The intersection structure of t-intersecting families

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Abstract

A family of sets is *t-intersecting* if any two sets from the family contain at least t common elements. Given a t-intersecting family of r-sets from an n-set, how many distinct sets of size k can occur as pairwise intersections of its members? We prove an asymptotic upper bound on this number that can always be achieved. This result can be seen as a generalization of the Erdős-Ko-Rado theorem.

1 Introduction

For integers $1 \leq r \leq n$ we define $[n] := \{1, 2, \dots, n\}$ and $[n]^{(r)} = \{A \subseteq [n] : |A| = r\}$. In this short note we give an asymptotically sharp answer to the following extremal question: if $\mathcal{A} \subseteq [n]^{(r)}$ is t-intersecting and $t \leq k \leq r$ then how many distinct sets of size k can occur as pairwise intersections of members of $\mathcal{A}$?

For example if $\mathcal{A} \subseteq [n]^{(6)}$ is 2-intersecting then how many 4-sets can occur as pairwise intersections of members of $\mathcal{A}$? (The answer in this case is at most $21\binom{n}{2} + O(n)$ and this can achieved.)

Let $\mathcal{A} \subseteq [n]^{(r)}$ be t-intersecting and $t \leq k \leq r$. The family of *k-intersections* of $\mathcal{A}$ is

$$\mathcal{A}\langle k\rangle = \{C \in [n]^{(k)} : \exists A, B \in \mathcal{A} \text{ such that } A \cap B = C\}.$$

If $\mathcal{A}$ is t-intersecting then trivially the smallest pairwise intersections of members of $\mathcal{A}$ have size t. These play an important role when trying to bound the size of $|\mathcal{A}\langle k\rangle|$ in general. In particular it is useful to define the following quantity for $1 \leq t \leq r$

$$\alpha_t^{(r)} = \max_{n \geq r}\{|\mathcal{A}\langle t\rangle| : \mathcal{A} \subseteq [n]^{(r)} \text{ is } t\text{-intersecting}\}.$$

The fact that $\alpha_t^{(r)}$ is well-defined follows from a result of Lovász [2] which implies that $\alpha_1^{(m)}$ exists for all $m \geq 1$. It is easy to then check that $\alpha_t^{(r)} \leq \alpha_1^m$, for $m = \binom{r}{t}$, and so $\alpha_t^{(r)}$ exists for all $1 \leq t \leq r$. The following general upper bound for $\alpha_t^{(r)}$ was given in [3].

Theorem 1 *If $1 \leq t < r$ then*

$$\alpha_t^{(r)} \leq \frac{1}{2}\binom{2r-t}{r-t}\binom{r}{t}.$$

2 Result

Theorem 2 *If $1 \leq t \leq k \leq r \leq n$ and $\mathcal{A} \subseteq [n]^{(r)}$ is t-intersecting then*

$$|\mathcal{A}\langle k\rangle| \leq \alpha_t^{(r-k+t)}\binom{n}{k-t} + O(n^{k-t-1}). \tag{1}$$

Moreover this bound can always be achieved.

Proof. We first prove the upper bound using induction on k. The result holds trivially for $k = t$ so suppose $k \geq t+1$. For $i \in [n]$ define

$$\mathcal{A}'_i = \{A \backslash \{i\} : i \in A \text{ and } A \in \mathcal{A}\}.$$

If $V = \bigcup_{A \in \mathcal{A}\langle t\rangle} A$, then $|V| \leq t\alpha_t^{(r)} = O(1)$ by Theorem 1 (as n tends to infinity). Note that if $i \in [n]\backslash V$ then $\mathcal{A}'_i$ is a t-intersecting family of $(r-1)$-sets and hence by our inductive hypothesis for $k-1$

$$\sum_{i \in [n]\backslash V} |\mathcal{A}'_i\langle k-1\rangle| \leq \alpha_t^{(r-k+t)}\binom{n}{k-t-1}(n-|V|). \tag{2}$$

We now consider how often a set $A \in \mathcal{A}\langle k\rangle$ is counted in the left-hand side of (2). Partition $\mathcal{A}\langle k\rangle$ as $\mathcal{A}\langle k\rangle = \mathcal{B}\dot{\cup}\mathcal{C}$, where

$$\mathcal{B} = \{A \in \mathcal{A}\langle k\rangle : |A \cap V| \leq t\} \quad \text{and} \quad \mathcal{C} = \{A \in \mathcal{A}\langle k\rangle : |A \cap V| \geq t+1\}.$$

If $A \in \mathcal{B}$ then A is counted at least $k-t$ times in the left-hand side of (2) while

$$|\mathcal{C}| \leq \binom{|V|}{k}\binom{n-|V|}{k-t-1} = O(n^{k-t-1}).$$

Hence we have

$$|\mathcal{A}\langle k\rangle| \leq \alpha_t^{(r-k+t)}\binom{n}{k-t} + O(n^{k-t-1}).$$

The upper bound then follows by induction on k.

The following simple construction shows that the upper bound (1) can always be achieved. Let $\mathcal{B}$ be a t-intersecting family of $(r-k+t)$-sets satisfying $|\mathcal{B}\langle t\rangle| = \alpha_t^{(r-k+t)}$. If $m = |\cup_{B \in \mathcal{B}} B|$ then we may suppose that $\mathcal{B} \subseteq [m]^{(r-k+t)}$. Let $n \geq m+k-t$ and consider the family

$$\mathcal{A} = \{A \in [n]^{(r)} : \exists B \in \mathcal{B} \text{ such that } B \subseteq A\}. \tag{3}$$

Since $\mathcal{B}$ is t-intersecting so $\mathcal{A}$ is also t-intersecting.

To see that the family $\mathcal{A}$ gives equality in (1) consider any k-set $C = D \cup E$, where $D \in \mathcal{B}\langle t \rangle$ and $E \in \{m+1, m+2, \ldots, n\}^{(k-t)}$. By definition of $\mathcal{B}\langle t \rangle$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 = D$. Moreover $B_1 \cup E, B_2 \cup E \in \mathcal{A}$ and hence $C = D \cup E = (B_1 \cup E) \cap (B_2 \cup E) \in \mathcal{A}\langle k \rangle$. Hence for each $D \in \mathcal{B}\langle t \rangle$ and each $E \in \{m+1, \ldots, n\}^{(k-t)}$ we obtain a unique set $C = D \cup E \in \mathcal{A}\langle k \rangle$. Thus

$$\begin{aligned} |\mathcal{A}\langle k \rangle| &\geq \alpha_t^{(r-k+t)} \binom{n-m}{k-t} \\ &= \alpha_t^{(r-k+t)} \binom{n}{k-t} + O(n^{k-t-1}). \end{aligned}$$

The first part of this theorem then implies that equality holds in (1) for $\mathcal{A}$. □

3 Remarks

The case $r = k$ of Theorem 2 says that if $\mathcal{A} \subseteq [n]^{(r)}$ is t-intersecting then

$$|\mathcal{A}| \leq \binom{n}{r-t} + O(n^{r-t-1}).$$

This is essentially a version of the Erdős-Ko-Rado theorem [1].

For $t = 1$ we can prove a stronger result (see [4]). Namely, if $\mathcal{A} \subseteq [n]^{(r)}$ is intersecting then either $\mathcal{A}$ is constructed as in the example (3) given above, or $\mathcal{A}$ has far fewer k-intersections than this example.

For the bound in Theorem 2 to be explicitly calculated we need to know the value of $\alpha_t^{(r-k+t)}$. This is an important problem in its own right. The values that are currently known are: $\alpha_t^{(t)} = 1$, $\alpha_t^{(t+1)} = \binom{t+2}{2}$, $\alpha_1^{(3)} = 7$, $\alpha_1^{(4)} = 16$ and $\alpha_2^{(4)} = 21$. An upper bound for $\alpha_t^{(r)}$ is given by Theorem 1 while the best known bounds for $\alpha_1^{(r)}$ are due to Tuza [5].

Theorem 3 (Tuza [5]) *If $r \geq 4$ then*

$$2\binom{2r-4}{r-2} + 2r - 4 \leq \alpha_1^{(r)} \leq \binom{2r-1}{r-1} + \binom{2r-4}{r-1}.$$

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