A note on an identity of Andrews

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Abstract

In this note we use the q-exponential operator technique on an identity of Andrews.

1 Introduction

The following formula is equivalent to an identity of Andrews (see [3] or [1]):

$$\begin{aligned} & d\sum_{n=0}^{\infty}\frac{(q/bc,acdf;q)_n}{(ad,df;q)_{n+1}}(bd)^n - c\sum_{n=0}^{\infty}\frac{(q/bd,acdf;q)_n}{(ac,cf;q)_{n+1}}(bc)^n \\ = \quad & d\frac{(q,qd/c,c/d,abcd,acdf,bcdf;q)_\infty}{(ac,ad,cf,df,bc,bd;q)_\infty}. \end{aligned} \tag{1}$$

Liu [3] showed it can be derived from the Ramanjan ${}_1\psi_1$ summation formula by the q-exponential operator techniques. In this short note, again using the q-exponential operator technique on it, we obtain a generalization of this identity. We have

Theorem 1.1. *Let* $0 <| q |< 1$. *Then*

$$\begin{aligned} & d\sum_{n=0}^{\infty}\frac{(q/bc,q/ce,acdf;q)_n}{(ad,df;q)_{n+1}(q^2/bcde;q)_n}q^n - c\sum_{n=0}^{\infty}\frac{(q/bd,q/de,acdf;q)_n}{(ac,cf;q)_{n+1}(q^2/bcde;q)_n}q^n \\ = \quad & d\frac{(q,qd/c,c/d,abcd,acdf,bcdf,acde,cdef,bcde/q;q)_\infty}{(ac,ad,cf,df,bc,bd,ce,de,abc^2d^2ef/q;q)_\infty}. \end{aligned} \tag{2}$$

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2 The proof of the Theorem

The q-difference operator and the q-shift operator η are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and

$$\eta\{f(a)\} = f(aq),$$

respectively. In [2] Chen and Liu construct the operator

$$\theta = \eta^{-1}D_q.$$

Based on these, they introduce a q-exponential operator:

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}.$$

For $E(b\theta)$, there hold the following operator identities.

$$E(b\theta)\left\{(at;q)_\infty\right\} = (at, bt; q)_\infty, \tag{3}$$

$$E(b\theta)\left\{(as, at;q)_\infty\right\} = \frac{(as, at, bs, bt; q)_\infty}{(abst/q;q)_\infty}. \tag{4}$$

Applying

$$(q/a;q)_n = (-a)^{-n} q^{\binom{n+1}{2}} \frac{(q^{-n}a;q)_\infty}{(a;q)_\infty}, \tag{5}$$

we rewrite (1) as

$$\begin{aligned}
& d\sum_{n=0}^{\infty} \frac{(acdf;q)_n}{(ad, df;q)_{n+1}} \left(-\frac{d}{c}\right)^n q^{\binom{n+1}{2}} \cdot \left\{(q^{-n}bc, bd;q)_\infty\right\} \\
& -c\sum_{n=0}^{\infty} \frac{(acdf;q)_n}{(ac, cf;q)_{n+1}} \left(-\frac{c}{d}\right)^n q^{\binom{n+1}{2}} \cdot \left\{(q^{-n}bd, bc;q)_\infty\right\} \\
= \; & d\frac{(q, qd/c, c/d, acdf;q)_\infty}{(ac, ad, cf, df;q)_\infty} \cdot \left\{(abcd, bcdf;q)_\infty\right\}.
\end{aligned} \tag{6}$$

Applying $E(e\theta)$ to both sides of the equation with respect to the variable b gives

$$\begin{aligned}
& d\sum_{n=0}^{\infty} \frac{(acdf;q)_n}{(ad, df;q)_{n+1}} \left(-\frac{d}{c}\right)^n q^{\binom{n+1}{2}} \cdot E(e\theta)\left\{(q^{-n}bc, bd;q)_\infty\right\} \\
& -c\sum_{n=0}^{\infty} \frac{(acdf;q)_n}{(ac, cf;q)_{n+1}} \left(-\frac{c}{d}\right)^n q^{\binom{n+1}{2}} \cdot E(e\theta)\left\{(q^{-n}bd, bc;q)_\infty\right\} \\
= \; & d\frac{(q, qd/c, c/d, acdf;q)_\infty}{(ac, ad, cf, df;q)^\infty} \cdot E(e\theta)\left\{(abcd, bcdf;q)_\infty\right\}.
\end{aligned} \tag{7}$$

Again, applying the results (3) and (4) of Chen and Liu, we have

$$E(e\theta)\left\{(q^{-n}bc, bd; q)_\infty\right\} = \frac{(q^{-n}bc, bd, q^{-n}ce, de; q)_\infty}{(q^{-n}bcde/q; q)_\infty}, \tag{8}$$

$$E(e\theta)\left\{(q^{-n}bd, bc; q)_\infty\right\} = \frac{(q^{-n}bd, bc, q^{-n}de, ce; q)_\infty}{(q^{-n}bcde/q; q)_\infty} \tag{9}$$

and

$$E(e\theta)\left\{(abcd, bcdf; q)_\infty\right\} = \frac{(abcd, bcdf, acde, cdef; q)_\infty}{(abc^2d^2ef/q; q)_\infty}. \tag{10}$$

Substituting these three identities into (7) and then using

$$(q^{-n}a; q)_\infty = (-a)^n q^{-\binom{n+1}{2}}(q/a; q)_n (a; q)_\infty, \tag{11}$$

we have

$$\begin{aligned}
& d\frac{(bd, de, bc, ce; q)_\infty}{(bcde/q; q)_\infty}\sum_{n=0}^{\infty}\frac{(q/bc, q/ce, acdf; q)_n}{(ad, df; q)_{n+1}(q^2/bcde; q)_n}q^n \\
& -c\frac{(bc, ce, bd, de; q)_\infty}{(bcde/q; q)_\infty}\sum_{n=0}^{\infty}\frac{(q/bd, q/de, acdf; q)_n}{(ac, cf; q)_{n+1}(q^2/bcde; q)_n}q^n \\
= \; & d\frac{(q, qd/c, c/d, acdf, abcd, bcdf, acde, cdef; q)_\infty}{(ac, ad, cf, df, abc^2d^2ef/q; q)_\infty}.
\end{aligned} \tag{12}$$

Hence we get

$$\begin{aligned}
& d\sum_{n=0}^{\infty}\frac{(q/bc, q/ce, acdf; q)_n}{(ad, df; q)_{n+1}(q^2/bcde; q)_n}q^n - c\sum_{n=0}^{\infty}\frac{(q/bd, q/de, acdf; q)_n}{(ac, cf; q)_{n+1}(q^2/bcde; q)_n}q^n \\
= \; & d\frac{(q, qd/c, c/d, abcd, acdf, bcdf, acde, cdef, bcde/q; q)_\infty}{(ac, ad, cf, df, bc, bd, ce, de, abc^2d^2ef/q; q)_\infty}.
\end{aligned} \tag{13}$$

The proof is completed.

References

[1] G. E. Andrews, *Ramanujan's "lost" notebook. I. Partial θ-functions,* Adv. in Math. 41 (1981), 137-172 .

[2] W. Y. C. Chen - Z. G. Liu, *Parameter augmentation for basic hypergeometric series, I,* in: B. E. Sagan, R. P. Stanley (Eds.), Mathematical Essays in honor of Gian-Carlo Rota, Birkauser, Basel. 1998, pp. 111-129

[3] Z. G. Liu, *Some operator identities and q-series transformation formulas,* Discrete Math. 265(2003), 119-139.