New lower bound for multicolor Ramsey numbers for even cycles

Tomasz Dzido*

Institute of Mathematics, University of Gdańsk,
Wita Stwosza 57, 80-952 Gdańsk, Poland.
tdz@math.univ.gda.pl

Andrzej Nowik†

Institute of Mathematics, University of Gdańsk,
Wita Stwosza 57, 80-952 Gdańsk, Poland.
nowik@manta.univ.gda.pl

Piotr Szuca‡

Institute of Mathematics, University of Gdańsk,
Wita Stwosza 57, 80-952 Gdańsk, Poland.
pszuca@radix.com.pl

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Abstract

For given finite family of graphs $G_1, G_2, \ldots, G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph on n vertices with k colors then there is always a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. We give a lower bound for $k-$color Ramsey number $R(C_m, C_m, \ldots, C_m)$, where $m \geq 4$ is even and C_m is the cycle on m vertices.

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. By K_m we denote the complete graph on m vertices, and by C_m we denote the cycle of length m. For given graphs $G_1, G_2, \ldots, G_k, k \geq 2$, the *multicolor*

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Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with k colors, then it always contains a monochromatic copy of G_i colored with i, for some $1 \leq i \leq k$. We denote such a number by $R_k(G)$ if $G = G_1 = G_2 = \cdots = G_k$. Here in, we consider only 3-color Ramsey number $R_3(G)$ (i.e. we color the edges of the complete graph K_n with color red, blue and green.) A 3-coloring of K_n is called a $(G;n)_3$-coloring if it contains neither a red G nor a blue G nor a green G, $(G;n)_k$-coloring is defined analogously. We refer the reader to [6] for a survey.

2 The Ramsey numbers for even cycles

Up to now, there have been known only two exact values for 3-color Ramsey numbers for even cycles. More precisely, in [2] it was proved that $R_3(C_4) = 11$, and $R_3(C_6) = 12$ was showed in [7] with a help of the computer support. When talking about lower bounds, let us recall that Graham *et al.* [5] proved that for any k and m, $R_k(C_{2m}) \geq (k-1)(m-1)+1$. This bound was improved to $(k+1)m - k + 1$ in [3]. Finally, recall that Figaj and Łuczak proved the following theorem.

Theorem 1 ([4]). *For any constants* $\alpha_1, \alpha_2, \alpha_3 > 0$*,*

$$R\left(C_{2\lfloor\alpha_1 n\rfloor}, C_{2\lfloor\alpha_2 n\rfloor}, C_{2\lfloor\alpha_3 n\rfloor}\right) = \left(\alpha_1 + \alpha_2 + \alpha_3 + max\{\alpha_1, \alpha_2, \alpha_3\} + o(1)\right) n$$

while $n \to \infty$.

Consequently, notice that if $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $n = m$ we obtain that

$$R_3(C_{2m}) = (4 + o(1))\, m.$$

In this paper, our main result is the following theorem.

Theorem 2. *For all integers* $m \geq 2$ *and an odd integer* $k \geq 1$*,*

$$R_k(C_{2m}) \geq (k+1)m.$$

Proof. We shall give a k-coloring of all edges of a complete graph $G'' = K_n$ on $n = (k+1)m - 1$ vertices which is a $(C_{2m};n)_k$-coloring. The situation for $k=1$ is obvious, so we may assume that $k \geq 3$.

Let $k \geq 3$ be an odd integer. Using a fact that $\chi'(K_{k+1}) = k$ when k is odd (see e.g. [8]), color properly edges of the complete graph K_{k+1} with k colors. "Blow-up" the coloring $m-1$ times, i.e. replace each vertex of K_{k+1} by the set G_i $(1 \leq i \leq k+1)$ of $m-1$ vertices and each colored edge by a complete monochromatic bipartite graph $K_{m-1,m-1}$ of an appropriate color (See Fig. 1 for illustration.) Formally, consider the complete graph G on $k+1$ vertices. Let $c\colon V \to \{1, \ldots, k+1\}$ be a proper edge-coloring of graph G. For a vertex $i \in V$, where $i \in \{1, \ldots, k+1\}$, let G_i denote a complete graph on $m-1$ vertices.

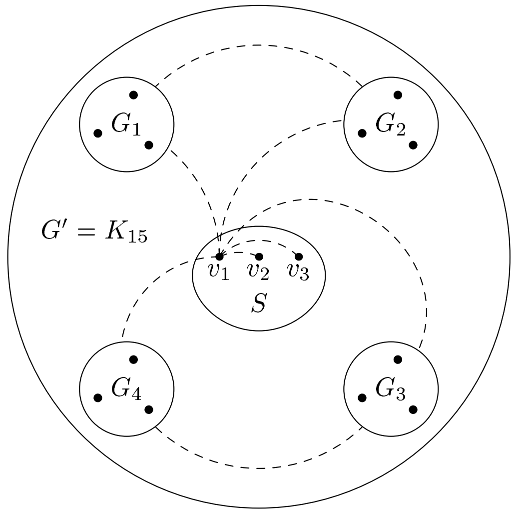

Figure 1: An illustration of coloring from the proof of Theorem 2 for the case $m = 4$ and $k = 3$. To make the picture readable there are shown only edges colored with color 1: ones which join v_1 with each G_i, and edges joining G_1 with G_2, and G_3 with G_4. Edges which join v_2 with each G_i, G_2 with G_3, and G_1 with G_4 are colored with color 2. Edges which join v_3 with each G_i, G_1 with G_3 and G_2 with G_4 are colored with color 3. Subgraphs G_1, G_2, G_3 and G_4 replace "blown-up" vertices of the complete graph K_4.

Let $G'(V', E')$ be a complete graph with the set of vertices

$$V' = \bigcup_{i=1}^{k+1} V\left(G_i\right).$$

The coloring c' of the graph G' is as follows:

$$c'\left(\{p,q\}\right) = \begin{cases} c\left(x,y\right) & \text{if } p \in V(G_x), \text{ and } q \in V(G_y), \text{ and } x \neq y; \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, such a graph G' contains no monochromatic path of more than $2m-2$ vertices.

Next, extend graph G' to the graph G'' by adding k new vertices $S = \{v_1, \ldots, v_k\}$. Now, color all edges between v_i and V' with the color i, and for any pair $\{i, j\}$ such that $j > i > 0$, color edge $\{v_i, v_j\}$ with the color i (thus there are no monochromatic cycles in the subgraph induced by S.)

More formally, let $G''(V'', E'')$ be a complete graph with the set of vertices $V'' = V' \cup S$. The coloring c'' of G'' is as follows:

$$c''\left(\{p,q\}\right) = \begin{cases} i & \text{if } p = v_i \in S, \text{ and } q \in V\left(G_j\right), \text{ and } 1 \leq j \leq k+1; \\ i & \text{if } p = v_i \in S, \text{ and } q = v_j \in S, \text{ and } 1 \leq i < j \leq k; \\ c'\left(\{p,q\}\right) & \text{otherwise.} \end{cases}$$

It remains to show that in G'' there are no monochromatic cycles of length at least $2m$. Suppose, contrary to our claim, that G'' contains a cycle C of color d longer than

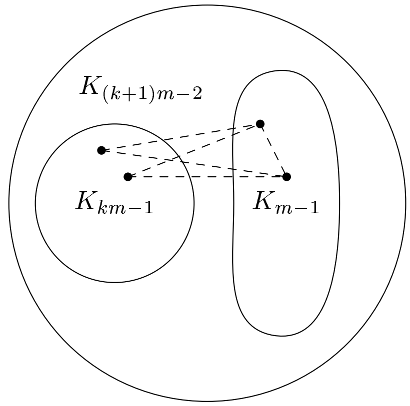

Figure 2: An illustration of coloring from the proof of Corollary 1. Edges of K_{km-1} are colored with $k-1$ colors without monochromatic cycle of length $2m$. All edges from the graph K_{m-1} and a bipartite graph $K_{km-1,m-1}$ are colored with color k (dashed line denote edges assigned with color k.)

$2m-1$. Since in G' there is no monochromatic path of length greater than $2m-2$, we have

$$C \cap S \neq \emptyset.$$

Next, the only vertex from S which is adjacent by an edge of color d with G is v_d, hence

$$C \cap S = \{v_d\}.$$

Since v_d can be contained only once in the cycle C, this implies that the cardinality of the set

$$\{i \ : \ C \cap G_i \neq \emptyset\}$$

is at most 2. Thus the length of C is less than $2m$, a contradiction. □

Corollary 1. *For all integers $m \geq 2$ and an even integer $k \geq 2$,*

$$R_k(C_{2m}) \geq (k+1)m - 1.$$

Proof. Let $n = (k+1)m - 2$. By Theorem 2, there exists $(C_{2m}; km-1)_{k-1}$-coloring of a complete subgraph K_{km-1} of K_n. A $(C_{2m}; n)_k$-coloring of K_n is obtained by assigning the last color k to all remaining edges (See Fig. 2.) Indeed, on the contrary suppose that there exists a monochromatic cycle of length $2m$. This cycle has the last k-th color. The number of vertices from K_{km-1} which belong to cycle is at least $m+1$ and is greater than the number of such vertices from K_{m-1}. The maximal possible number of edges between K_{km-1} and K_{m-1} is $2m-2$. Thus there exists an edge contained in the graph K_{km-1}, what is impossible. □

The following corollary is straightforward:

Corollary 2. *For all integers $m \geq 2$,*

$$R_3(C_{2m}) \geq 4m.$$

In particular, notice that we obtain $R_3(C_8) \geq 16$. Moreover, by using upper bound for Ramsey number for even cycles ([5, Section 5.7, Theorem 10]), we have $R_3(C_8) \leq 2412$, and by using known upper bound for Ramsey number for symmetric bipartite graph $K_{4,4}$ ([1]), we have $R_3(C_8) \leq 648$.

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