Outerplanar Crossing Numbers, the Circular Arrangement Problem and Isoperimetric Functions

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Abstract

We extend the lower bound in [15] for the outerplanar crossing number (in other terminologies also called convex, circular and one-page book crossing number) to a more general setting. In this setting we can show a better lower bound for the outerplanar crossing number of hypercubes than the best lower bound for the planar crossing number. We exhibit further sequences of graphs, whose outerplanar crossing number exceeds by a factor of $\log n$ the planar crossing number of the graph. We study the circular arrangement problem, as a lower bound for the linear arrangement problem, in a general fashion. We obtain new lower bounds for the circular arrangement problem. All the results depend on establishing good isoperimetric functions for certain classes of graphs. For several graph families new near-tight isoperimetric functions are established.

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1 Introduction

This paper is a sequel to our paper with Shahrokhi [15]. We use similar notation as in that paper: $G = (V(G), E(G))$ denotes a graph and d_v denotes the degree of $v \in V$. A *drawing* of G is a placement of the vertices into distinct points of the plane and a representation of edges uv by simple continuous curves connecting the corresponding points and not passing through any point corresponding to a vertex other than u and v. A *crossing* is a common interior point of two edges of G. We also assume that any two curves representing the edges of G have at most one interior point in common and that two curves incident to the same vertex do not cross. Let $\mathrm{cr}(G)$ denote the *crossing number* of G, i.e. the minimum number of crossings over all possible drawings of G in the plane with the above properties (see [14] or [20]).

An important application area of crossing numbers is automated graph drawing. We know that the number of crossings influences the aesthetical properties and readability of graphs [6, 12].

An *outerplanar (also called circular or convex) drawing* of G places the vertices on a circle and draws the edges as straight-line segments. The outerplanar crossing number of G is the minimum number of pairs of crossing edges over all outerplanar drawings of G.

Let $\nu_1(G)$ denote the outerplanar [10] crossing number of G. There are other notations and terminologies used for this quantity. In [15] we used the term convex crossing number and notation $\mathrm{cr}^*(G)$. As the outerplanar drawing is topologically equivalent to the one-page drawing, we returned to the one-page book crossing number notation $\nu_1(G)$ in accordance with [14].

In our paper with Shahrokhi [15] we showed the following upper bound for the outerplanar crossing number through a divide-and-conquer algorithm:

Theorem 1. $\nu_1(G) = O\left((\mathrm{cr}(G) + \sum_{v \in V} d_v^2) \log |V|\right)$.

We also showed on the example of the grid $P_n \times P_n$, where P_n is the path on n vertices, that Theorem 1 is the best possible, since $\nu_1(P_n \times P_n) = \Theta(n^2 \log n)$. This example hinges on a general lower bound for $\nu_1(G)$ that we are going to present now.

We say that $f(x)$ is an *isoperimetric function* for G, if for any k-vertex subset U of V and $k \leq |V|/2$, there are at least $f(k)$ edges between U and $V \setminus U$. We require that $f(0) = 0$. Define the difference function of f, denoted by Δf as

$$\Delta f(i) = f(i+1) - f(i)$$

for any $i = 0, 1, ..., \lfloor \frac{|V|}{2} \rfloor - 1$, and set

$$\Delta^2 f(i) = (\Delta(\Delta f))(i) = \Delta f(i+1) - \Delta f(i),$$

for any $i = 0, 1, ..., \lfloor \frac{|V|}{2} \rfloor - 2$. In [15] we have found the following lower bound for the outerplanar crossing number.

Theorem 2. *Assume that $f(x)$ is an isoperimetric function for $G = (V, E)$, $|V| = n$ and Δf is non-negative and decreasing till $\lfloor \frac{n}{2} \rfloor - 1$. Then*

$$\nu_1(G) \geq -\frac{n}{8} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 2} f(j)\Delta^2 f(j) - \frac{1}{2} \sum_{v \in V} d_v^2. \tag{1}$$

In this paper we show an extension of Theorem 2 (Theorem 3), which allows three different kind of improvement on Theorem 2. We will make use of all of them in Section 6. In particular, one extension relaxes the condition that $\Delta(f)$ is decreasing till $\lfloor \frac{|V|}{2} \rfloor - 1$. This extension enables us to establish a near tight lower bound for the outerplanar crossing number of the hypercube. Note that Theorem 2 cannot be applied to the hypercube, since its isoperimetric function is *not* increasing till $\lfloor |V|/2 \rfloor - 1$. We also show in Subsections 6.1 and 6.5 that even if the isoperimetric function *is* increasing till this point, our extension can improve the lower bound of Theorem 2 by a constant multiplicative factor.

In addition to the square lattice $P_n \times P_n$, [15] showed that "fat finite chunks" of the hexagonal lattice graph also have a logarithmic gap between $\mathrm{cr}(G) + \sum_v d_v^2$ and the outerplanar crossing number. The proof in [15] traced back this problem to that of the square lattice by ad hoc methods. Now we give a more direct proof using Theorem 2, by establishing isoperimetric functions.

In [15] the logarithmic gap between $\mathrm{cr}(G)+\sum_v d_v^2$ and the outerplanar crossing number was shown *only* for sparse graphs. We provide here two families of *dense* graphs (Theorems 11, 12) with the same logarithmic gap. The basic tool is providing new isoperimetric functions for these graphs (Theorems 7, 8).

We study the *circular arrangement problem*, which sets a lower bound for the usual linear arrangement problem. Recall that the linear arrangement problem requires the placement of the vertices of the graph into integer positions, and minimizes the sum of edge lengths over all placements. The circular arrangement problem requires the placement of the vertices of the graph into equidistant positions on a circle of perimeter $|V(G)|$, and minimizes the sum of lengths of paths on the circle, into which the edges of the graph G are embedded on the circle, over all placements; and edges are embedded onto the shorter side of the circle. The circular arrangement problem has been introduced recently by Ching-Jung Guu [3], who solved in his thesis the circular arrangement problem for the hypercube, and Bezrukov and Schroeder [1], who showed that for trees the solutions for the linear arrangement problem and the circular arrangement problem are the same. The *generalized F-linear arrangement* and *generalized F-circular arrangement problems* assume a given function $F(x)$, and instead of summing up edge length, sum up $F(x)$ evaluated at the edge lengths. Probably the first occurence of the generalized F-linear arrangement problem was in the paper of Crimmins, Horwitz, and Palermo [5], who solved this problem in the case of $F(x) = x^2$ for the hypercube. Juvan and Mohar [9] studied the generalized F-linear arrangement problem for $F(x) = x^p$ for $p > 0$, and in particular for $p = 1, 2$, and developed heuristics.

We show how to adapt our method of Theorems 2 and 3 to prove new lower bounds for the circular arrangement problem. These lower bounds are particularly good when

$F(x)$ is near $x^{1/2}$.

2 A Better Lower Bound

Theorem 3. *Assume that we have a family of graphs $G = G_n$ on n vertices for infinitely many n such that $f(x) = f_n(x)$ is an isoperimetric function for $G = (V, E)$, $f(0) = 0$ and $f(x) > 0$ for $x \geq 1$. For the sequences $0 \leq s = s(n) \leq \lfloor \frac{n}{2} \rfloor - 1$ and $0 \leq s_0 = s_0(n) \leq s(n)$, assume that Δf is non-negative and decreasing till s and for each $s \leq x \leq \lfloor \frac{n}{2} \rfloor$ we have $f(x) \geq f(s_0)$. Define*

$$m_{f,s}(l) = \frac{f(l)}{f(\min\{\lfloor \frac{l}{2} \rfloor + 1, s+1\})} \quad \text{and} \quad \kappa(f, s_0, s) = \min_{s_0 \leq l \leq \lfloor \frac{n}{2} \rfloor} m_{f,s}(l). \tag{2}$$

Assume that $\kappa = \kappa_n$ has a universal upper bound as $n \to \infty$, and in addition, as $n \to \infty$,

$$ns_0 = o(|E(G)|) \quad or \quad f(s_0)|E(G)| = o\Big(\sum_{\substack{uv \in E: \\ l(u,v) \geq s_0}} f(l(u,v)) \Big). \tag{3}$$

Then we have

$$\nu_1(G) \geq -\kappa(f, s_0, s)(1 - o(1)) \cdot \frac{n}{8} \sum_{j=1}^{s-1} f(j)\Delta^2 f(j) - \frac{1}{2} \sum_{v \in V} d_v^2. \tag{4}$$

Proof. We follow the proof in [15] and modify it where it is necessary. Let D be a outerplanar drawing of G. Without loss of generality we may assume that vertices in D are placed on the perimeter of the unit circle in equidistant positions. Label the vertices by $0, 1, 2, ..., n-1$ according to their cyclic order. For simplicity, we will often identify a vertex with its corresponding integer. For $u, v \in V$ define the *distance* $l(u, v)$ between them by

$$l(u, v) = \min\{|u - v|, n - |u - v|\}. \tag{5}$$

Let us observe that (3) implies that

$$\begin{aligned} \sum_{\substack{uv \in E: \\ l(u,v) < s_0}} f(l(u,v)) \leq \sum_{\substack{uv \in E: \\ l(u,v) < s_0}} f(s_0) &\leq f(s_0) \min\Big(ns_0, |E(G)| \Big) \\ &= o\Big(\sum_{\substack{uv \in E \\ l(u,v) \geq s_0}} f(l(u,v)) \Big), \end{aligned} \tag{6}$$

where the last equality follows from the fact that for $x \geq s_0$ we have $f(x) \geq f(s_0)$.

For any $uv \in E$, let $c(u, v)$ denote the number of crossings of the edge uv with other edges in D, and observe, as in [15], that $c(u, v) \geq f(l(u, v)) - d_u - d_v$. Let $c(D)$ denote

the number of crossings in the drawing D. We conclude that

$$\begin{aligned} c(D) &= \frac{1}{2}\sum_{uv\in E} c(u,v) \geq \frac{1}{2}\sum_{uv\in E}(f(l(u,v)) - d_u - d_v) \\ &= \frac{1}{2}\sum_{uv\in E} f(l(u,v)) - \frac{1}{2}\sum_{v\in V} d_v^2. \end{aligned} \tag{7}$$

We say that edge $uv \in E$ in the drawing D covers a vertex i if the unique shortest path between u and v (using only the edges on the boundary of the convex n-gon) contains i. If the shortest path is not unique (this happens if $n = 2l(u,v)$), then we pick arbitrarily one of the two shortest paths, and declare its vertices be covered by the uv edge. (Note that an edge covers its endpoints.) For any edge $e = uv$ and any vertex i define $load_{u,v}(i)$, as

$$load_{u,v}(i) = \begin{cases} \Delta f\Big(\min\{l(u,i), l(i,v)\}\Big) & \text{if } e \text{ covers } i, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Let $i \in V$. For $0 \leq t$, $E_{i,t}$ to be the set of all edges $uv \in E$ covering vertex i in D such that $\min\{l(i,u), l(i,v)\} \leq t$. Observe that $E_{i,j-1} \subseteq E_{i,j}$. Note that for any $i \in V$, and any $uv \in E_{i,j} \setminus E_{i,j-1}$, we have that i is at distance j from one of u and v, and at distance at least j from the other one. Therefore, for any $i \in V$, and any $uv \in E_{i,j} \setminus E_{i,j-1}$, we have $load_{u,v}(i) = \Delta f(j)$, according to the definition of the load. Let k_t denote $\sum_{i\in V} |E_{i,t}|$.

It is easy to see that for any $uv \in E$

$$\sum_{i\in V:\ uv\in E_{i,s}} load_{u,v}(i) \leq 2 \sum_{j=0}^{\min\left(\left\lfloor \frac{l(u,v)}{2}\right\rfloor, s\right)} \Delta f(j) = 2f\left(\min\left(\left\lfloor \frac{l(u,v)}{2}\right\rfloor, s\right)+1\right). \tag{9}$$

(The inequality uses the fact that $\Delta f \geq 0$ till s.)

We have

$$\begin{aligned} \sum_{uv\in E}\sum_{i:\ uv\in E_{i,s}} load_{u,v}(i) &= \sum_{i\in V}\sum_{j=0}^{s}\sum_{uv\in E_{i,j}\setminus E_{i,j-1}} load_{u,v}(i) \\ &= \sum_{i\in V}\sum_{uv\in E_{i,0}} load_{u,v}(i) + \sum_{j=1}^{s}\sum_{i\in V}\sum_{uv\in E_{i,j}\setminus E_{i,j-1}} load_{u,v}(i) \\ &= k_0\Delta f(0) + \sum_{j=1}^{s}(k_j - k_{j-1})\Delta f(j), \end{aligned}$$

where the last equality is obtained by observing that the number of terms in the sum $\sum_{i\in V}\sum_{uv\in E_{i,j}\setminus E_{i,j-1}} load_{u,v}(i)$, is $k_j - k_{j-1}$. It follows that

$$\sum_{uv\in E} 2f\left(\min\left(\left\lfloor \frac{l(u,v)}{2}\right\rfloor, s\right)+1\right) \geq \sum_{j=0}^{s-1} k_j(\Delta f(j) - \Delta f(j+1)) + k_s\Delta f(s). \tag{10}$$

Note that up to (10) we did not use the assumption that Δf is decreasing, we used only the fact that Δf is non-negative till s. Since $k_s \Delta f(s) \geq 0$, we can drop the last term from the lower bound in (10). We also argue—as in [15]—that for all $j \leq n/2$,

$$k_j \geq \frac{1}{2} n f(j). \tag{11}$$

To see this, consider any j consecutive integers $i, i+1, ..., i+j-1$. Then at least $f(j)$ edges leave this j-set, and those edges must cover either i or $i+j-1$. We may have counted some cases twice, since a vertex i is an endpoint of two intervals of j, and if an edge goes from the first interval to the second, then this edge is counted twice covering i.

Definition (2) implies that for $l(u,v) \geq s_0$

$$f\left(\min(\left\lfloor \frac{l(u,v)}{2} \right\rfloor, s)+1\right) \leq \frac{1}{\kappa(f,s_0,s)} f\Big(l(u,v)\Big). \tag{12}$$

Combining (6), (12), and the universal boundedness of $\kappa = \kappa_n$, we obtain

$$\sum_{uv \in E} f\left(\min(\left\lfloor \frac{l(u,v)}{2} \right\rfloor, s)+1\right) \leq \frac{1+o(1)}{\kappa(f,s_0,s)} \sum_{uv \in E} f\Big(l(u,v)\Big). \tag{13}$$

We conclude using (10), (11), (13) and (7), that

$$\begin{aligned} -\frac{n}{2} \sum_{j=0}^{s-1} f(j) \Delta^2 f(j) &\leq 2 \sum_{uv \in E} f\left(\min(\left\lfloor \frac{l(u,v)}{2} \right\rfloor, s)+1\right) \\ \leq \frac{2+o(1)}{\kappa(f,s_0,s)} \sum_{uv \in E} f\Big(l(u,v)\Big) &\leq \frac{4+o(1)}{\kappa(f,s_0,s)} \Big(c(D) + \frac{1}{2} \sum_v d_v^2\Big). \end{aligned}$$

Note that in the first inequality we used (11) and the condition that $\Delta f(j) - \Delta f(j+1) \geq 0$, i.e. that Δf is decreasing till s. This, together with the fact that $f(0) = 0$, finishes the proof. □

Of course, the choice $s_0 = 0$ is always possible, but then $l = 1$ in (2) does not allow to have $\kappa > 1$, which is our goal to obtain improvement over (1) by (4). Alternatively, if $f(x)$ is a smooth function, one can get a more convenient estimation than in Theorem 3.

Theorem 4. *Under the assumptions of Theorem 3, with additional assumptions that for an $s \leq \lfloor \frac{n}{2} \rfloor - 2$ on the interval $(0, s+1)$ f' and f'' exist, $f' \geq 0$, $f'' \leq 0$ and increasing, one can change the right hand side of (4) to*

$$\nu_1(G) \geq -\frac{n}{8} \kappa(f,s_0,s)(1-o(1)) \int_0^{s-1} f(x) f''(x+3) dx - \frac{1}{2} \sum_{v \in V} d_v^2. \tag{14}$$

Proof. Recall Taylor's formula with remainder:

$$\begin{aligned} f(j+2) &= f(j+1)+f'(j+1)+\frac{1}{2}f''(\xi) \qquad (j+1<\xi<j+2), \\ f(j) &= f(j+1)-f'(j+1)+\frac{1}{2}f''(\eta) \qquad (j<\eta<j+1), \\ & \Delta f(j)-\Delta f(j+1) = -\frac{1}{2}(f''(\xi)+f''(\eta)) > -f''(j+2). \end{aligned}$$

Now

$$-\sum_{j=1}^{s-1} f(j)\Delta^2 f(j) \geq -\sum_{j=1}^{s-1} f(j)f''(j+2), \tag{15}$$

and for any $1 \leq j \leq s-1$, one has

$$-f(j)f''(j+2) \geq -\int_{j-1}^{j} f(x)f''(x+3)dx. \tag{16}$$

(14) is easily obtained from (4) using (15), and (16). □

Even if $f(x)$ is not as smooth as required above, Theorem 4 still applies with slight modification, if we allow appropriate error terms arising at the "bad" points of $f(x)$. There are many ways to handle this problem. We may relax the additional assumptions of Theorem 4 as follows. f' and f'' may be undefined in a bounded number of points in $(0, s+1)$, but $f' \geq 0$, $f'' \leq 0$ and f'' must be increasing in each of the subintervals between the special points. Assume that $\max_{[1,s+1]} |f_n(x)\Delta^2 f_n(x)|$ and $\max_{[1,s+1]} |f_n(x)f_n''(x+3)|$ are little-oh the right-hand side of (14) as $n \to \infty$. Then Theorem 4 still holds. This relaxation of Theorem 4 allows to prove Theorems 11 and 12 by integrating the piecewise smooth isoperimetric functions from Theorems 7 and 8. Similar relaxation can be given for the conditions of Theorem 5 as well.

3 Circular Arrangement Problem

We define the generalized F-linear arrangement problem as follows. Let us be given a non-negative and increasing real function $F(x)$. Let h be a bijection between $V(G)$ and the set of integers $\{1, 2, ..., |V|\}$. Define the *generalized F-linear arrangement value* as

$$L_F(h, G) = \sum_{uv \in E(G)} F(|h(u) - h(v)|). \tag{17}$$

The *generalized F-linear arrangement problem* asks for

$$L_F(G) = \min_h L_F(h, G) = \min_h \sum_{uv \in E(G)} F(|h(u) - h(v)|). \tag{18}$$

We define similarly the generalized F-circular arrangement problem as follows. Let us be given a non-negative and increasing real function $F(x)$. Let h be a bijection between

$V(G)$ and points $\{1, 2, ..., |V|\}$ placed equidistantly on a circle in this order. Define the *generalized F-circular arrangement value* as

$$L_F^o(h, G) = \sum_{uv \in E(G)} F(l(h(u), h(v))), \tag{19}$$

where l is the distance function defined in (5). The *generalized F-linear arrangement problem* asks for

$$L_F^o(G) = \min_h L_F^o(h, G) = \min_h \sum_{uv \in E(G)} F(l(h(u), h(v))). \tag{20}$$

It is clear that $L_F^o(h, G) \leq L_F(h, G)$, and consequently $L_F^o(G) \leq L_F(G)$. Therefore, it is of interest to set lower bounds on the generalized F-circular arrangement problem.

Theorem 5. *Assume that we have a family of graphs $G = G_n$ on n vertices for infinitely many n, $f(x) = f_n(x) \geq 0$ is an isoperimetric function for $G = (V, E)$. Assume that $F(x) > 0$ for $x \geq 1$ and $F(0) = 0$, a function not dependent on n. For the sequences $0 \leq s = s(n) \leq \lfloor \frac{n}{2} \rfloor - 1$ and $0 \leq s_0 = s_0(n) \leq s(n)$, assume that ΔF is non-negative and decreasing till ∞. Define*

$$m_{F,s}(l) = \frac{F(l)}{F(\min\{\lfloor \frac{l}{2} \rfloor + 1, s + 1\})} \quad \text{and} \quad \kappa(F, s_0, s) = \min_{s_0 \leq l \leq \lfloor \frac{n}{2} \rfloor} m_{F,s}(l).$$

Assume that $\kappa = \kappa_n$ has a universal upper bound as $n \to \infty$, and in addition, as $n \to \infty$, $ns_0 = o(|E(G)|)$ or $F(s_0)|E(G)| = o\Big(\sum_{\substack{uv \in E: \\ l(u,v) \geq s_0}} F(l(u, v))\Big)$. Then we have

$$L_F^o(G) \geq -\kappa(F, s_0, s)(1 - o(1)) \cdot \frac{n}{4} \sum_{j=0}^{s-1} f(j) \Delta^2 F(j). \tag{21}$$

Furthermore, with additional assumptions that for an $s(n) \leq \lfloor \frac{n}{2} \rfloor - 2$, on the interval $(0, s + 1)$ F' and F'' exist, $F' \geq 0$, $F'' \leq 0$ and increasing, one can change the right hand side of (21) to

$$-\kappa(F, s_0, s)(1 - o(1)) \cdot \frac{n}{4} \int_1^{s-1} f(x) F''(x + 3) dx. \tag{22}$$

Proof. Mutatis mutandis, we follow the proof of Theorem 3. We define the *load* as in (8), but with Δ_F instead of Δ_f. Formula (9) is substituted by

$$\sum_{i \in V : uv \in E_{i,s}} load_{u,v}(i) \leq 2 \sum_{j=0}^{\min\{\lfloor \frac{l(u,v)}{2} \rfloor, s\}} \Delta_F(j) = 2F\left(\min\left(\left\lfloor \frac{l(u, v)}{2} \right\rfloor, s\right) + 1\right). \tag{23}$$

We obtain

$$\sum_{uv \in E} F(l(u, v)) \geq \kappa(F, s_0, s)(1 - o(1)) \sum_{uv \in E} F\left(\min\left(\left\lfloor \frac{l(u, v)}{2} \right\rfloor, s\right) + 1\right)$$

as we obtained (13). Using (23), the partial summation leading to (10), and formula (11) as it is, we obtain:

$$\frac{1}{2}\sum_{uv\in E}\sum_{i\in V:\ uv\in E_{i,s}} load_{u,v}(i) \geq -\frac{n}{4}\sum_{j=0}^{s-1} f(j)\Delta^2 F(j).$$

Finally, (22) is easily obtained from (21) using Taylor's formula with remainder, like in the argument in the previous section. □

4 Citing Isoperimetric Inequalities

It is clear that the complete graph K_n has isoperimetric function

$$f(x) = x(n-x). \tag{24}$$

For the hypercube Q_n on 2^n vertices, Chung et al. [4] established the isoperimetric function

$$f(x) = x(n - \log_2 x). \tag{25}$$

Bollobás and Leader [2] established the isoperimetric function

$$f(x) = \sqrt{2x} \tag{26}$$

for the $n \times n$ grid $P_n \times P_n$, i.e. the Cartesian product of two n-vertex paths P_n. More generally, they established the isoperimetric function $f(x)$

$$f(x) = \begin{cases} \sqrt{2x} & \text{if } x \leq \frac{1}{2}n^2; \\ n & \text{if } \frac{1}{2}n^2 \leq x \leq \frac{kn}{2} \end{cases} \tag{27}$$

for the $n \times k$ grid $P_n \times P_k$, i.e. the Cartesian product of an n-vertex path with a k-vertex path, for $n \leq k$. Tillich [21] has made a substantial study of isoperimetric inequalities in Cartesian product graphs. For K_p^n, the n-th Cartesian power of the complete graph K_n, he proved

$$f(x) = (p-1)x(n - \log_p x), \tag{28}$$

which gives back the isoperimetric inequality (25) for the hypercube for $p = 2$. For the Cartesian power P^n of the Petersen graph P, Tillich [21] provided two isoperimetric functions that are incomparable:

$$f_1(x) = 2x(5^n - x)/5^n \tag{29}$$

$$f_2(x) = 2x(n - \log_5 x). \tag{30}$$

Recall the definition of the edge-forwarding index $\pi(G)$ of a graph G. For every ordered pair of vertices (a, b), where $a \neq b \in V(G)$, assign a path of G connecting a to b. The congestion of an edge is the number of paths using this edge, and the congestion of the

path system is the maximum congestion of edges. The edge-forwarding index $\pi(G)$ of the graph G is the minimum congestion of any such path system. Now, we have immediately from the definition the following isoperimetric function:

$$f(x) = \frac{2x(n-x)}{\pi(G)}, \tag{31}$$

where n is the number of vertices. Note that the edge-forwarding index is also studied under the name optimal integral concurrent multicommodity flow.

5 Proving Relevant Isoperimetric Inequalities

In this section we prove isoperimetric inequalities for one sparse and two dense graphs closely related to the grid. By applying Theorem 2 we get tight lower bounds in section 6.

First, we are going to study a relative of the grid, which we call a *rhombus of hexagons*, see Fig. 1.

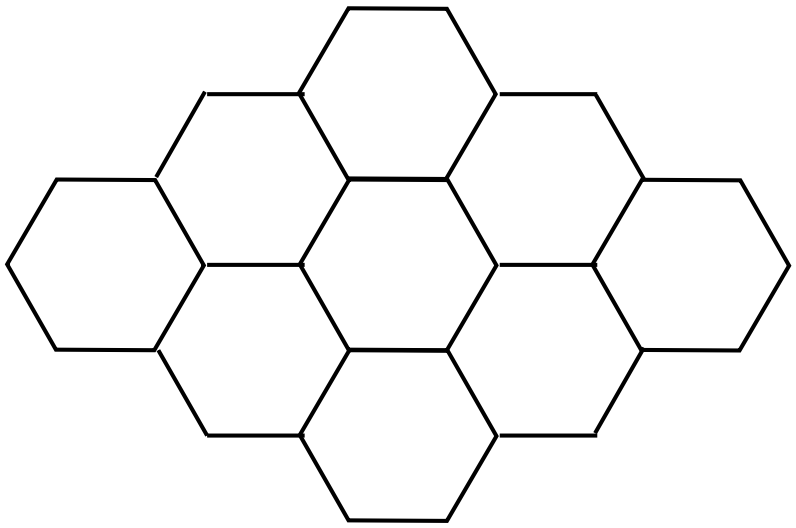

Figure 1: The 3×3 rhombus of hexagons.

Theorem 6. *The following is an isoperimetric function for a rhombus of hexagons with $n > n_0$ vertices:*

$$f(x) = \frac{1}{3}\sqrt{x}. \tag{32}$$

Proof. Observe that by contracting all horizontal edges in the rhombus of hexagons, we obtain an ordinary square grid. With two exceptions at the corners of the grid, vertices of the square grid have two pre-images, and the exceptional vertices have just one.

Let us be given a subset X of vertices of the rhombus of hexagons, with $x = |X|$, $x \le n/2$. Assume that $x = a + 2b + c$, where a denotes the number of vertices from X which occur alone on a horizontal edge, b denotes the number of pairs of vertices from X which occur on the same horizontal edge, and c denotes the number of exceptional vertices from X.

If $a \ge \frac{1}{3}\sqrt{x}$, then we have a horizontal edges between X and $\overline{X}$ in the rhombus of hexagons, and we are at home. Therefore we may assume $a \le \frac{1}{3}\sqrt{x} \le \frac{1}{3}\sqrt{n/2}$; and as we noted above, $c \le 2$.

Let X' denote the set of vertices in the square grid, which have at least one pre-image in X. Clearly $|X'| = a + b + c$. It is easy to check that edges between X' and $\overline{(X')}$ correspond to distinct edges between X and $\overline{X}$, except in the case shown in Fig. 2, where 2 edges in the square grid correspond to a single edge in the rhombus of hexagons.

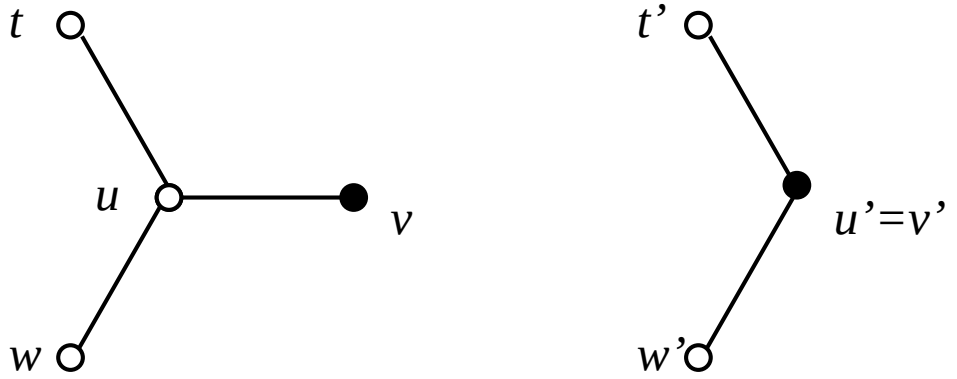

Figure 2: $t, u, w \in \overline{X},\ v \in X;\ t', w' \in \overline{(X')},\ u' = v' \in X'$. A single edge in $E(X, \overline{X})$ may correspond to two edges in $E(X', \overline{(X')})$.

Therefore, $|E(X, \overline{X})| \ge \frac{1}{2}|E(X', \overline{(X')})|$. The square grid has $\frac{n}{2} + 1$ vertices, and $|X'| \le \frac{n}{4} + \frac{1}{3}\sqrt{n/2} + 2$. Formula (26) applies to $|E(X', \overline{(X')})|$, if $|X'| \le \frac{n}{4}$; and since $|X'|$ can only slightly exceed $\frac{n}{4}$, for sufficiently large n, a slightly weaker formula holds, reducing 2 to 9/5: $|E(X', \overline{(X')})| \ge \sqrt{\frac{9}{5}(a + b + c)}$. Combining the observations yields $|E(X, \overline{X})| \ge \frac{1}{2}\sqrt{\frac{9}{5}(a + b + c)} \ge \frac{1}{2}\sqrt{\frac{9}{10}x} \ge \frac{1}{3}\sqrt{x}$. □

Next we study the isoperimetric inequality of two graphs closely related to the grid. Let $P(n, k)$ denote the graph that we obtain from an n-vertex path by joining vertices whose distance in the path is at most k. Let $G(n, k)$ denote the Cartesian product $P(n, k) \times P(n, k)$. We will find an isoperimetric function for $G(n, k)$ in the case when k is an even divisor of $2n$. To set lower bounds, we first study the isoperimetric problem in $G(\frac{k}{2}, \frac{k}{2}) = K_{\frac{k}{2}} \times K_{\frac{k}{2}}$.

Lemma 1. *Let $0 < c < 1/2$ and $2/3 \le d \le 1$. We put the vertices of $K_{\frac{k}{2}} \times K_{\frac{k}{2}}$ into $\frac{k}{2}$ rows and $\frac{k}{2}$ columns in the natural way. For any $X \subset V(K_{\frac{k}{2}} \times K_{\frac{k}{2}})$,*
(i) If a row S has the property, that $|X \cap S| \le (1-c)\frac{k}{2}$, then $|E(X \cap S, \overline{X} \cap S)| \ge \frac{ck}{2}|X \cap S|$.
(ii) If at least $c|X|$ elements of X fall into rows S such that $|X \cap S| \le (1-c)\frac{k}{2}$, then $|E(X, \overline{X})| \ge \frac{c^2 k}{2}|X|$.
(iii) If $|X| \le (1-2c)^3(\frac{k}{2})^2$, then $|E(X, \overline{X})| \ge \frac{c^2 k}{2}|X|$.
(iv) If $|X| \le \frac{1}{3}(\frac{k}{2})^2$, then $|E(X, \overline{X})| \ge \frac{k}{86}|X|$.
(v) If $|X| \le \frac{2}{3}(\frac{k}{2})^2$, then $|E(X, \overline{X})| \ge \frac{k}{501}|X|$.
(vi) If $|X| = d(\frac{k}{2})^2$, then $|E(X, \overline{X})| \ge \frac{k(1-d)}{86d}|X|$.

Proof. (i) is trivial, (ii) is immediate from (i). Parts (iv) and (v) easily follow from (iii) by setting $c = \frac{1}{2}[1 - (1/3)^{1/3}]$ and $c = \frac{1}{2}[1 - (2/3)^{1/3}]$, respectively. To show (iii), assume that the conditions of (ii) fail for the rows as well as for the colums (otherwise $|E(X, \overline{X})| \ge \frac{c^2 k}{2}|X|$). Then at least $(1-c)|X|$ vertices of X fall into rows S such that $|X \cap S| \ge (1-c)\frac{k}{2}$. Let a denote the number of rows S such that $|X \cap S| \ge (1-c) \cdot \frac{k}{2}$, and let A denote the subset of elements of X which lie on such rows. Similarly, at least $(1-c) \cdot |X|$ vertices of X fall into columns T such that $|X \cap T| \ge (1-c)\frac{k}{2}$. Let b denote the number of columns T with $|X \cap T| \ge (1-c)\frac{k}{2}$, and let B denote the subset of elements of X which lie on such columns. Clearly $|X| \ge (1-c)a\frac{k}{2}$ and $|X| \ge (1-c)b\frac{k}{2}$. Since the intersection of a rows and b columns contains ab vertices, by inclusion-exclusion we have:

$$|X| \ge |A| + |B| - ab \ge 2(1-c)|X| - \left(\frac{|X|}{(1-c)\frac{k}{2}}\right)^2$$

which implies $|X| \ge (1-2c)(1-c)^2\frac{k^2}{4} > (1-2c)^3\frac{k^2}{4}$, a contradiction.

To show (vi), note that $\frac{1-d}{d}|X| = |\overline{X}| \le \frac{1}{3}(\frac{k}{2})^2$, so from (iv) we obtain that $|E(X, \overline{X})| \ge \frac{k(1-d)}{d86}|X|$.

□

We also make use of the following rather technical result:

Lemma 2. *Put the vertices of $K_{\frac{k}{2}} \times K_k$ into $\frac{k}{2}$ rows and k columns in the natural way, and divide this graph into two disjoint $\frac{k}{2} \times \frac{k}{2}$ squares, A and B. Let X be a subset of the vertices such that $|X \cap A| \le \frac{k^2}{6}$ and $|X \cap B| > \frac{k^2}{6}$. Then the number of $(X, \overline{X})$-type edges not entirely in A is at least $\frac{k^3}{1032}$.*

Proof. Let $d = |X \cap B| \cdot \frac{4}{k^2}$, then by our assumptions $\frac{2}{3} < d \le 1$. By Lemma 1 (vi), $|E(X \cap B, \overline{X} \cap B)| \ge \frac{k^3(1-d)}{344}$. By construction, A and B share their rows. Let x_i (y_i) denote the proportion of elements from X in the i^{th} row of B (A). (With other words, the number of such elements is $\frac{kx_i}{2}$ and $\frac{ky_i}{2}$, respectively.) Then $\sum_{i=1}^{k/2} x_i = dk/2$, and $\sum_{i=1}^{k/2} y_i \le k/3$. We use the estimate

$$\sum_{i=1}^{k/2} x_i(1-y_i) \ge \sum_{i=1}^{k/2}(x_i - y_i) \ge (3d-2) \cdot \frac{k}{6},$$

and obtain that the number of $(X \cap B, \overline{X} \cap A)$ type edges is at least $\frac{k^2}{4} \sum_{i=1}^{k/2} x_i(1-y_i) \geq (3d-2)\frac{k^3}{24}$. Therefore the number of $(X, \overline{X})$ type edges not entirely in A is at least $\frac{k^3(1-d)}{344} + \frac{k^3(3d-2)}{24} = (126d-83)\frac{k^3}{1032} \geq \frac{k^3}{1032}$. □

Theorem 7. *Let k be an even divisor of $2n$. The following is an isoperimetric function for $G(n,k)$:*

$$f(x) = \begin{cases} \frac{k}{501} \cdot x & \text{if } 0 \leq x \leq (\frac{501k}{2064})^2, \\ \frac{k^2}{2064} \cdot \sqrt{x} & \text{if } (\frac{501k}{2064})^2 \leq x \leq n^2/2. \end{cases} \tag{33}$$

Proof. We make a $(2n/k) \times (2n/k)$ grid G' by cutting the original grid G into $(k/2) \times (k/2)$ squares. Let us be given a set X of vertices of G, $|X| = x$ $(0 \leq x \leq n^2/2)$. We have to find $f(x)$ edges in $G(n,k)$ leaving the set X. Every $v \in X$ falls into some $(2n/k) \times (2n/k)$ square in G'. Let us denote this square by S_v. Set the density of the square S_v as $d(S_v) = |X \cap S_v|/|S_v| = 4|X \cap S_v|/k^2$. We distinguish cases:

(i) v is a small vertex, (S_v is a small square) if $d(S_v) \leq \frac{2}{3}$

(ii) v is large vertex, (S_v is a large square) if $\frac{2}{3} < d(S_v)$.

Let x_S (x_L) denote the number of small (large) vertices of X. We have $x = x_S + x_L$. It follows from Lemma 1 (v) that the number of $(X, \overline{X})$ edges inside the small squares is at least $kx_s/501 = k(x - x_L)/501$.

Let us estimate the number of large squares of G'. Clearly their number n' falls between $x_L/(k/2)^2$ and $3x_L/[2(k/2)^2]$. According to the isoperimetric inequality for grids (26), the number of (large square, small square) pairs that share a common side is at least $\sqrt{2\min\{n', (2n/k)^2 - n'\}} \geq (2/k) \cdot \sqrt{2\min\{x_L, n^2 - 3x_L/2\}} \geq 2\sqrt{x_L}/k$. Lemma 2 tells us that at each such pair of squares there is at least $\frac{k^3}{1032}$ edges of type $(X, \overline{X})$ that are not inside the small square. Observe that every large square shares side edges with at most 4 small squares. Therefore, the number $(X, \overline{X})$ type edges not in a small square is at least $\frac{1}{4} \cdot \frac{k^2}{516} \cdot \sqrt{x_L}$. Consequently, the number of $(X, \overline{X})$ type edges is at least

$$T(x_L) = k\frac{x - x_L}{501} + \frac{k^2\sqrt{x_L}}{2064}. \tag{34}$$

As $T(x_L)$ is quadratic in $\sqrt{x_L}$, it is easy to see that $T(x_L) \geq \min\{T(0), T(x)\}$. $T(0) = kx/501$ and $T(x) = k^2\sqrt{x}/2064$, so $T(0) \leq T(x)$ when $x \leq \frac{501^2k^2}{2064^2}$, otherwise $T(0) > T(x)$, and so we arrived at (33). □

Let $H(n,k)$ denote the graph that we obtain by taking the $n \times n$ grid G in the plane, joining vertices whose distance is at most k

We establish an isoperimetric inequality for $H(n,k)$ when 3 divides k and k divides $3n$. As before, we start with 2 lemmas.

Lemma 3. *Let $0 \leq d \leq 1$. For any $X \subset V(K_{\frac{k}{3} \times \frac{k}{3}})$,*
(i) If $|X| \leq d(\frac{k}{3})^2$, then $E(X, \overline{X})| \geq \frac{(1-d)k^2}{9}|X|$.
(ii) If $|X| \leq \frac{2}{3}(\frac{k}{3})^2$, then $|E(X, \overline{X})| \geq \frac{k^2}{27}|X|$. □

Lemma 4. *Arrange the vertices of $K_{\frac{k}{3}\times\frac{2k}{3}}$ into $k/3$ rows and $2k/3$ columns and divide them into two squares, A and B. Let X be a subset of the vertices such that $|X\cap A|\le\frac{2k^2}{27}$ and $|X\cap B|>\frac{2k^2}{27}$. Then the number of $(X,\overline{X})$-type edges not in A is at least $4k^4/729$.*

Proof. Let d denote $9|X\cap B|/k^2$. Then, $|E(X\cap B,\overline{X}\cap B)|\ge\frac{d(1-d)k^4}{81}$. Also, $|E(X\cap B,\overline{X}\cap A)|\ge\frac{dk^2}{9}\cdot(\frac{k^2}{9}-|\overline{X}\cap A|)\ge\frac{dk^2}{9}\cdot\frac{k^2}{27}=\frac{dk^4}{243}$. The sum of these items is $k^4\frac{d+3d(1-d)}{243}\ge\frac{4k^4}{729}$. □

Theorem 8. *Let k,n be such that 3 divides k and k divides $3n$. The following is an isoperimetric function for $H(n,k)$:*

$$f(x)=\begin{cases}\frac{k^2x}{27} & \text{if } 0\le x\le\frac{k^2}{144},\\ \frac{k^3\sqrt{x}}{324} & \text{if } \frac{k^2}{144}\le x\le n^2/2.\end{cases}\tag{35}$$

Proof. We make a $(3n/k)\times(3n/k)$ grid G' by cutting the original grid G into $(k/3)\times(k/3)$ squares. Let us be given a set X of vertices of $H(n,k)$, $|X|=x$ $(0\le x\le n^2/2)$. We have to find $f(x)$ edges in $H(n,k)$ leaving the set X. Every $v\in X$ falls into some $(3n/k)\times(3n/k)$ square in G', we call this square S_v. Set the density of the square S_v as $d(S_v)=|X\cap S_v|/|S_v|=9|X\cap S_v|/k^2$. We distinguish cases:

(i) v is small vertex, (S_v is a small square) if $d(S_v)<2/3$

(ii) v is large vertex, (S_v is a large square) if $2/3\le d(S_v)$.

Let x_S and x_L denote the number of small (large) vertices of X. We have $x=x_S+x_L$. It follows from Lemma 3 that the number of $(X,\overline{X})$ type edges inside the small squares is at least $\frac{k^2(x-x_L)}{27}$.

Let us estimate the number of large squares of G'. Clearly their number n' falls between $x_L/(k/3)^2$ and $3x_L/[2(k/3)^2]$. According to the isoperimetric inequality for grids (26), the number of (small square, large square) pairs that share a side is at least $\sqrt{2\min\{n',(3n/k)^2-n'\}}\ge(3/k)\sqrt{x_L}$. As before, we get

$$|E(X,\overline{X})|\ge\frac{k^2(x-x_L)}{27}+\frac{k^3\sqrt{x_L}}{243}\ge\min\left(\frac{k^2x}{27},\frac{k^3\sqrt{x}}{324}\right),\tag{36}$$

and so we arrived at (35). □

We remark here about the tightness of the new isoperimetric inequalities studied in this Section. Theorem 6 is tight within a multiplicative factor of 3. To see this, consider a cut with a vertical line through k horizontal edges of the rhombus of hexagons, as the rhombus of hexagons is drawn as in Fig. 1. This cut separates k^2-1 vertices from the rest. For Theorems 7 and 8, an $\sqrt{x}\times\sqrt{x}$ subgrid shows the tightness of the isoperimetric functions within a constant multiplicative factor.

6 Specific Results

6.1 Warm-Up

The outerplanar crossing number of K_n is *exactly* $\binom{n}{4}$. This is a nice exercise that can be found in many texts in a somewhat different formulation. Let us see what kind of lower

bound our method can show. By (24), $f(x) = x(n-x)$, $f''(x) = -2$, $s = \lfloor n/2 \rfloor - 3$, and $\kappa(f, s_0, s) = \frac{4}{3} + o(1)$ by $s_0 = n^{1/3}$. Theorem 4 yields $\nu_1(K_n) \geq -\frac{n}{8}(\frac{4}{3} + o(1)) \int_1^{\lfloor n/2 \rfloor - 4} x(n-x)(-2)dx = \frac{n^4}{36}(1 + o(1))$, which is 2/3 of the truth. (On the other hand, Theorem 2 would yield only $\nu_1(K_n) \geq \frac{n^4}{48}$.)

6.2 Estimating with Edge-Forwarding Index

Theorem 9. *For any connected graph G, we have*

$$\nu_1(G) \geq \frac{n^4(1 - o(1))}{9\pi^2(G)} - \frac{1}{2}\sum_v d_v^2. \tag{37}$$

Note that the isoperimetric function (31) is only by a multiplicative factor of $\frac{2}{\pi(G)}$ off from the isoperimetric function (24) used in subsection 6.1. The multiplicative factor squares in that calculation, since we multiply with the second derivative. Therefore, we have for free from the calculations in subsection 6.1 that $\nu_1(G) \geq \frac{n^4}{36}(1 + o(1))(\frac{2}{\pi(G)})^2$, and Theorem 9 follows.

Note that exact values or good upper bounds for edge forwarding indices are known for many graph families, e.g. for the de Bruijn, Kautz, Butterfly and Cube-Connected-Cycles graphs [8, 16, 17], also for the star and complete transposition graphs [7] and, in general, for edge symmetric Cayley graphs [16, 17].
For comparison, let us see what kind of lower bound can be provided for $\mathrm{cr}(G)$ in terms of $\pi(G)$. Theorem 3.11 in [13] easily adapts to embedding of multigraphs. Embed $2K_n$ into G, where $2K_n$ denotes the complete graph on n vertices, where every edge comes with multiplicity two. Theorem 3.11 provides

$$\mathrm{cr}(G) \geq \frac{\mathrm{cr}(2K_n)}{\pi^2(G)} - \text{error term}. \tag{38}$$

However, it is well known that making k copies of all edges of a graph H to make multigraph kH, one has $\mathrm{cr}(kH) = k^2\mathrm{cr}(H)$, see [19]. Using this, (38) yields

$$\mathrm{cr}(G) \geq \frac{4\mathrm{cr}(K_n)}{\pi^2(G)} - \text{error term}. \tag{39}$$

It is known that $\mathrm{cr}(K_n) \leq (1 + o(1))\frac{n^4}{64}$ and equality is conjectured here [20]. Therefore, the lower bound for ν_1 in terms of edge forwarding index (37) is *always* better than the the lower bound for cr in terms of edge forwarding index (39).

6.3 Hypercubes and Powers of Complete Graphs

Theorem 10. *For the outerplanar crossing number of the $n-$dimensional hypercube $\nu_1(Q_n)$, we have $(.1914 + o(1)) \cdot 4^n \leq \nu_1(Q_n) \leq .5 \cdot 4^n$.*

Proof. Recall from [14] the drawing providing the upper bound. The drawing was constructed in the following way: put the vertices of Q_n on the spine of a one-page book according to the Gray-code order. Draw the edges in one page in the usual recursive way. The number of crossings in the drawing is bounded from above by $4^n/2$. By (25), $f(x) = x(n - \log_2 x)$, $f'(x) = n - \log_2 x - \frac{1}{\ln 2}$, $f''(x) = -\frac{1}{x \ln 2}$. Note that $f'(x) > 0$ only till $\frac{2^n}{e}$, so f is not increasing on the entire interval $[0, \frac{|V(G)|}{2} - 1]$; in other words Theorem 2 does not apply. However, Theorem 3 still applies. We choose $s = 2^n/e$. One obtains $\kappa(f, s_0, s) \geq 1 - o(1)$ by $s_0 = n^{1/3}$. Theorem 4 yields $\nu_1(Q_n) \geq (1 - o(1))\frac{2^n}{8} \int_1^{2^n/e-2} \frac{1}{(x+3)\ln 2} x(n - \log_2 x)dx = 4^n \frac{1}{4e(\ln 2)^2}(1 + o(1))$, which is more than $(.1914 + o(1)) \cdot 4^n$. □

The best lower bound for the crossing number of the hypercube is $\mathrm{cr}(Q_n) \geq \frac{1}{20}(1 - o(1))4^n$ [18]. This lower bound applies to $\nu_1(Q_n)$, but we are not aware of any improvement for ν_1. Analogues of Theorem 10 can be proved for K_p^n (as $Q_n = K_2^n$), using isoperimetric function (28) instead of (25)

6.4 Powers of Petersen Graphs

The isoperimetric function $f_1(x)$ in (29) is only a multiplicative factor $\frac{2}{5^n}$ off from the isoperimetric function of the complete graph K_{5^n}. Therefore, the calculations in subsection 6.1 apply and we obtain $\nu_1(P^n) \geq \frac{5^{4n}}{36} \cdot (\frac{2}{5^n})^2(1 - o(1)) = (\frac{1}{9} - o(1))5^{2n} \geq .1111 \cdot 5^{2n}$. The isoperimetric function $f_2(x)$ in (30) is rather similar to the isoperimetric function (25). $f_2(x)$ increases till $5^n/e$, and like in the proof of Theorem 10, we can take $s_0 = n^{1/3}$, $s = 5^n/e$, and obtain $\kappa = 1$. A calculation similar to the proof of Theorem 10 shows that $\nu_1(P^n) \geq \frac{5^{2n}}{e(\ln 5)^2}(1 - o(1)) \geq .1420 \cdot 5^{2n}$. The conclusion is, that the use of $f_2(x)$, which is *not* increasing till $5^n/2$, yields better results. The hypercube is not the only example where we have to deal with isoperimetric functions taking maximum before the middle, since K_p^n and $f_2(x)$ for P^n behaves similarly. However, the exact isoperimetric function of P^n is not known. The "natural" idea for improvement taking $f_3(x) = \max(f_1(x), f_2(x))$ fails, since this function is not concave down (see Fig. 5 in [21]). We have no idea how to find an $f_4(x) \leq f_3(x)$ that satisfies the conditions of Theorem 4 and maximizes the lower bound (14). Such an $f_4(x)$ might provide a better lower bound for ν_1 than any of $f_1(x)$, $f_2(x)$.

6.5 Theorem 3 is stronger than Theorem 2

As we have seen in the case of the hypercube, unlike Theorem 2, Theorem 3 remains applicable when the isoperimetric function is not increasing till $\lfloor \frac{|V(G)|}{2} \rfloor - 1$. When both theorems are applicable, one cannot expect more than a multiplicative constant gain by using Theorem 3, though, as we have seen above, we can obtain such a gain. In the two applications above, the optimal choice of s was always the maximum possible. A natural problem is the following: are there examples in which the optimal choice of s is not the maximum possible allowed by the conditions?

Consider the rectangular grid graph $P_n \times P_k$ with $k = \lfloor n\sqrt{\ln n} \rfloor$. Simple calculation based on (27) shows, that for any sufficiently small $\epsilon > 0$ the choice $s = \lfloor (1-\epsilon)n^2/2 \rfloor$ beats $s = \lfloor nk/2 \rfloor - 1$ in Theorem 3. Indeed, whatever is our choice for s, $\Delta^2 f(j) = 0$ for any $j > n^2/2$. Approximating the summation with integral in (4), we have

$$\nu_1(P_n \times P_k) \geq -\kappa(f, s_0, s)(1 - o(1)) \cdot \frac{nk}{8} \int_1^{\min(s, n^2/2)} f(x) f''(x) dx - \frac{1}{2} \sum_{v \in V} d_v^2. \quad (40)$$

If $s = \lfloor nk/2 \rfloor - 1$, we have to take $\kappa = 1$, since some $m_{f,s}(l)$ values in (2) are equal to 1 (say $l = 2n^2$), but $\kappa \geq 1$. However, for $s = \lfloor (1-\epsilon)n^2/2 \rfloor$ and any $s_0 \to \infty$, we can take $\kappa = \frac{1}{\sqrt{1-2\epsilon}}$, which is realized by $f(l)/f(s) = n/\sqrt{2s}$ for any $l > 2s$, according to (27). Our claim boils down to the comparison of the integrals

$$\frac{1}{\sqrt{1-2\epsilon}} \int_1^{(1-\epsilon)n^2/2} \frac{dx}{\sqrt{2x}} \quad \text{and} \quad \int_1^{n^2/2} \frac{dx}{\sqrt{2x}}.$$

The first integral is asymptotically bigger as $n \to \infty$, and this confirms our claim. We still have to show that we can select an $s_0 \to \infty$ for $s = \lfloor (1-\epsilon)n^2/2 \rfloor$ so that we satisfy (3). We claim that any $s_0 = \sqrt{\ln n}$ suffices. We have

$$f(s_0)|E(G)| < \sqrt{2s_0} \cdot 2nk = 2s_0 n^2 \ln n = 2n^2 (\ln n)^{3/4}. \quad (41)$$

On the other hand, $P_n \times P_k$ contains a $P_n \times P_n$ subgraph. Observe that for $x \leq n^2/2$ the *same* $f(x) = \sqrt{2x}$ is isoperimetric function for both $P_n \times P_k$ and $P_n \times P_n$. We already know [15] for $P_n \times P_n$ that

$$\sum_{e \in E(P_n \times P_n)} f(l(e)) = \Omega(n^2 \ln n). \quad (42)$$

Formula (42) must be valid if the summation is extended for $e \in E(P_n \times P_k)$. Furthermore, (42) must still be valid, if restricted for edges of length at least s_0:

$$\sum_{e \in E(P_n \times P_k):\ l(e) \geq s_0} f(l(e)) = \Omega(n^2 \ln n). \quad (43)$$

The reason is that the last two summations differ by at most $|E(P_n \times P_k)| \leq 2nk = 2n^2\sqrt{lnn}$ terms, and each term is at most $f(s_0) = \sqrt{2}(\ln n)^{1/4}$, and hence the two summations differ by $O(n^2(\ln n)^{3/4}) = o(n^2 \ln n)$. Finally, observe that (41) and (43) together prove the second case of (3) for the graph $P_n \times P_k$.

6.6 Rhombus of Hexagons

In [15] we did not establish an isoperimetric function for the rhombus of hexagons, instead, we used an ad hoc technique (a variant of embedding) to show that the outerplanar crossing number of the rhombus of hexagons is $\Omega(n^2 \log n)$. Based on the isoperimetric function in (32), Theorem 2 or 3 yields this result immediately.

6.7 Some Dense Geometric Graphs

Consider $G(n,k)$ the graph defined in Section 5. We claim that $\mathrm{cr}(G(n,k)) = \Theta(k^4n^2)$ for $1 \le k \le n$. The upper bound follows from a natural grid-like drawing, and the lower bound follows from a bisection width argument (see [11]), since the (1/3-2/3) bisection width of $G(n,k)$ is at least $\Omega(k^2n)$, which immediately follows from Theorem 7, for any $k \le n$. We show here that $\nu_1(G(n,k))$ is bigger by a $\log n$ factor than $\mathrm{cr}(G(n,k))$.

Theorem 11. *For every $\epsilon > 0$ and $k \le n^{1-\epsilon}$, as $n \to \infty$, we have*

$$\nu_1(G(n,k)) = \Theta(k^4n^2 \log n). \tag{44}$$

Proof. The upper bound follows from Theorem 1. The lower bound follows from Theorem 2 and the isoperimetric inequality in Theorem 7, (33). □

Consider $H(n,k)$ the graph defined in section 5. It is known that $\mathrm{cr}(H(n,k)) = \Theta(n^2k^6)$. The upper bound follows from the natural straight line drawing of the graph, with infinitesimally small modification, to make sure that edges do not pass through vertices. The lower bound follows from a bisection width argument (see [11]), since the (1/3-2/3) bisection width of $H(n,k)$ is at least $\Omega(k^3n)$, which immediately follows from Theorem 8, for any $k \le n$. We show here that $\nu_1(H(n,k))$ is bigger than $\mathrm{cr}(H(n,k))$ by a $\log n$ factor.

Theorem 12. *For every $\epsilon > 0$ and $k \le n^{1-\epsilon}$, as $n \to \infty$, we have*

$$\nu_1(H(n,k)) = \Theta(k^6n^2 \log n). \tag{45}$$

Proof. The upper bound follows from Theorem 1. The lower bound follows from Theorem 2 and the isoperimetric inequality in Theorem 8, (35). □

Remark 1. *Note that Theorems 11, 12 fail for $k = n$, and therefore a condition like $k \le n^{1-\epsilon}$ is necessary.*

6.8 Linear Arrangement in Norms less than 1

Observe that $F(x) = x^\alpha$ with $0 < \alpha < 1$ satisfies all the requirements of Theorem 5. For simplicity, we abbreviate $L^o_F(G)$ by $L^o_\alpha(G)$ ($L_F(G)$ by $L_\alpha(G)$) for $F(x) = x^\alpha$. We show as an application

Theorem 13. *For the grid $G = P_n \times P_n$, we have*

$$L_\alpha(G) = \begin{cases} \Theta(n^2) & \text{if } 0 < \alpha < 1/2; \\ \Theta(n^2 \log n) & \text{if } \alpha = 1/2; \\ \Theta(n^{2\alpha+1}) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Proof. Apply the isoperimetric inequality (26) and formula (22) from Theorem 5 to obtain the lower bound for $L^o_\alpha(G)$, and hence for $L_\alpha(G)$. The upper bound is given by the following recursive layout of the grid. Divide the grid $P_n \times P_n$ into four subgrids $P_{\frac{n}{2}} \times P_{\frac{n}{2}}$ by removing $2n$ edges, lay down the the subgrids recursively and add the $2n$ edges back. We get the recurrence relation: $L_\alpha(P_n \times P_n) \leq 4L_\alpha(P_{\frac{n}{2}} \times P_{\frac{n}{2}}) + 2n^{2\alpha+1}$. The solution of the recurrence relation is

$$L_\alpha(P_n \times P_n) = \begin{cases} O(n^2) & \text{if } 0 < \alpha < 1/2; \\ O(n^2 \log n) & \text{if } \alpha = 1/2; \\ O(n^{2\alpha+1}) & \text{if } 1/2 < \alpha < 1. \end{cases}$$

□

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