty cluster in the definition of at least one empty core.*

Proof: First we show that the exceptional constructions do use a backwards empty cluster in the definition of an empty core. If $n = 3k+1$, the column to the right of the one with the top vertex is an expanded color set, and thus a small empty core. Add the unique blue vertex to get a backwards empty cluster. If $n = 3k+2$, the non-red vertices in the column to the right of one without a vertex in the top row form an expanded color set, and thus a small empty core. Add the red vertex in the column to the right to get a backwards empty cluster.

Conversely, we must show that no other hypergraphs can use a backwards empty cluster in the definition of an empty core. In order to be used in the definition of an empty core, an empty cluster must contain an entire expanded color set. The top vertex of a backwards empty cluster is in a different expanded color set from the rest of the vertices, so the expanded color set would have to consist of the other k vertices.

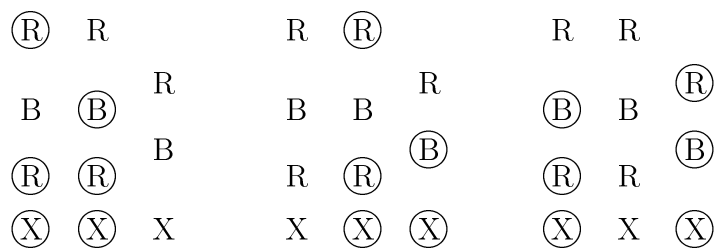
Suppose that all but the lowest of these k vertices are red. If there is a blue vertex in the column to the right, then it would also be in the expanded color set, a contradiction. As there are no blue vertices in the top row, there can be no blue vertex in the column with the k vertices. A backwards empty cluster requires that there be one blue vertex in the column to the left of the k vertices, and either in the top row or higher than the red vertices in the second row. The structure of Construction 2.1 forces the blue vertex to be in a particular column, and prevents it from being in the top row. If there are $n = 3k+1$ vertices, this gives an exceptional construction. If there are $n = 3k+2$ vertices, the top row of the column with the k vertices has another red vertex, so the backwards empty cluster does not contain the entire expanded color set.

Otherwise, all but the lowest of the k vertices are blue. If there is a red vertex in the column to the left, then it is also in the expanded color set, so the k vertices do not contain the entire expanded color set. If there are at least three red vertices, then every column has a red vertex. If $n = 3k+1$, then there are at least three red vertices. If $n = 3k+2$, then there are at least two red vertices. The only way to avoid having a third red vertex is if all vertices outside of the top row are blue, which gives an exceptional construction. $\square$

Lemma 3.16 *If not an exceptional construction, the empty unions are precisely the sets consisting of a color set and all vertices below the color set.*

Proof: Lemma 3.15 excludes the third type of small empty clusters, as defined in Lemma 3.8, from affecting the empty cores. Lemma 3.13 excludes the second type. By Lemma 3.5, each way to pick a large color set and a vertex in each of its columns below the color set gives a large empty cluster. Similarly, by Lemma 3.8, the same construction with a small color set gives a small empty cluster. These contain the entire expanded color set, so the empty clusters that contain an empty core consist precisely of the corresponding color set plus one additional vertex in each column that is below the color set. We can choose any vertex below the last red one in the left column or the last blue one in the right column to be in an empty cluster containing the empty core, so the union of all such empty clusters gives us the empty union as described in the lemma. □

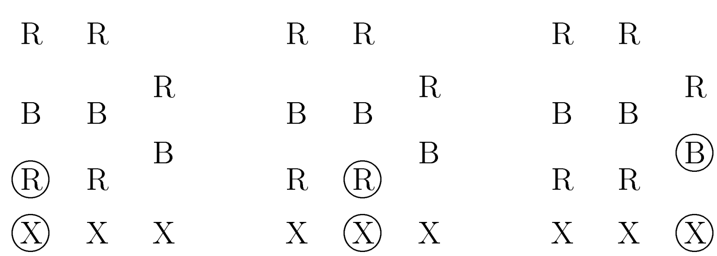
The next diagram shows the empty unions of our running example.



Lemma 3.17 *If not an exceptional construction, a set of $j \geq 2$ vertices forms a column leg exactly if they are all in the same column, the bottom j vertices in the column, all the same color (except the uncolored bottom vertex), and the $(j+1)$-th vertex from the bottom either does not exist or is the opposite color.*

Proof: By Lemma 3.16, there is exactly one empty union for each pair of columns. An intersection of two empty unions thus has exactly one column in common. The intersection consists of all vertices in the column which are either in or below both color sets. This excludes exactly the vertices that are above a vertex in the column of the opposite color, leaving the statement of the lemma. □

The next diagram shows the three column legs for our example.



Lemma 3.18 *No exceptional construction is isomorphic to any other construction.*

Proof: Two exceptional constructions cannot be isomorphic to each other, as no two have the same number of vertices.

There are only two cases to check. We know the empty cores from Lemma 3.13 and the empty clusters from Lemmas 3.8 and 3.5. We can readily compute the empty unions from this information, and then the column legs.

The following diagram shows both of the empty unions that differ from the description of Lemma 3.16, as found in the proof of Lemma 3.15. The vertices in the empty core have a box, while the rest of the vertices in the empty union are circled. The reason for the discrepancy is a backwards empty cluster consisting of the empty core plus the circled vertex that is the only one in its column.

$$\begin{array}{ccc} \mathrm{R} & & \\ \mathrm{B} & & \\ & \boxed{\mathrm{R}} & \mathrm{R} \\ \mathrm{R} & \boxed{\mathrm{R}} & \mathrm{R} \\ \mathrm{X} & \boxed{\mathrm{X}} & \mathrm{X} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{R} & \mathrm{R} & \\ \boxed{\mathrm{B}} & \mathrm{B} & \mathrm{B} \\ \boxed{\mathrm{B}} & \mathrm{B} & \mathrm{B} \\ \boxed{\mathrm{X}} & \mathrm{X} & \mathrm{X} \end{array}$$

If $n = 3k+1$, the exceptional construction has column legs of sizes $k+1$, k, and $k-1$. The only way for any other construction on $3k+1$ vertices to have a column leg of size $k+1$ is for the entire column with the top vertex to be red, in which case, all vertices are red. This construction has column legs of sizes $k+1$, k, and k, so it is not isomorphic to the exceptional one.

If $n = 3k+2$, the exceptional construction has column legs of sizes $k+1$, k, and k. In order for a normal construction to have a column leg of $k+1$ vertices, an entire column with one of the top two vertices must be red. The other column with one of the top two vertices must then have a column leg of size k, so all but the top vertex must be blue. As $k \geq 3$, this gives one column with at least two blue vertices and another with zero, which is impossible. $\square$

Lemma 3.19 *If not an exceptional construction, a column foot consists of all vertices of a column leg that are indistinguishable from the lowest vertex of the column.*

Proof: If the vertices of a column leg comprise an indistinguishable set, then the number of edges containing a particular vertex of the column leg and no other vertices of it is the same for every vertex. The associated column foot thus consists of the entire column leg.

Otherwise, the column leg has two vertices that are distinguishable. By Lemma 3.17, all vertices of the column leg except the lowest are the same color. Suppose without loss of generality that they are blue. Let u and v be two vertices of the column leg that are distinguishable, and assume without loss of generality that u is higher than v. There must be two vertices a and b such that either abu or abv is an edge, but not both. Assume without loss of generality that b is not higher than a.

Let C be the column containing u and v, D be the column to the right of C, and E be the column to the left of C. The following table lists the various possibilities for the columns of a and b. The entries of each row are the column of a, the column of b, and the conditions for abu to form an edge.

a	b	conditions
C	C	a and b different colors
C	D	a blue
C	E	a red
D	C	(a red and a below u) or (a, b both blue)
D	D	a red and a above u
D	E	always
E	C	a and b both red
E	D	always
E	E	a blue

If abu forming an edge does not depend on u, then either abu and abv both form an edge or else neither do. We can thus restrict our attention to the DC and DD lines. The only way for abu to form an edge and abv not is if a is red, in column D, below u, and above v, and b is below a and in column C, as given on line DC. Note that this makes b part of the column leg. The only way for abv to form an edge and abu not is if a is red, in column D, below u, and above v, and b is below a and in column D, as given on line DD.

In both cases, in order for u and v to be distinguishable, there must be a red vertex below u, above v, and in the column D. Conversely, if there is such a red vertex, then we can take a to be this vertex and b to be the bottom vertex of column D, and get that abv is an edge while abu is not, so u and v are distinguishable. The bottom vertex of a column is below all colored vertices, so the set of vertices indistinguishable from the bottom one of a column leg is precisely the set of vertices of the column leg below all red vertices in column D.

Note that line DD gives an edge with only one vertex in the column leg while line DC does not, so the latter does not matter when computing a column foot. For the DD case, vertices of the column leg below all red vertices in column D are in more edges with exactly one vertex in the column leg than vertices that are above some red vertex of column D. By the definition of the column foot, the vertices it contains are the vertices of the column leg that are below all red vertices in column D. As we have seen, this is precisely the set of vertices indistinguishable from the lowest in the column. $\square$

This diagram shows the three column feet in our example.

R	R		R	R		R	R	
		R			R			R
B	B		B	B		B	B	
		B			B			B
R	R		R	R		R	R	
X	X	X	X	X	X	X	X	X

Now we have a way to essentially pick out the bottom row of a construction. The basic idea of the proof is that we remove the bottom row and proceed by induction.

Lemma 3.20 *If two non-exceptional constructions are isomorphic, then if the bottom row of each were removed, they would still be isomorphic.*

Proof: It is clear from the definition of column feet that they are preserved by isomorphisms. Pick a vertex from each column foot of one construction. The isomorphism sends these to a vertex from each column foot of the other construction. By Lemma 3.19, a vertex of a column foot is either the lowest vertex of the column or indistinguishable from it. If the latter, there is an automorphism of the hypergraph that interchanges the chosen vertex of the column foot with the lowest one of the column. Composing these as necessary with the isomorphism between the two hypergraphs, we get an isomorphism that preserves the bottom row. This induces an isomorphism between the two hypergraphs with their respective bottom rows removed. □

Lemma 3.21 *There is an empty core containing $k+2$ vertices if and only if the second bottom row is of mixed color. There is at most one empty core with $k+2$ vertices.*

Proof: By Lemma 3.13, each empty core is an expanded color set, which is a color set plus at most two vertices. In order to contain $k+2$ vertices, it must be a large color set and have two additional vertices. In order to have two additional vertices, the left column must have a red vertex in its second lowest row, and the right column must have a blue vertex in this row. This can only happen if the second bottom row is of mixed color. In addition, for this to happen more than once would require at least four vertices in the second lowest row.

If the second bottom row has vertices of both colors, then at some point it has a red vertex with a blue vertex to its right. The pair of columns of these two vertices has a corresponding color set. By Lemma 2.3, if we remove the bottom two rows, at least $k-2$ vertices of the color set remain. If we remove the bottom row, at most k vertices of the color set remain. Since the second bottom row adds two such vertices, both of these bounds are sharp, so the color set has k vertices. By Lemma 3.13, adding the two vertices in the bottom row gives an empty core with $k+2$ vertices. □

Lemma 3.22 *No two constructions are isomorphic.*

Proof: By Lemma 3.18, we can disregard the exceptional cases. We use induction on k. The base case of $k = 2$ is Lemma 3.7.

For the inductive step, by Lemma 3.20, if two hypergraphs are isomorphic, then removing their bottom row of each leaves two isomorphic hypergraphs. By the inductive hypothesis, this requires the new hypergraphs to be the same construction. As such, in the original constructions, only the second bottom rows can differ. It thus suffices to show that if two hypergraphs of Construction 2.1 are identical except for their second bottom rows, then they are not isomorphic.

We can use Lemma 3.21 to distinguish between the second bottom row being of solid color or mixed colors. We can distinguish between it being all red or all blue by counting the number of column legs of size at least three; one color will have at least two such column legs, and the other at most one.

If the number of blue vertices in the second and third rows from the bottom added together is even, then the number of column legs of length at least three is odd; otherwise it is even. This can distinguish between a hypergraph with one blue vertex in the second bottom row and one with two blue vertices there.

That leaves as the only possible pairs of isomorphic constructions the cases where the vertex colorings are exactly the same, the second bottom row is of mixed colors, and the two hypergraphs differ on which color of vertices in this row is higher. These two constructions differ only in a single edge, which consists of a red vertex in the second bottom row, a blue vertex immediately to its left, and the vertex in the bottom row of the column containing one of the first two vertices.

The second bottom row is of mixed color, so by Lemma 3.21, there is a unique empty core that consists of a large empty cluster. The two columns for this empty core are the same for both constructions in question, as by Lemma 3.13, empty cores depend only on the colors of the vertices. They are not the same pair of columns as contain the edge that varies between the two constructions, as the second bottom row has two vertices of this edge, of which the left is blue and the right is red. Hence, the empty core with $k + 2$ vertices has one column in common with the edge that varies between the two constructions.

The empty core with $k + 2$ vertices contains the two lowest vertices in each of its columns. The edge that varies between the two constructions has the bottom vertex in one of its two columns, and which column varies by construction. Thus, the edge has one vertex of the empty core with $k + 2$ vertices in one construction and two vertices in the other. The number of edges with exactly two vertices in common with the empty core with $k + 2$ vertices then differs by one between the two constructions, so they are not isomorphic. $\square$

Finally, we can combine the above lemmas to prove Theorem 1.3.

Proof of Theorem 1.3: We have counted that Construction 2.1 gives $\frac{1}{2}(6)^{k-1}$ hypergraphs if $n = 3k + 1$ and 6^{k-1} hypergraphs if $n = 3k + 2$. By Theorem 2.4, all of them attain the bound of Turán's conjecture. By Lemma 3.22, no two of them are isomorphic. $\square$

4 Concluding remarks

Kostochka [5] observed that many constructions for the cases $n = 3k+1$ and $n = 3k+2$ could be obtained by removing one or two vertices from a construction for $n = 3(k+1)$. Removing two vertices often gives a hypergraph that does not attain the bound of Conjecture 1.1, however, and there can be many ways to remove one or two vertices and obtain the same hypergraph.

Kostochka did not compute how many non-isomorphic hypergraphs that attain the bound that he could construct in this manner. From the results of this paper, one can readily show that he had $k2^{k-3}$ hypergraphs if $n = 3k+2$ and $(k^2+3k-2)2^{k-4}$ hypergraphs if $n = 3k+1$. These are both far shy of the roughly 6^k hypergraphs we have found.

Everything in Construction 2.1 is, however, a section hypergraph of one given by Kostochka. The easiest way to show this is by giving each vertex of the intended construction its own row in Kostochka's construction, and filling in the empty spots such that all vertices in a row are the same color. This can be done more efficiently to start with a smaller hypergraph of Kostochka's construction and remove fewer vertices, but usually requires removing more than one fourth of the vertices. Conversely, it can be shown that every section hypergraph of one of Kostochka's constructions either is isomorphic to one of Construction 2.1, or else does not attain the bound of Conjecture 1.1.

Another question is whether these are all of the hypergraphs that attain the bound of Turán's (3, 4)-conjecture. If one could list all such hypergraphs and show that none permit another edge to be added, that would prove the conjecture. Indeed, it is easy to show that adding one more edge to any hypergraph in Construction 2.1 would create a set of four vertices with four edges.

The answer is that Construction 2.1 does not give all of the possible hypergraphs that attain the bound, but it might give most of them. On up to six vertices, the hypergraph that attains the conjecture is unique. Spencer [6] computed all such hypergraphs on at most twelve vertices by a brute force computer search. He found that Construction 2.1 does give all of the hypergraphs that attain the bound on eight, nine, or twelve vertices.

On seven vertices, there are 4 hypergraphs that attain the bound, of which we have three. If there are 10 vertices, Spencer found 20 hypergraphs, of which we have 18. If there are 11 vertices, Spencer found 40 hypergraphs, of which we have 36. It is unclear whether these few additional hypergraphs lead to very large classes of hypergraphs on more vertices that greatly outnumber the ones given in this paper.

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