A Note about Bezdek's Conjecture on Covering an Annulus by Strips

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Abstract

A closed plane region between two parallel lines is called a strip. András Bezdek posed the following conjecture: *For each convex region K there is an $\varepsilon > 0$ such that if εK lies in the interior of K and the annulus $K \backslash \varepsilon K$ is covered by finitely many strips, then the sum of the widths of the strips must be at least the minimal width of K.* In this paper, we consider problems which are related to the conjecture.

1 Introduction and Basic Definitions

A closed plane region between two parallel lines at distance d is called a strip of width d. For each direction θ, $0 \leq \theta \leq \pi$, a convex region M has two parallel supporting lines and the distance between them is denoted by $\omega(\theta)$. The minimum $\omega(\theta)$ is called the minimal width of M. In the case of a triangle, the minimal width is the altitude on the longest side.

Let O denote the origin of the plane E^2. For a given convex set K and $\varepsilon > 0$, let εK denote a homothetic copy of K consisting of all points X such that $\overrightarrow{OX} = \varepsilon \overrightarrow{OY}$, where $Y \in K$.

Tarski [5] conjectured and Bang [1] proved that if a convex region K can be covered by a finite collection of strips, then the sum of the widths of the strips must be at least

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the minimal width of K. András Bezdek [2] posed the following conjecture and proved two theorems:

Conjecture. ([2]) For each convex region K there is an $\varepsilon > 0$ such that if εK lies in the interior of K and the annulus $K \backslash \varepsilon K$ is covered by finitely many strips, then the sum of the widths of the strips must be at least the minimal width of K.

Theorem A. ([2]) Let C be the unit square and let ε be equal to $1 - 1/\sqrt{2} \approx 0.29$. If εC lies in the interior of the square C and the annulus $C \backslash \varepsilon C$ is covered by finitely many strips, then the sum of the widths is at least 1.

Theorem B. ([2]) The conjecture is true for each polygon whose incircle is tangent to two of its parallel sides. In particular, it is true for regular polygons with an even number of sides.

White and Wisewell obtained in 2007 the following result:

Theorem C. ([6]) Let P be a convex polygon. If there is a minimal-width chord of P that meets a vertex and divides the angle at that vertex into two acute angles, then for every $\varepsilon > 0$ an ε-scaled copy of P can be removed so that the resulting annulus can be covered by strips of total width strictly less than the minimal width of P.

We would like to make the following remarks here. In the previous version of this article which is in the References of the paper [6] by White and Wisewell, we claimed the following result as a counter example to Bezdek's conjecture:

Proposition The conjecture is not true for any equilateral triangle.

Clearly this proposition is a special case of Theorem C ([6]). But our three-sentence proof is quite elementary and independent of [6]. Maybe it's worthy of consideration. Here is the proof.

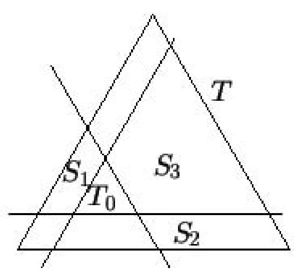

Figure 1:

Proof. It's an elementary fact that the sum of the distances from any point in an equilateral triangle T to its three sides is equal to the minimal width of T. Let the width of T be ω and $T_0 = \varepsilon T \subset T$, then the annulus $T \backslash T_0$ is covered by the strips S_1, S_2, S_3, as shown in Fig.1. But the sum of the three strips' widths is equal to $(1 - \varepsilon)\omega$ which is strictly less than ω. $\square$

2 Main Results

In this paper we consider more problems which are related to the conjecture. We need a few lemmas for our further discussion.

Lemma 1. *([2],[3]) Let $S_1, S_2, \cdots, S_n$ be a finite number of strips such that S_i is of width d_i. Let M be a centrally symmetric convex polygon of $2n$ sides such that each pair of opposite sides are of length d_i and perpendicular to one of the strips S_i. Assume that the origin O is the center of M. Denote by u_i the vector which is perpendicular to S_i and has magnitude $\frac{1}{2}d_i$. Then one of the $2n$ points, say P, of the form $\varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_n u_n$(where each ε_i can be ± 1) does not belong to the interior of any strip $S_i (i = 1, \cdots, n)$.*

Moreover, if no two of the strips are parallel and no three of the boundary lines pass through the same point, then the strips cannot cover a neighborhood of the point P and thus they cannot cover a neighborhood of M.

Lemma 2. *[4] Let M be a centrally symmetric convex polygon with perimeter l. $\triangle ABC$ is a circumscribed triangle of M, then the minimal width w of $\triangle ABC$ is not greater than $\frac{l}{2}$.*

Theorem 1. *Let T be a triangle with minimal width w and $S_1, S_2, \cdots, S_n$ be n strips such that S_i is of width d_i and $\sum_{i=1}^{n} d_i = d$. If $d < w$, for any ε with $0 < \varepsilon < 1 - (d/w)$, εT lies in the interior of T, then the annulus $T \backslash \varepsilon T$ can not be covered by the n strips.*

Proof. Without loss of generality, let $T = \triangle ABC$ with $|BC| \geq |CA| \geq |AB|$ and $T_0 = \varepsilon T$. Assume that BC is horizontal. Obviously, the minimal width w of T is the altitude on BC.

Assume that the n strips are in general position, otherwise, we expand the covered parts by moving the strips slightly so that the sum of the widths is still less than w, while the strips are in general position. Consider the centrally symmetric convex polygon M of $2n$ sides which corresponds to the n strips as in Lemma 1, that is, each pair of opposite sides of M are perpendicular to one of the strips S_i and of length d_i. It's easy to see that for any given triangle T, each convex region M has a circumscribed triangle T_1 similar to T. So let $T_1 = \triangle A_1B_1C_1$ be the circumscribed triangle of M with B_1C_1 horizontal and T_1 be similar to T. By Lemma 2, $w_1 = w(T_1) \leq \frac{l}{2} = \frac{1}{2}(2\sum_{i=1}^{n} d_i) = d < w$. As T is a triangle with minimal width w, $\triangle ABC \simeq \triangle A_1B_1C_1$, we can translate T_1 with the inscribed polygon M until the angles $\angle BAC$ and $\angle B_1A_1C_1$ coincide, and it follows that T_1 lies in T and $BC \parallel B_1C_1$. Now denote the distance between B_1C_1 and BC by $\varepsilon_1 = w - w_1 \geq w - d$. Take a positive number ε which is slightly less than $1 - d/w$. Then add an additional strip S_{n+1} which is horizontal with width d_{n+1} satisfying $\varepsilon_1 > d_{n+1} > \varepsilon$. If M has a pair of horizontal sides, then we choose a direction sufficiently close to horizontal. So the inner triangle T_0 can be covered by S_{n+1} while the strips $S_1, S_2, \cdots, S_{n+1}$ remain in general position.

Denote by M_1 the convex polygon corresponding to the given $n+1$ strips. A pair of vertical sides of length smaller than ε_1 are added to the sides of M to get the polygon

M_1. Thus M_1 can also be translated into the interior of T. According to Lemma 1, M_1 can not be covered by $S_1, S_2, \cdots, S_{n+1}$. $M_1 \subseteq T$, T_0 is covered by S_{n+1}, and so $T \backslash T_0$ can not be covered by $S_1, S_2, \cdots, S_n$. $\square$

We denote by $c(\varepsilon K)$ any copy of εK obtained by translating or rotating εK, and if $c(\varepsilon K) \subset K$, $K \backslash c(\varepsilon K)$ is also called an annulus. Denote by $P(u, v, \beta)$ a parallelogram with two adjacent sides u, v $(u \leq v)$ and the smaller angle β.

For the proof of our third result, we need another three lemmas besides Lemma 1:

Lemma 3. *Let M be a centrally symmetric convex polygon with perimeter l. If $P(u, v, \beta)$ is a circumscribed parallelogram of M, then $u \leq \frac{l}{2\sin\beta}, v \leq \frac{l}{2\sin\beta}$.*

Proof. It's easy to get a circumscribed parallelogram $ABCD$ of M with $|AB| = |CD| = u, |DA| = |BC| = v$ and smaller angle $\angle ABC$ equal to β. Let E be the common point of the side AB and M, and F be the common point of the side CD and M. Then it's obvious that $|EF| < \frac{l}{2}$ and $v \sin\beta \leq |EF|$, so $v \leq \frac{l}{2\sin\beta}$. In the same way, we get $u \leq \frac{l}{2\sin\beta}$. $\square$

By the law of sines, we have:

Lemma 4. *Among all the triangles with a side and its opposite angle given, the isosceles one has the largest perimeter.*

Lemma 5. *Let parallelogram $P = P(u, v, \beta)$ be the circumscribed parallelogram of a quadrilateral $EFGH$ whose perimeter is less than l, then the perimeter of P is less than $\sqrt{\frac{2}{1-\cos\beta}}l$.*

Proof. Let A, B, C, D be the four vertices of the parallelogram $P(u, v, \beta)$, and E, F, G, H lie on AB, BC, CD, DA respectively. If $|AH| = |AE| = a_1$, by the law of cosines, we have $2a_1^2 + 2a_1^2 \cos\beta = |EH|^2$, hence $a_1 = \sqrt{\frac{1}{2(1+\cos\beta)}}|EH| < \sqrt{\frac{1}{2(1-\cos\beta)}}|EH|$. For the given $|EH|$ and $\pi - \beta$, by Lemma 4, $|AH| + |AE| \leq 2a_1 < \sqrt{\frac{2}{1-\cos\beta}}|EH|$. Similarly we obtain $|BE| + |BF| < \sqrt{\frac{2}{1-\cos\beta}}|EF|$, $|FC| + |CG| < \sqrt{\frac{2}{1+\cos\beta}}|FG| < \sqrt{\frac{2}{1-\cos\beta}}|FG|$, $|HD| + |DG| < \sqrt{\frac{2}{1-\cos\beta}}|HG|$. So $p(ABCD) < \sqrt{\frac{2}{1-\cos\beta}}p(EFGH) < \sqrt{\frac{2}{1-\cos\beta}}l$. $\square$

On the basis of Theorem A [2], we obtain the following result which may be considered as a generalization in some sense.

Theorem 2. *For a given parallelogram $P = P(u, v, \beta)$, assume that $\varepsilon = (1 - \sqrt{\frac{1+\cos\beta}{2}})u$ and $P_0 = c(\varepsilon P)$ is any copy of εP such that the angle from the longer side of P to the perpendicular line of the longer side of P_0 is β(see Fig.2). If P_0 lies in the interior of P and the annulus $P \backslash P_0$ is covered by finitely many strips, then the sum of the width of strips is at least $u \sin\beta$.*

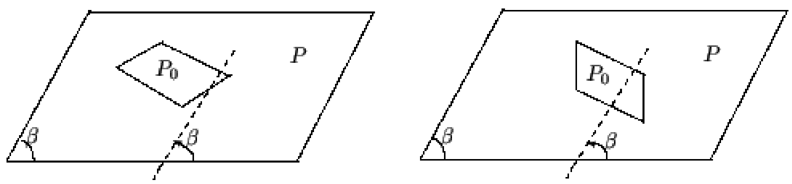

Figure 2:

Proof. Let $S_1, S_2, \cdots, S_n$ be the strips with widths $d_1, d_2, \cdots, d_n$ which cover $P \backslash P_0$. Assume that P has a pair of horizontal sides. As in the proof of Theorem 1, assume that the strips are in general position. Suppose, on the contrary, that $d_1 + d_2 + \cdots + d_n < u \sin \beta$ and we show that the union of strips S_i does not contain the annulus $P \backslash P_0$. Let M be the centrally symmetric convex $2n-$gon corresponding to the n strips and satisfying that each pair of opposite sides are of length d_i and perpendicular to one of the strips S_i. It's easy to get a circumscribed parallelogram $P_1 = P_1(u_1, v_1, \beta)$ of M. Translate P_1 along with the inscribed polygon M until the left upper vertices of the parallelograms P and P_1 coincide. Since the perimeter of M is less than $2u \sin \beta$, by Lemma 3, we have $u_1 \leq \frac{2u \sin \beta}{2 \sin \beta} = u, v_1 \leq u \leq v$ and it follows that P_1 lies in P. Choose a vertex of M on each side of P and connect them counterclockwise. The quadrilateral obtained has a perimeter less than $2u \sin \beta$ as well. By Lemma 5, the perimeter of P_1 is less than $2u \sin \beta \sqrt{\frac{2}{1-\cos \beta}}$. So one of the sides of P_1 is less than $\frac{u \sin \beta \sqrt{\frac{2}{1-\cos \beta}}}{2} = u \sqrt{\frac{1+\cos \beta}{2}}$. Without loss of generality, assume that $u_1 < u \sqrt{\frac{1+\cos \beta}{2}}$. Add an additional strip S_{n+1} such that

1. The angle from the horizontal side of P to the boundary line of S_{n+1} is $\frac{\pi}{2} + \beta$;

2. Its width d_{n+1} satisfies $(u - u_1) \sin \beta > d_{n+1} > (1 - \sqrt{\frac{1+\cos \beta}{2}}) u \sin \beta$;

3. It covers the inner parallelogram P_0 of the annulus, while the strips $S_1, S_2, \cdots, S_{n+1}$ remain in general position.

Denote by M_1 the convex $2(n+1)$-polygon corresponding to the given $n+1$ strips. A pair of sides of length less than $(u - u_1) \sin \beta$ and parallel to the shorter sides of P are added to the sides of M to get the polygon M_1. Thus M_1 can be translated into the interior of P as well. According to Lemma 1, M_1 can not be covered by $S_1, S_2, \cdots, S_{n+1}$. $M_1 \subseteq P$, P_0 is covered by S_{n+1}, and so $P \backslash P_0$ can not be covered by $S_1, S_2, \cdots, S_n$, a contradiction. So $\sum_{i=1}^{n} d_i \geq u \sin \beta$. □

Acknowledgements

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