# New infinite families of almost-planar crossing-critical graphs

**Petr Hliněný***

Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Republic

hlineny@fi.muni.cz

Submitted: Aug 6, 2007; Accepted: Jul 30, 2008; Published: Aug 4, 2008
Mathematics Subject Classification: 05C10, 05C62

**Abstract**

We show that, for all choices of integers $k > 2$ and $m$, there are simple 3-connected $k$-crossing-critical graphs containing more than $m$ vertices of each even degree $\leq 2k-2$. This construction answers one half of a question raised by Bokal, while the other half asking analogously about vertices of odd degrees at least 7 in crossing-critical graphs remains open. Furthermore, our newly constructed graphs have several other interesting properties; for instance, they are almost planar and their average degree can attain any rational value in the interval $\left[3+\frac{1}{5}, 6-\frac{8}{k+1}\right)$.

**Keywords:** crossing number, graph drawing, crossing-critical graph.

## 1 Introduction

We assume that the reader is familiar with basic terms of graph theory. In a *drawing* of a graph $G$ the vertices of $G$ are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and no three edges cross in a common point. The *crossing number* $\mathrm{cr}(G)$ of a graph $G$ is the minimum number of crossing points of edges in a drawing of $G$ in the plane.

For $k \geq 1$, we say that a graph $G$ is *$k$-crossing-critical* if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G-e) < k$ for each edge $e \in E(G)$. It is important to study crossing-critical graphs in order to understand structural properties of the crossing number problem. The only 1-crossing-critical graphs are, by the Kuratowski theorem, subdivisions of $K_5$ and $K_{3,3}$. The first construction of an infinite family of 2-crossing-critical simple 3-connected graphs was by

*Supported by the Institute for Theoretical Computer Science, project 1M0545.

Kochol [8] (Figure 8), improving previous construction by Širáň [12]. Many more crossing-critical constructions have appeared since.

It has been noted by D. Bokal (personal communication and preprint of [2]) that typical constructions of infinite families of simple 3-connected $k$-crossing-critical graphs create bounded numbers (wrt. $k$) of vertices of degrees other than $3, 4, 5$, or $6$. Actually, the existence of such 2-crossing-critical families with many degree-5 vertices has been established by Bokal only recently. Bokal's natural question thus was, what about occurrence of other vertex degree values in infinite families of $k$-crossing-critical graphs? We positively answer one half of his question in Theorem 3.1 and Proposition 2.1;

- namely we construct, for all $k > 2$, infinite families of simple 3-connected almost-planar $k$-crossing-critical graphs which contain arbitrary numbers of vertices of each *even degree* $4, 6, 8, \ldots, 2k - 2$.

The analogous question about occurrence of vertices of odd degrees $\geq 7$ in $k$-crossing-critical graphs remains open, and it appears to be significantly harder than the even case. One should also note that a (still open) question about the existence of an infinite family of simple 5-regular crossing-critical graphs was raised long before by Richter and Thomassen [9].

Usual constructions of crossing-critical graphs use an approach that can be described as a "Möbius twist"—they create graphs embeddable on a Möbius band which thus have to be twisted for drawing in the plane. We offer a quite different approach in Section 2, which extends our older construction [4], resulting in graphs that are *almost-planar* (sometimes called "near planar"), i.e. they can be made planar by deleting just one edge. As an easy corollary of this new and very flexible construction;

- we also produce almost-planar crossing-critical families with any prescribed average degree from $\left[3 + \frac{1}{5},\, 6 - \frac{8}{k+1}\right)$,

see in Theorem 4.1 and Corollaries 4.2, 4.3.

## 2 "Belt" constructions

An illustrating example of crossing-critical graphs constructed in our older work [4] is shown in Figure 1. The construction in [4] used vertices of degrees 4 or 3, and now we generalize it to allow more flexible structure and, particularly, vertices of arbitrary even degrees.

For easier notation, we (in the coming definitions) consider embeddings in the plane $\mathcal{P}$ with removed open disc $\mathcal{X}$. We say that a closed curve (loop) $\gamma$ is of *type-$\mathcal{X}$* if the homotopy type of $\gamma$ in $\mathcal{P} \setminus \mathcal{X}$ is to "wind once around $\mathcal{X}$". Having two loops $\gamma, \delta$ of type-$\mathcal{X}$, we write $\gamma \preceq \delta$ if $\gamma$ separates $\mathcal{X}$ from $\delta \setminus \gamma$ (meaning $\gamma$ is "*nested*" inside $\delta$).

**Crossed belt graphs.** A plane graph $F_0$ is a *plane $k$-belt* graph if it can be constructed as a connected edge-disjoint union of $k$ embedded "*belt*" cycles $C_1 \cup C_2 \cup \cdots \cup C_k = F_0$, where all $C_1, \ldots, C_k$ are of type-$\mathcal{X}$ nested as $C_1 \preceq C_2 \preceq \cdots \preceq C_k$.

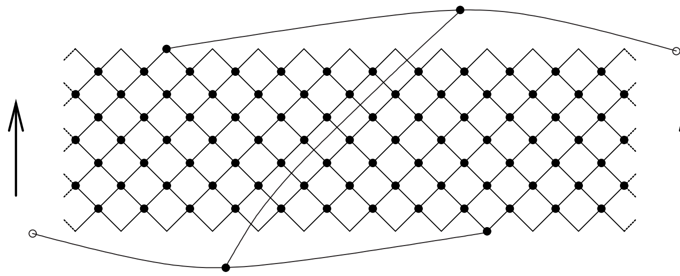

Figure 1: A simple 3-connected almost-planar 8-crossing-critical graph [4]. (The "grid-belt" is wraps around a cylinder without twist.)

A path $R \subseteq F_0$ connecting a vertex $p$ of $C_1$ to a vertex $q$ of $C_k$ is *radial* if, for each $1 < i \leq k$, $R$ intersects $C_i \cup \cdots \cup C_k$ in a subpath (with one end $q$). Informally, a radial path of $F_0$ has to "proceed straight across $F_0$" from $C_1$ to $C_k$. A vertex of $F_0$ is *accumulation* if its degree is at least 6 in $F_0$, i.e. if it is contained in at least three of the cycles $C_1, \ldots, C_k$.

Furthermore, a planar $k$-belt graph is *proper* if there are four distinct vertices $s_1, t_1 \in V(C_1) \setminus V(C_2)$ and $s_2, t_2 \in V(C_k) \setminus V(C_{k-1})$, and the following is true:

(B1) No radial path of $F_0$ starting in $s_1$ or $t_1$ contains an accumulation vertex. In particular, no accumulation vertex exists on the cycle $C_k$.

(B2) Let $P_2, P_2' \subseteq C_k$ be the two paths with the ends $s_2, t_2$ on $C_k$. Then every radial path of $F_0$ strating in $s_1$ (in $t_1$) hits $C_k$ first in an internal vertex of $P_2$ (of $P_2'$, respectively).

(B3) Let $P_1, P_1' \subseteq C_1$ be analogously the two paths with the ends $s_1, t_1$ on $C_1$. There exist collections of $k$ pairwise disjoint radial paths in $F_0$, all disjoint from $s_1, t_1$ and all starting on $P_1$ (on $P_1'$, respectively).

A graph $F$ is a *crossed $k$-belt* if it is $F = F_0 \cup S_0 \cup S_1 \cup S_2$, where

- $F_0$ is a proper planar $k$-belt graph as above;
- $S_1$ is a path with the ends $s_1, t_1$ internally disjoint from $F_0$ and $S_2$ is a path with the ends $s_2, t_2$ internally disjoint from $F_0 \cup S_1$; and
- $S_0$ is a path disjoint from $F_0$, connecting a vertex of $S_1$ to one of $S_2$.

This lengthy definition is illustrated in Figure 2. Notice that a crossed 1-belt graph is always a subdivision of $K_{3,3}$, and that removing an edge of $S_0$ from a crossed $k$-belt graph leaves it planar. Particularly, the graph in Figure 1 is a crossed 8-belt graph without accumulation vertices, and we call this special case a "*square-grid*" 8-belt graph. We aim to show that crossed $k$-belt graphs are $k$-crossing-critical with the exception of $k = 2$. (This exception is remarkable in view of successful research progress into the structure of 2-crossing-critical graphs.)

For better understanding we first discuss the conditions (B1), (B2) and (B3) imposed on our graphs. (B1) is generally unavoidable, as a nontrivial (counter)example violating

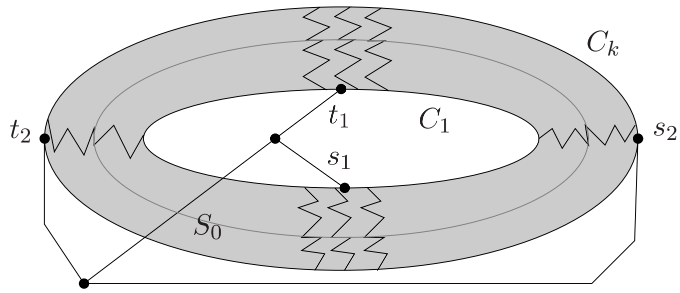


Figure 2: An illustration of the definition of a *crossed $k$-belt* graph. (The "zig-zag" lines are examples of radial paths as discussed in the definition.)

(B1) in Figure 3 shows. The other two conditions are, on the other hand, necessary mainly due to our inductive proof in the next section. (B2) establishes the base cases $k = 1, 3$ of the induction—violating (B2), one could easily construct planar graphs for $k = 1$ or graphs of crossing number 2 for $k = 3$. Perhaps, (B2) might not be necessary for higher values of $k$, but without Lemma 3.3 we could hardly start our induction. Finally, (B3) gives a sort of "sufficient interconnection" between the cycles $C_1, \ldots, C_k$ (we obviously cannot allow those to be disjoint), and then (B3) is the key ingredience in the inductive step in Theorem 3.1.

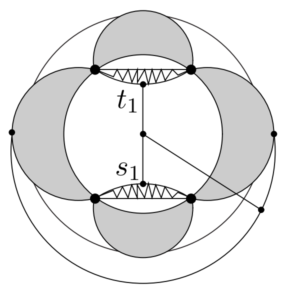


Figure 3: A sketch of a graph similar to crossed $k$-belt (with four "bad" accumulation vertices) which has crossing number 13 for large values of $k$.

The cruical property which motivated our construction, and which (in half) answers the aforementioned question of Bokal, is stated now:

**Proposition 2.1** *Let $k > 3$ be an integer. For every integer $m$ there is a crossed $k$-belt graph which is simple 3-connected and which contains more than $m$ vertices of each of degrees $\ell = 4, 6, 8, \ldots, 2k - 2$.*

*Proof.* In this case a picture is worth more than thousand words. Figure 4 shows local modifications of the square-grid 8-belt graph which produce accumulation vertices of degrees 14 and 12 while preserving its simplicity and connectivity. It is straightforward to generalize this picture to any $k > 3$ and all degrees $\ell = 6, 8, \ldots, 2k - 2$. Starting from a sufficiently large square-grid $k$-belt graph $F$, we can produce in this way $F'$ with

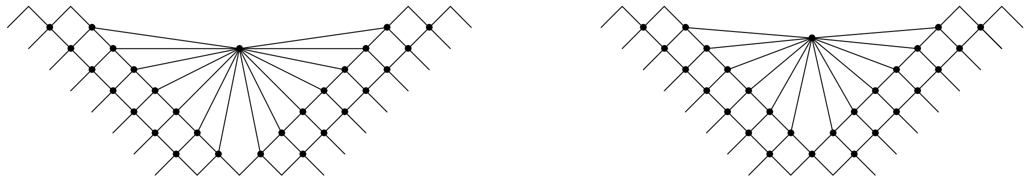

Figure 4: Examples of accumulation vertices.

arbitrarily many accumulation vertices of each degree $\ell = 6, 8, \ldots, 2k-2$, all of which are "sufficiently far" from the vertices $s_1, t_1$ as in the condition (B1). □

# 3 Crossing-criticality

We continue to use the notation from the definition of $k$-belt graphs also in this section. Now we come to the main result of our paper.

**Theorem 3.1** *For $k \geq 3$, every crossed $k$-belt graph is $k$-crossing-critical.*

*Proof.* Let $F$ be our $k$-belt graph, considered with notation as in the definition above. In one direction, by a straightforward induction we argue that any crossed $k$-belt graph, $k \geq 1$, can be drawn such that the only crossings occur between the path $S_0$ and each of the belt cycles $C_1, \ldots, C_k$ once. This is trivial for $k = 1$. For $k > 1$, we draw a $(k-1)$-belt subgraph $F' \subset F$ from Lemma 3.2 with $k-1$ crossings between $S_0$ and each of the belt cycles $C_2, \ldots, C_k$, in a way that one end of $S_0$ is inside the set $\mathcal{X}$ (see the definition of type-$\mathcal{X}$ in Section 2) and the other end of $S_0$ is in the face of $C_k$ not with $\mathcal{X}$. By definition the remaining cycle $C_1$ is nested inside each cycle $C_i$, $i > 1$, and so to obtain an analogous drawing of (whole) $F$ it is enough to add one more crossing of $S_0$ with $C_1$ since $C_1$ is also of type-$\mathcal{X}$. Furthermore, using analogous arguments, it is easy to verify that deleting any edge $e$ of $F$ allows us to draw $F - e$ with fewer than $k$ crossings.

Conversely, we assume an arbitrary drawing $\mathcal{F}$ of $F$, and we want to prove that $\mathcal{F}$ has at least $k$ edge crossings. There are two possibilities—either $C_1$ is drawn uncrossed in $\mathcal{F}$, or some edge of $C_1$ is crossed in $\mathcal{F}$. In the first case, assuming $k \geq 4$, we will argue that $\mathrm{cr}(\mathcal{F}) \geq k$ straight away.

Let $Q_1, \ldots, Q_k$ and $R_1, \ldots, R_k$ be the collections of disjoint radial paths established in (B3), ordered such that $Q_1$ and $R_1$ are the closest ones to $s_1$. Also using (B3), there exist $Q_0$ a radial path starting in $s_1$ and $R_0$ a radial path starting in $t_1$, none of $Q_0, R_0$ intersecting more than one of $Q_1, \ldots, Q_k$ and $R_1, \ldots, R_k$. Then there exist $k-2$ pairwise edge-disjoint paths $T_i \subseteq (Q_i \cup C_{i+2} \cup R_i) - V(R_0)$ for $i = 1, 2, \ldots, k-2$ in $F$, such that each $T_i$ intersects $C_1$ in two single vertices ($T_i$-ends) which separate $s_1$ from $t_1$ on $C_1$. Notice that these $T_i$ need not actually use sections of $Q_i$ or $R_i$ if closer accumulation vertices between $C_1$ and $C_{i+2}$ exist (still respecting (B1) ), but in this particular setting such paths $T_i$ always exist. Their key properties are that $T_1, \ldots, T_{k-2}$ are internally disjoint from $C_1$, and that all of them intersect $Q_0 - V(C_1)$.

Analogously, we obtain two more such edge-disjoint paths $T_{k-1} \subseteq (Q_{k-1} \cup C_k \cup R_{k-1}) - V(Q_0)$ and $T_k \subseteq (Q_k \cup C_{k-1} \cup R_k) - V(Q_0)$, both intersected by $R_0 - V(C_1)$. Thus all $T_1, \ldots, T_k$ belong to the same connected component of $F - V(C_1)$ as $C_k \cup Q_0 \cup R_0$ does, where $C_k$ is disjoint from $C_1$ by (B1). Furthermore, $S_1 - s_1 - t_1$ also belongs to the component with $C_k$. So, if $C_1$ is drawn uncrossed in $\mathcal{F}$, then all $S_1$ and $T_1, \ldots, T_k$ are drawn in the same face of $C_1$, and hence $S_1$ has to cross each of the edge-disjoint paths $T_1, \ldots, T_k$ by Jordan's curve theorem, witnessing $\mathrm{cr}(\mathcal{F}) \geq k$.

Otherwise, there is an edge $f$ of $C_1$ which is crossed in $\mathcal{F}$. We apply Lemma 3.2 to $F$ and $f$, so obtaining a crossed $(k-1)$-belt subgraph $F'$ of $F - f$, and conclude by induction that $\mathrm{cr}(\mathcal{F}) \geq 1 + \mathrm{cr}(F') = 1 + (k-1) = k$ if the claim holds true in the base case $k = 3$. Hence we can finish the proof of the theorem with further Lemma 3.3 which takes care of $k = 3$. □

**Lemma 3.2** *Let $F$ be a crossed $k$-belt graph as above, and choose any $f \in E(C_1)$. Then $F - f$ contains a crossed $(k-1)$-belt subgraph $F'$ having $C_2, \ldots, C_k$ as its collection of belt cycles.*

*Proof.* We refer to the notation in the definition of belt graphs. Let $s_1', t_1'$ denote vertices of $C_1 \cap C_2$ connected across $C_1 - V(C_2) - f$ to $s_1, t_1$, respectively. Then $s_1', t_1' \notin C_3$ thanks to (B1). Notice that for at least one of $s_1', t_1'$ we have a choice of two possibilities at each "side" of $s_1$ or $t_1$, and so we can ensure that not both $s_1', t_1'$ intersect the same one collection of radial paths from (B3).

Let $F_0'$ denote the subgraph of $F$ induced on $V(C_2) \cup \cdots \cup V(C_k)$, and let path $S_1'$ be the prolongation of $S_1$ on $C_1 - f$ with the ends $s_1', t_1'$. We claim that $F' = F_0' \cup S_1' \cup S_2 \cup S_0$ is a crossed $(k-1)$-belt graph: The properties (B1) and (B2) are easily inherited by $F'$ since radial paths starting in $s_1'$ or $t_1'$ form a subset of those starting in $s_1$ or $t_1$. (B3) is then satisfied thanks to our choice of $s_1'$ or $t_1'$ above. □

**Lemma 3.3** *Any crossed 3-belt graph is 3-crossing-critical.*

*Proof.* We adapt some of the ideas of Theorem 3.1 to this special case of $k = 3$. Let $\mathcal{F}$ be again a drawing of $F$. Say, if both cycle $C_1$ and $C_3$ are crossed in $\mathcal{F}$, then this case accounts for two distinct crossings—even if $C_1$ crossed $C_3$, these two disjoint cycles would have to cross twice. So let $f \in E(C_1)$ and $f' \in E(C_3)$ be edges of distinct crossings in $\mathcal{F}$. We can now successively apply Lemma 3.2 to $F$ and $f$, then $f'$. The result is a 1-belt graph $F'' \supset C_2$ (avoiding the crossings on $f, f'$) which is a subdivision of nonplanar $K_{3,3}$ thanks to (B2), and hence we conclude $\mathrm{cr}(\mathcal{F}) \geq 2 + 1 = 3$ in this case.

The other possible case is that $C_1$ or $C_3$ is uncrossed in $\mathcal{F}$. Considering uncrossed $C_1$, we turn the definition of a 3-belt graph $F$ into a symmetric one by establishing the following properties:

(B1+) There is clearly no accumulation vertex at all in $F$.

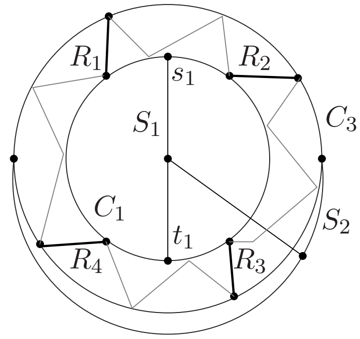


Figure 5: The "core" scheme of a crossed 3-belt graph, cf. (B2+).

(B2+) Let $P_1, P_1' \subseteq C_1$ and $P_2, P_2' \subseteq C_3$ be the paths as in (B2) and (B3) above. There are pairwise disjoint paths $R_1, R_2, R_3, R_4 \subseteq C_2$ connecting internal vertices, in order, of $P_1$ to $P_2$, of $P_1'$ to $P_2$, of $P_1'$ to $P_2'$, and of $P_1$ to $P_2'$. This fact follows rather easily from previous (B2) and (B3) when $k = 3$. See in Figure 5.

Analogously to Theorem 3.1, there are paths $T_1 \subseteq R_1 \cup C_3 \cup R_2$ and $T_2 \subseteq R_3 \cup C_3 \cup R_4$ such that the ends of each one $T_1$ or $T_2$ separate $s_1$ from $t_1$ on $C_1$. Again, the paths $T_1, T_2$ must be drawn in the same face of the uncrossed cycle $C_1$ in $\mathcal{F}$ as the path $S_1$ is, and hence they account for two crossings on $S_1$. If, moreover, the cycle $C_3$ is uncrossed in $\mathcal{F}$, then we get by symmetry another two crossings on $S_2$, and conclude $\mathrm{cr}(\mathcal{F}) \geq 2 + 2 = 4$. Hence $C_3$ has got some crossings, and if such a crossing is not with $S_1$, we are done again as $\mathrm{cr}(\mathcal{F}) \geq 2 + 1 = 3$. So it remains to consider that the only two crossings on $C_3$ are those with $S_1$, and then another crossing with $S_2$ or $C_2$ must exist on $S_1$ as well. Thus $\mathrm{cr}(\mathcal{F}) \geq 3$. □

# 4 Average degrees

Although the main motivation for our $k$-belt construction of crossing-critical graphs was to answer a part of Bokal's [2, Section 6, preprint] question, the critical graph families we obtain are so rich and flexible that they deserve further consideration and applications.

We look here at one particular question studied in a series of papers [11, 10, 2]: Salazar constructed infinite families of $k$-crossing-critical graphs with average degree equal to any rational in the interval $[4, 6)$. Then Pinontoan and Richter [10] extended this to the interval $(3.5, 4)$, and finally Bokal [2] has found $k$-crossing-critical families for any rational average degree in the interval $(3, 6)$. (Average degrees $\leq 3$ or $> 6$ cannot occur for infinite families, and the average degree 6 remains an open case.)

Using our construction and Theorem 3.1, we duplicate Salazar's result in Theorem 4.1 within the restricted subclass of almost-planar crossing-critical graphs, and further extend this in the subsequent corollaries.

**Theorem 4.1** *For every odd $k > 3$ there are infinitely many simple 3-connected crossed $k$-belt graphs with the average degree equal to any given rational value in the interval* $\left[4,\, 6 - \frac{8}{k+1}\right)$.

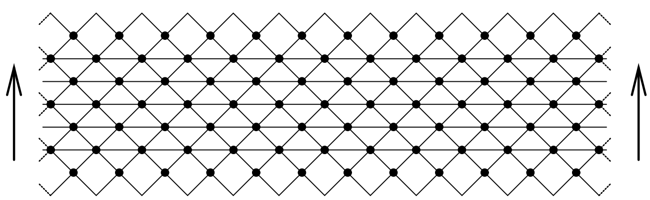

Figure 6: An approach to a plane 13-belt graph with accumulation vertices of degree 6.

*Proof.* Figure 6 illustrates a construction of a plane graph $F_1$ that fulfills all conditions of the definition of a plane 13-belt graph except (B1). *Splitting* of a vertex is a simple-graph inverse (not necessarily unique) of the edge-contraction operation. Figure 7 shows details of two "splitting" operations which can be applied to any accumulation vertex of $F_1$. These both preserve simplicity and 3-connectivity of $F_1$, and can be used to eventually construct a proper 13-belt graph from $F_1$.

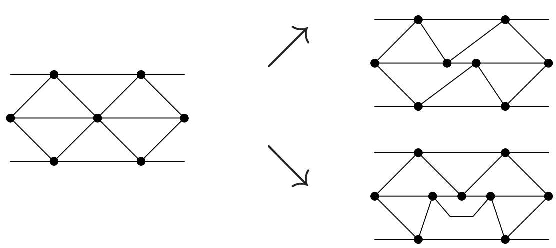

Figure 7: Details of *single-split* (top) and *double-split* (bottom) operations in the graph from Figure 6.

The construction of $F_1$ from Figure 6 can easily be generalized for any odd $k > 3$. Let $\ell$ be the length of the $C_1$-cycle in $F_1$, and let the number of accumulation vertices from $F_1$ that are single-split during the construction of $F_0$ be $m$ and the number of double-split accumulation vertices be $m'$. Admissible values of $m$ and $m'$ in our construction are at most the total number of accumulation vertices $m+m' \leq \ell(k-3)/2$, and at least $m \geq 4k^2$ since it is enough to single-split $2k^2$ accumulation vertices from $F_1$ near each of $s_1, t_1$ to satisfy (B1) of a proper $k$-belt graph.

An easy calculation shows that $F_0$ has $\ell(k+1)/2 + m + 2m'$ vertices, and so $F$ has $\ell(k+1)/2 + m + 2m' + 6$ vertices. The average degree of $F$ is

$$d_{avg}(F) = \frac{6k\ell - 2\ell + 4m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12} = 6 - \frac{8\ell + 8m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12}\,. \tag{1}$$

Now choose any rational $d_{avg} \in \left[4,\, 6 - \frac{8}{k+1}\right)$. Then setting $d_{avg} = 6 - \frac{p}{q} = 6 - \frac{cp}{cq}$ in (1) gives a system of two linear equations in two unknowns $\ell, m$ and parameters $k, c, m'$, which is nonsingular for each $k \neq 1$. Its solution is

$$\ell = \frac{c}{4k-4}(4q-p) - \frac{m'+3}{k-1}\,, \qquad m = \frac{cp}{8} - \frac{12(m'+3)}{8} - \ell\,.$$

The expressions show that choosing our parameters as $m'+3 = 2(k-1)$ and $c = c'\cdot 8(k-1)$ leads always to integer values of $\ell$ and $m$ as

$$\ell = c'(8q-2p) - 2\,, \qquad m = c'\big((k+1)p - 8q\big) - 3k + 5\,. \tag{2}$$

By the choice $6 - \frac{p}{q} \in \big[4,\, 6 - \frac{8}{k+1}\big)$ it is easy to show in (2) that always $m + m' \leq \ell(k-3)/2 - 3$, and since $(k+1)p - 8q > 0$ it follows that for sufficiently large choices of $c'$ we get also $m \geq 4k^2$. Thus we get from (2) an infinite sequence of admissible pairs $\ell, m$ (note fixed $k$ and $m' = 2k-5$), defining each one a crossed $k$-belt graph $F$ with average degree exactly $6 - \frac{p}{q}$ as needed. This holds for any fixed odd $k > 3$. □

Our restriction to odd values of $k$ was just for our comfort. We can easily overcome it using a powerful "zip-product" construction of Bokal [1, 2]. In our *restricted case*; having two simple graphs $G_1, G_2$ with cubic vertices $u_i \in V(G_i)$ and their neighbors denoted by $r_i, s_i, t_i$, the *zip product* $G$ of $G_1$ and $G_2$, according to the chosen vertices $u_1, u_2$ and their neighbors, is the disjoint union of $G_1 - u_1$ and $G_2 - u_2$ with added three edges $r_1r_2$, $s_1s_2$, $t_1t_2$. A cubic vertex $u_1$ in $G_1$ with the neighbors $r_1, s_1, t_1$ has *two coherent bundles* if there are two vertices $v, w \in V(G_1 - u_1)$ such that there exist six pairwise edge-disjoint paths, three of them from $v$ and the other three from $w$ to each of $r_1, s_1, t_1$. We shall use Bokal's [2, Theorem 21];

- if the above graphs $G_i$, $i = 1, 2$ are $k_i$-crossing-critical where $\mathrm{cr}(G_i) = k_i$, and $u_i$ have two coherent bundles in $G_i$, then their zip product $G$ is $(k_1 + k_2)$-crossing-critical.

**Corollary 4.2** *For every $k \geq 5$ there are infinitely many simple 3-connected almost-planar $2k$-crossing-critical graphs with the average degree equal to any given rational value in the interval $\big[4,\, 6 - \frac{8}{k+1}\big)$.*

*Proof.* We take two disjoint copies $G_1, G_2$ of a graph resulting from Theorem 4.1. It is easy to check that the (unique) cubic vertex $v_1$ of $G_1$, which is a neighbor of $s_1, t_1$ as in Figure 2, has two coherent bundles. (This fact is implicitly contained already in [2, Section 6].) Let $f_1$ denote the edge of $v_1$ not incident with $s_1, t_1$, and let $v_2, f_2$ be the corresponding elements in $G_2$. Recall that $G_i - f_i$ is planar. Then the zip product $G$ of $G_1$ and $G_2$ at $v_1, v_2$, matching edges $f_1, f_2$ into $f$ of $G$, is $2k$-crossing-critical by [2], and $G - f$ is planar. To achieve the same average degree of the product as that of $G_1$, we finally double-split one more accumulation vertex in $G_1$. □

Furthermore, we can lower the average degree of almost-planar crossing-critical graphs down to 3.2. For that we recall an old construction of Kochol [8]: His 3-connected 2-crossing-critical graphs consist of $2m + 1$ copies of a pentagon joined together as in Figure 8. Notice that also these graphs are almost-planar—just delete the marked edge $f$, and their average degree equals $3 + \frac{1}{5}$. They can be nicely combined with our construction in Theorem 4.1 using zip product, too.

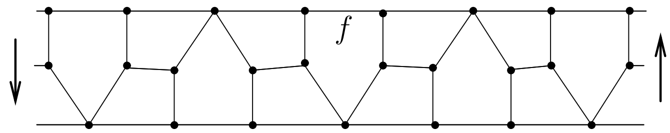


Figure 8: The 2-crossing-critical family of Kochol [8]; with "twisted" winding around a Möbius band.

**Corollary 4.3** *For every $k \geq 12$ (odd $k \geq 7$) there are infinitely many simple 3-connected almost-planar $k$-crossing-critical graphs with the average degree equal to any given rational value in the interval $\left(3 + \frac{1}{5}, 4\right)$.*

*Proof.* We consider first odd $k \geq 7$, and denote by $F_1$ the graph sketched in Figure 6, made as a union of $k - 2$ cycles with the first cycle of length $\ell$. Then we construct a simple 3-connected crossed $(k - 2)$-belt graph $F$ from $F_1$ after double-splitting $\ell + 1$ accumulation vertices of $F_1$ and single-splitting remaining accumulation vertices. Hence $F$ has $n = (k - 1)\ell + (\ell + 1) + 6 = k\ell + 7$ vertices and degree sum $4n - 6$ (note that all vertices of $F$ are of degree 4 except six of degree 3). We again denote by $v_1$ the cubic vertex of $F$, which is a neighbor of $s_1, t_1$ as in Figure 2.

We also denote by $G$ Kochol's graph (Figure 8) on $10m - 5$ vertices, and by $w$ one end the edge $f$ in $G$. It is again easy to check that $w$ has two coherent bundles in $G$, and so we may apply zip product here: Let $H$ be the result of the zip product of $F$ and $G$ at $v_1, w$, such that $H$ is almost-planar and $(k - 2 + 2)$-crossing-critical by [2]. A direct calculation shows that $H$ has $k\ell + 7 + 10m - 5 - 2 = k\ell + 10m$ vertices and its degree sum is $4(k\ell + 7) - 6 + 32m - 16 - 6 = 4k\ell + 32m$. Hence expressing its average degree as

$$\frac{4k\ell + 32m}{k\ell + 10m} = 4 - \frac{p}{q}$$

leads to an equation

$$m \cdot (8q - 10p) = \ell \cdot kp\,,$$

which clearly has infinitely many admissible integral solutions $\ell, m$ for all choices of $4 - \frac{p}{q} \in \left(3 + \frac{1}{5}, 4\right)$.

On the other hand, for even $k \geq 12$ we may apply an analogous construction starting from the graphs of Corollary 4.2. $\square$

# 5 Additional remarks

First, we remind readers that our Theorem 3.1 gives an answer only to a half of the question originally asked by Bokal, and so we repeat the other part which remains open:

**Question 5.1 (Bokal)** *For which odd values of $d \geq 7$ are there infinite families of simple 3-connected $k$-crossing-critical graphs having arbitrarily many vertices of degree $d$?*

Second, although our subsequent results in Section 4 are not quite new, they bring some interesting advantages over previous [2, 10, 11]. Prominently, we are constructing such crossing-critical graphs as almost-planar which was not the case of previous constructions. Our construction works with all (not too small) values of $k$, and not only with sporadic large $k$'s as, say [11], and we approach the upper–boundary value of 6 with much smaller values of $k$ than [2]. Though, in connection with Corollary 4.3 it is interesting to ask the next.

**Question 5.2** *Do there exist infinite families of almost-planar $k$-crossing-critical graphs with average degree below $3 + \frac{1}{5}$?*

Third, we have shown [5] that all $k$-crossing-critical graphs have path-width bounded in $k$. This result has been followed by a conjecture of Richter and Salazar; that $k$-crossing-critical graphs have bandwidth bounded in $k$. The close relation of this conjecture to our topic appears clear when one notices a positive answer would imply that maximal degree of $k$-crossing-critical graphs is bounded in $k$. We, however, are not strong supporters of it (particularly since an analogous claim for the projective plane is false [6]), and so we ask:

**Question 5.3** *Do $k$-crossing-critical graphs have maximal degree bounded by a function of $k$?*

One may, as well, ask whether can all $k$-crossing-critical graphs be "nicely characterized"? Recent signals suggest that such a characterization is not far in the case of $k = 2$, but values of $k > 3$ appear hopeless. At least one could hope an asymptotic characterization of almost-planar crossing-critical is feasible. In this relation the following question occurs naturally. (We note that for non-critical graphs, the questioned claim is false [3, 7].)

**Question 5.4** *Is it true that for every almost-planar $k$-crossing-critical graph $G$ there is an optimal drawing of $G$ with all the crossings concentrated on one edge of $G$?*

## Acknowledgments

The author would like to express his thanks to Banff International Research Station in Canada and the organizers of workshop #06w5067 for a very nice meeting where the idea of this papers has been born. Furthermore, the author thanks to an anonymous referee whose questions helped to uncover a hidden bug in the original version of this paper.

# References

[1] D. Bokal, *On the crossing number of Cartesian products with paths*, J. of Combinatorial Theory ser. B 97 (2007), 381–384.

[2] D. Bokal, *Infinite families of crossing-critical graphs with prescribed average degree and crossing number*, preprint, 2006.

[3] C. Gutwenger, P. Mutzel, R. Weiskircher, *Inserting an edge into a planar graph*, Algorithmica 41 (2005), 289–308.

[4] P. Hliněný, *Crossing-critical graphs and path-width*, In: Graph Drawing, 9th Symposium GD 2001, Vienna Austria, September 2001; Lecture Notes in Computer Science 2265, Springer Verlag 2002, 102–114.

[5] P. Hliněný, *Crossing-critical graphs have bounded path-width*, J. of Combinatorial Theory ser. B 88 (2003), 347–367.

[6] P. Hliněný and G. Salazar, *Stars and Bonds in Crossing-Critical Graphs*, submitted.

[7] P. Hliněný and G. Salazar, *On the Crossing Number of Almost Planar Graphs*, In Graph Drawing 2006; Lecture Notes in Computer Science 4372, Springer 2007, 162–173.

[8] M. Kochol, *Construction of crossing-critical graphs*, Discrete Math. 66 (1987), 311–313.

[9] R.B. Richter, C. Thomassen, *Minimal graphs with crossing number at least $k$*, J. of Combinatorial Theory ser. B 58 (1993), 217–224.

[10] R.B. Richter, B. Pinontoan, *Crossing Numbers of Sequences of Graphs II: Planar Tiles*, Journal of Graph Theory 42 (2003), 332–341.

[11] G. Salazar, *Infinite families of crossing-critical graphs with given average degree*, Discrete Math. 271 (2003), 343–350.

[12] J. Širáň, *Infinite families of crossing-critical graphs with a given crossing number*, Discrete Math. 48 (1984), 129–132.