Fans and Bundles in the Graph of Pairwise Sums and Products

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Abstract

Let $G_+^\times$ be the graph on the vertex-set the positive integers $\mathbb{N}$, with n joined to m if $n \neq m$ and for some $x, y \in \mathbb{N}$ we have $x+y=n$ and $x \cdot y = m$. A pair of triangles sharing an edge (*i.e.*, a K_4 with an edge deleted) and containing three consecutive numbers is called a 2-fan, and three triangles on five numbers having one number in common and containing four consecutive numbers is called a 3-fan. It will be shown that $G_+^\times$ contains 3-fans, infinitely many 2-fans and even arbitrarily large "bundles" of triangles sharing an edge. Finally, it will be shown that $\chi\big(G_+^\times\big) \geq 4$.

1 Motivation

If we colour the positive integers $\mathbb{N}$ with finitely many colours, then, by Ramsey's Theorem, we find two one-to-one sequences $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that each set

$$\big\{x_n + x_m : n, m \in \mathbb{N},\ n \neq m\big\} \ \ \textit{and} \ \ \big\{y_n \cdot y_m : n, m \in \mathbb{N},\ n \neq m\big\}$$

is monochromatic. On the other hand, it is known (*cf.* [HS, Chapter 17.2]) that one can colour the positive integers with finitely many colours in such a way that there is no one-to-one sequence $\langle x_n \rangle_{n=1}^\infty$ such that

$$\big\{x_n + x_m : n, m \in \mathbb{N},\ n \neq m\big\} \cup \big\{x_n \cdot x_m : n, m \in \mathbb{N},\ n \neq m\big\}$$

is monochromatic. However, it is not known if for any finite colouring of $\mathbb{N}$ there are distinct x and y in $\mathbb{N}$, such that $x+y$ and $x \cdot y$ are monochromatic (see also [HS, Question 17.18]). Moreover, it is not even known if there are x and y in $\mathbb{N}$, not both equal to 2, such that $x+y$ and $x \cdot y$ are monochromatic.

Let us state this problem in terms of graphs: Let $G_+^\times = (\mathbb{N}, E)$ be the graph on the vertex-set the positive integers $\mathbb{N}$, with $(n,m) \in E$ if $n \neq m$ and for some $x, y \in \mathbb{N}$ we have $n = x + y$ and $m = x \cdot y$. Notice that for all $n \in \mathbb{N}$, $(n, n+1) \in E$ (this is just because $n = 1 \cdot n$ and $n + 1 = 1 + n$). If we colour $\mathbb{N}$, then a **monochromatic edge** of $G_+^\times$ is an edge (n,m) such that n has the same colour as m. Now, the question reads as follows: If we colour $\mathbb{N}$ with finitely many colours, does $G_+^\times$ has a monochromatic edge?

If there would be a 2-colouring of $\mathbb{N}$ such that $G_+^\times$ has just finitely many monochromatic edges, then we could easily construct a finite colouring of $\mathbb{N}$ such that no edge of $G_+^\times$ is monochromatic. But it is not hard to show that $G_+^\times$ has arbitrarily "large" triangles, and therefore, for any 2-colouring of $\mathbb{N}$, $G_+^\times$ has arbitrarily "long" monochromatic edges. In fact, for any positive integer m there are $x_0, y_0, x_1, y_1, x_2, y_2 \in \mathbb{N}$ such that $m < \min\{x_0, y_0, x_1, y_1, x_2, y_2\}$ and $x_0 + y_0 = x_1 + y_1$, $x_1 \cdot y_1 = x_2 + y_2$ and $x_0 \cdot y_0 = x_2 \cdot y_2$. To see this, fix some positive integer m and let $a \in \mathbb{N}$ be such that $a > m$. Let h be any positive integer and define x_0, x_1, x_2, and n as follows: $x_0 = 2ha^2$, $x_1 = a$, $x_2 = ha$, and $n = 4ha^2 - h - a$. Further, let $y_0 = n - x_0 = 2ha^2 - h - a$, $y_1 = n - x_1 = 4ha^2 - h - 2a$, and $y_2 = x_1 \cdot y_1 - x_2 = 2a(2ha^2 - h - a)$. Now, by definition we have $x_0 + y_0 = x_1 + y_1$ and $x_1 \cdot y_1 = x_2 + y_2$, and in addition we also get $x_0 \cdot y_0 = 2ha^2(2ha^2 - h - a) = x_2 \cdot y_2$, and $m < \min\{x_0, y_0, x_1, y_1, x_2, y_2\}$.

A pair of triangles sharing an edge (*i.e.*, a K_4 with an edge deleted) and containing 3 consecutive numbers is called a 2-fan, and three triangles on 5 numbers having one number in common and containing 4 consecutive numbers is called a 3-fan. In the sequel it will be shown that $G_+^\times$ contains 3-fans, infinitely many 2-fans and even arbitrarily large "bundles" of triangles sharing an edge, and an algorithm is provided to generate such "bundles". Further, it will be shown that $\chi\big(G_+^\times\big) \geq 4$.

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2 Fans

For a positive integer ℓ, an ***ℓ*-fan** is a subgraph of $G_+^\times$ of the following type:

$$\begin{tikzcd}
n \arrow[r] \arrow[drr] & n-1 \arrow[r] \arrow[dr] & n-2 \arrow[r] \arrow[d] & \cdots \arrow[r] & n-\ell \arrow[dll] \\
 & & m & &
\end{tikzcd}$$

where an arrow from n to m indicates that there are $x, y \in \mathbb{N}$ with $x + y = n$ and $x \cdot y = m$ respectively.

The following result provides a characterization of ℓ-fans.

THEOREM 2.1. For $\ell, n, m \in \mathbb{N}$ with $n > \ell$, the integers $n, n-1, \ldots, n-\ell, m$ are the vertices of an ℓ-fan if and only if there are positive integers a and b with $b \le a$ such that $n = a + b + 2ab + \ell$, $m = a(a+1)b(b+1)$, and for each $t \in \{1, 2, \ldots, \ell - 1\}$, $\sqrt{(a+b+t+1)^2 + 4abt}$ is an integer.

Proof. Sufficiency: For each $i \in \{0, 1, \ldots, \ell\}$ let

$$k_i = \frac{(n-i) - \sqrt{(a+b+\ell-i)^2 + 4ab(\ell-1-i)}}{2} .$$

Then, by assumption, for $i \in \{0, 1, \ldots, \ell-2\}$, k_i is rational with denominator at most 2, and by definition, for $i \in \{0, 1, \ldots, \ell\}$, $k_i(n-i-k_i) = m$. So, to complete the proof of the sufficiency we only need to show that each k_i is an integer. For this one has by direct computation that $k_\ell = (a+1)b$ and $k_{\ell-1} = ab$. For $i \in \{0, 1, \ldots, \ell-2\}$, if one had that k_i were a fraction with denominator 2 (in lowest terms) one would conclude from the equation $k_i(n-i-k_i) = m$ that m is a fraction with denominator 4, which is impossible since m is an integer.

Necessity: Given $i \in \{0, 1, \ldots, \ell\}$. Since we have an arrow from $n-i$ to m, there is a positive integer $k_i \le (n-i)/2$ such that $k_i(n-i-k_i) = m$. Solving this equation for k_i and using the fact that $k_i \le (n-i)/2$ we get

$$k_i = \frac{(n-i) - \sqrt{(n-i)^2 - 4m}}{2} . \tag{1}$$

Let $c = \sqrt{(n-\ell+1)^2 - 4m}$ and let $d = \sqrt{(n-\ell)^2 - 4m}$. Since $k_{\ell-1}$ and k_ℓ are integers, c and d are integers as well. Further we have $c^2 - d^2 = 2(n-\ell) + 1$, which shows that $c - d$ is odd. Notice also that $n > \ell$, so $c - d > 1$. Since every square is the sum of consecutive odd numbers we have $c^2 = \sum_{j=1}^{c}(2j-1)$ and $d^2 = \sum_{j=1}^{d}(2j-1)$, and therefore

$$c^2 - d^2 = \sum_{j=d+1}^{c} (2j-1) = \sum_{j=1}^{c-d} (2d + 2j - 1) .$$

Let $b = (c-d-1)/2$ and let $a = d + b$, then

$$\begin{aligned} c^2 - d^2 &= \sum_{j=1}^{2b+1} (2a - 2b + 2j - 1) \\ &= (2a - 2b - 1)(2b+1) + (2b+1)(2b+2) \\ &= (2b+1)(2a+1) . \end{aligned}$$

Since $c^2 - d^2 = 2(n-\ell) + 1$ we get $n = a + b + 2ab + \ell$, and since $d^2 = (n-\ell)^2 - 4m$ and $d = a - b$ we get $m = a(a+1)b(b+1)$. Finally, substituting the values of n and m in (1) we get that for each $i \in \{0, 1, \ldots, \ell\}$,

$$k_i = \frac{(n-i) - \sqrt{(a+b+\ell-i)^2 + 4ab(\ell-1-i)}}{2} ,$$

which implies that for each $t \in \{1, 2, \ldots, \ell-1\}$, $\sqrt{(a+b+t+1)^2 + 4abt}$ is an integer. $\dashv$

Theorem 2.1 shows that each ℓ-fan $n, n-1, \ldots, n-\ell, m$ is characterized by a tuple (a, b), and accordingly, let us call (a, b) the **characteristic** of the ℓ-fan $n, n-1, \ldots, n-\ell, m$. Further, let us say that an ℓ-fan $n, n-1, \ldots, n-\ell, m$ is of **type** d if $d = a - b$, where (a, b) is its characteristic.

Certain types of 2-fans. In the following we will see that for $d = 0, 1, 2, 3$, the graph $G_+^\times$ contains infinitely many 2-fans of type d.

PROPOSITION 2.2. $G_+^\times$ contains infinitely many 2-fans of type 0.

Proof. By Theorem 2.1, $n, n-1, n-2, m$ is a 2-fan of type 0 if and only if $n = 2(a + a^2 + 1)$, $m = a^2(a+1)^2$ and $\sqrt{(2a+2)^2 + 4a^2}$ is an integer. Now, $\sqrt{(2a+2)^2 + 4a^2} = 2\sqrt{a^2 + (a+1)^2}$, which is an integer if and only if there is an integer c such that

$$a^2 + (a+1)^2 = c^2 ,$$

or in other words, if $(a, a+1, c)$ is a Pythagorean triple. It is well-known that there are infinitely many Pythagorean triples of this form (see [Sl, A001652 & A001653]). Now, for any Pythagorean triple $(a, a+1, c)$ we have $a^2 + (a+1)^2 = c^2$, which implies $(2a+1)^2 = 2c^2 - 1$. This equation holds for all pairs a_j, c_j, where

$$a_0 = 0, \quad a_1 = 3, \quad a_j = 6a_{j-1} - a_{j-2} + 2 ,$$

and

$$c_0 = 1, \quad \frac{c_j}{2c_{j+1}} + \frac{c_{j+1}}{2c_j} + \frac{2}{c_j c_{j+1}} = 3$$

(*cf.* [Sl, A001652 & A001653]). So, for any positive integer j, $n_j, n_j - 1, n_j - 2, m_j$ is a 2-fan of type 0, where $n_j = 2(a_j + a_j^2 + 1)$ and $m_j = a_j^2(a_j+1)^2$. By the way, also c_j is involved, namely $k_0^j = (c_j - 1)^2/2$, where $k_0^j(n_j - k_0^j) = m_j$. $\dashv$

PROPOSITION 2.3. $G_+^\times$ contains infinitely many 2-fans of type 1.

Proof. By Theorem 2.1, $n, n-1, n-2, m$ is a 2-fan of type 1 if and only if $n = 2a^2 + 1$, $m = a^2(a^2 - 1)$ and $\sqrt{8a^2+1}$ is an integer. Now, $8a^2 + 1$ is odd, and thus, if $8a^2 + 1$ is a square, then there is a t such that $8a^2 + 1 = (2t+1)^2$. Consequently we get $8a^2 = 4t^2 + 4t = 4t(t+1)$, which implies

$$a^2 = \frac{t(t+1)}{2} ,$$

or in other words, a^2 is a triangular number. The numbers a_j such that a_j^2 is triangular we get by the following recursion (*cf.* [Sl, A001109]):

$$a_0 = 0, \quad a_1 = 1, \quad a_j = 6a_{j-1} - a_{j-2} .$$

So, for any integer $j \geq 2$, $n_j, n_j - 1, n_j - 2, m_j$ is a 2-fan of type 1, where $n_j = 2a_j^2 + 1$ and $m_j = a_j^2(a_j^2 - 1)$. $\dashv$

PROPOSITION 2.4. $G_+^\times$ contains infinitely many 2-fans of type 2.

Proof. By Theorem 2.1, $n, n-1, n-2, m$ is a 2-fan of type 2 if and only if $n = 2a(a-1)$, $m = (a-2)(a-1)a(a+1)$ and $\sqrt{8a(a-1)}$ is an integer. Now, $8a(a-1) = 16a(a-1)/2$, and thus, if $8a(a-1)$ is a square, then so is $a(a-1)/2$ is a square as well, but $a(a-1)/2$ is a triangular number. In other words, $8a(a-1)$ is a square if and only if the triangular number $a(a-1)/2$ is a square. The numbers a_j such that $a_j(a_j-1)/2$ is a square are given by the following recursion (*cf.* [Sl, A055997]):

$$a_0 = 1, \quad a_1 = 2, \quad a_j = 6a_{j-1} - a_{j-2} - 2\,.$$

So, for any integer $j \geq 2$, n_j, n_j-1, n_j-2, m_j is a 2-fan of type 2, where $n_j = 2a_j(a_j-1)$ and $m_j = (a_j-2)(a_j-1)a_j(a_j+1)$. ⊣

PROPOSITION 2.5. $G_+^\times$ contains infinitely many 2-fans of type 3.

Proof. By Theorem 2.1, $n, n-1, n-2, m$ is a 2-fan of type 3 if and only if $n = 2a(a-2)-1$, $m = (a-3)(a-2)a(a+1)$ and $\sqrt{8a^2-16a+1}$ is an integer. Now, if $8a^2-16a+1 = c^2$ for some odd integer $c = 2t+1$, then $8a^2-16a = 4t(t+1)$, and thus, a^2-2a is a triangular number. In other words, $8a^2-16a+1$ is a square if and only if a^2-2a is triangular. If we set $\tilde{a} = a-1$, then $\tilde{a}^2 - 1 = a^2-2a$, and thus, $8a^2-16a+1$ is a square if and only if $\tilde{a}^2-1$ is triangular. The numbers $\tilde{a}_j$ such that $\tilde{a}_j^2 - 1$ is triangular are given by the following recursion (*cf.* [Sl, A006452]):

$$\tilde{a}_0 = 1, \quad \tilde{a}_1 = 2, \quad \tilde{a}_2 = 4, \quad \tilde{a}_3 = 11, \quad \tilde{a}_j = 6\tilde{a}_{j-2} - \tilde{a}_{j-4}\,.$$

So, if we put $a_j = \tilde{a}_j + 1$, then for any integer $j \geq 2$, n_j, n_j-1, n_j-2, m_j is a 2-fan of type 3, where $n_j = 2a_j(a_j-2)-1$ and $m_j = (a_j-3)(a_j-2)a_j(a_j+1)$. ⊣

On 3-fans. In order to find 3-fans, we have to find positive integers a and b with $b \leq a$ such that $\sqrt{(a+b+2)^2+4ab}$ and $\sqrt{(a+b+3)^2+8ab}$ are simultaneously integers.

The following table gives a complete list of 3-fans for $1 \leq a \leq 10^4$:

a	b	type	n	m
14	8	6	249	15120
51	35	16	3659	3341520
54	15	39	1692	712800
99	48	51	9654	23284800
132	24	108	6495	10533600
143	84	59	24254	147026880
160	81	79	26164	171097920
224	77	147	34800	302702400
260	35	225	18498	85503600
299	216	83	129686	4204418400

a	b	type	n	m
344	285	59	196712	9673606800
407	299	108	244095	14895223200
440	116	324	102639	2633510880
450	48	402	43701	477338400
527	350	177	369780	34183749600
531	220	311	234394	13734761040
539	299	240	323163	26108082000
615	185	430	228353	13035884400
666	224	442	299261	22388788800
714	63	651	90744	2058376320
1025	511	514	1049089	275145292800
1064	80	984	171387	7342876800
1104	594	510	1313253	431156325600
1196	340	856	814819	165981095280
1287	425	862	1095665	300118618800
1295	230	1065	597228	89169141600
1420	836	584	2376499	1411933224240
1512	99	1413	300990	22647794400
2013	1679	334	6763349	11435712251040
2024	308	1716	1249119	390071959200
2024	1547	477	6265830	9815146941600
2070	120	1950	498993	62246804400
2133	559	1574	2387389	1424902358880
2184	510	1674	2230377	1243641344400
2484	868	1616	4315579	4656048400080
2716	2115	601	11493514	33025198686480
2750	143	2607	789396	155783628000
3024	402	2622	2434725	1481966085600
3311	2925	386	19375589	93853333173600
3375	3267	108	22058895	121648679064000
3401	1855	1546	12622969	39834817061760
3564	168	3396	1201239	360739098720
4047	999	3048	8090955	16365873744000
4224	3654	570	30876873	238345275168000
4514	2448	2066	22107509	122185454317920
4524	195	4329	1769082	782405442000
4575	902	3673	8258780	17051846011200
4895	805	4090	7886653	15549807873600
5031	930	4101	9363624	21919345353360
5301	3535	1766	37486909	351317029583520
5642	224	5418	2533485	1604625422400
5719	5300	419	60632422	919072558404000

a	b	type	n	m
6062	5775	287	70027940	1225977990098400
6083	644	5439	7841634	15372786789360
6496	3249	3247	42220756	445647993336000
6699	4887	1812	65487615	1072156830544800
6930	255	6675	3541488	3135517862400
7314	4263	3051	62370744	972527330895120
7749	5312	2437	82338440	1694904550416000
8280	795	7485	13174278	43390366437600
8400	288	8112	4847091	5873549068800
8463	1359	7104	23012259	132390968935680
8700	3552	5148	61817055	955336972867200
9176	6699	2477	122955926	3779539748661600
9295	1701	7594	31632589	250155109844640

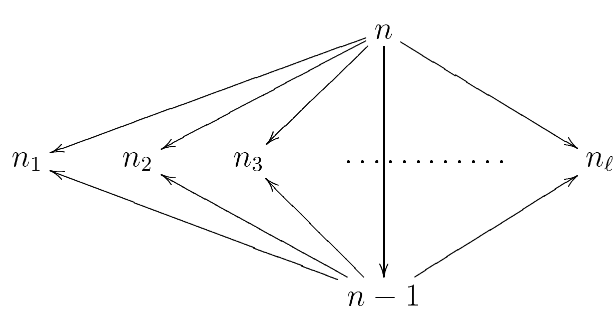
Thus, it seems that $G_+^\times$ contains quite a lot of 3-fans, and it is very likely that it contains infinitely many 3-fans. It might even be possible that $G_+^\times$ contains infinitely many 3-fans of a certain type. A short look at the table might suggest that the type 108 is a good candidate, but at least for $a \le 10^9$, there are just the three 3-fans of type 108 given in the table.

On the other hand, I could not find a single 4-fan in $G_+^\times$, even though there is no obvious reason why 4-fans should not exist. However, for $1 \le a \le 10^6$, there are at least no 4-fans of type less than or equal to 5000.

3 Bundles

A set of triangles sharing an edge we call a bundle of triangles or just a **bundle**. If a bundle contains ℓ triangles, then we call it an **ℓ-bundle**. In the following we will show that $G_+^\times$ contains ℓ-bundles for arbitrarily large integers ℓ.

THEOREM 3.1. The graph $G_+^\times$ contains ℓ-bundles for arbitrarily large integers ℓ. In particular, for any positive integer ℓ, the graph $G_+^\times$ contains subgraphs of the following type:



where an arrow from n to m indicates that there are $x, y \in \mathbb{N}$ with $x+y=n$ and $x \cdot y = m$ respectively.

Proof. Let $\ell \in \mathbb{N}$ be given and let $b \in \mathbb{N}$ be any odd integer with $b > 1$. Let $n = (b^\ell+1)^2/2$ and note that $n \in \mathbb{N}$ since b^ℓ is odd. Further, let $m = n - 1$ and for each $i \in \{1, 2, \ldots, \ell\}$ let

$$\begin{aligned} k_i &= \frac{b^{2\ell} - b^{2\ell-i} + 2b^\ell - 2b^{\ell-i} + b^i - 1}{4} \\ &= \frac{(b^{2\ell-i} + 2b^{\ell-i} + 1)(b^i - 1)}{4}\,, \\ h_i &= \frac{b^{2\ell} - b^{2\ell-i} + 2b^\ell - 2b^{\ell-i} - b^i + 1}{4} \\ &= \frac{(b^{2\ell-i} + 2b^{\ell-i} - 1)(b^i - 1)}{4}\,, \quad \text{and} \\ n_i &= k_i(m - k_i)\,. \end{aligned}$$

Then $b^i - 1$, $b^{2\ell-i} + 2b^{\ell-i} + 1$, and $b^{2\ell-i} + 2b^{\ell-i} - 1$ are even, thus h_i and k_i are integers for each i. Also, since $b > 1$ we have $k_i > h_i > 0$ for each i.

Next we claim that $k_1 < k_2 < \ldots < k_\ell < m/2$ (and consequently $n_1 < n_2 < \ldots < n_\ell$). Indeed, for $i \in \{1, 2, \ldots, \ell - 1\}$ we get

$$4k_{i+1} - 4k_i = (b^{2\ell-i-1} + 2b^{\ell-i-1} + b^i)(b - 1) > 0$$

and

$$k_\ell = \frac{b^{2\ell} + 2b^\ell - 3}{4} < \frac{b^{2\ell} + 2b^\ell - 1}{4} = \frac{m}{2}\,.$$

To complete the proof we observe that for each $i \in \{1, 2, \ldots, \ell\}$ we have

$$k_i(m - k_i) = \frac{b^{4\ell} - b^{4\ell-2i} + 4b^{3\ell} - 4b^{3\ell-2i} + 4b^{2\ell} - 4b^{2\ell-2i} - b^{2i} + 1}{16} = h_i(n - h_i)\,.$$

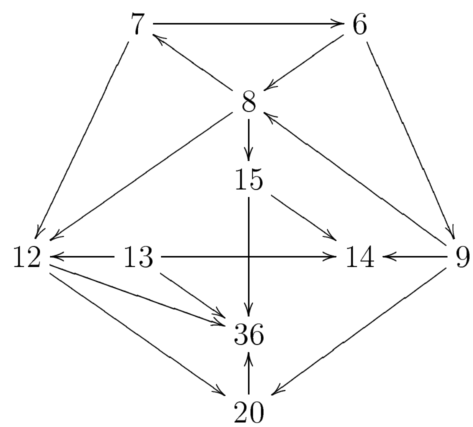
⊣

4 On $\chi\big(G^\times_+\big)$ and $\chi\big(\dot{G}^\times_+\big)$

In this section we will see that the chromatic number of $G^\times_+$ is at least 4. Moreover, even if we delete the edges of the form $(2x, x^2)$, it can still not be 3-coloured. We show this by giving two subgraphs with chromatic number 4, which we found with the help of Prolog.

Proposition 4.1. $\chi\big(G^\times_+\big) \geq 4$.

Proof. Let G_{10} be the subgraph of $G^\times_+$ induced by the 10 vertices 6, 7, 8, 9, 12, 13, 14, 15, 20, and 36:



where an arrow from n to m indicates that there are $x, y \in \mathbb{N}$ with $x+y=n$ and $x \cdot y = m$ respectively.

Assume towards a contradiction that $\chi(G_{10}) = 3$. So, let us colour the vertices of G_{10} with three colours, say x, y, and z. Without loss of generality, let us colour 8 with colour x and 6 with colour y, denoted $[8, x]$ and $[6, y]$, respectively. Now, the edge $(7, 6)$ makes it impossible to colour 7 with y. To keep the notation short, let us write this in the form $\mathbf{7} : (7,6) \to \neg y$. Further, we have $\mathbf{7} : (8,7) \to \neg x$, and thus, together with $\mathbf{7} : (7,6) \to \neg y$, this implies $[7, z]$.

Consequently we get the following:

$\mathbf{12} : (7,12) \to \neg z$; $\mathbf{12} : (8,12) \to \neg x$, thus $[12, y]$;
$\mathbf{9} : (9,8) \to \neg x$; $\mathbf{9} : (6,9) \to \neg y$, thus $[9, z]$;
$\mathbf{20} : (9,20) \to \neg z$; $\mathbf{20} : (12,20) \to \neg y$, thus $[20, x]$;
$\mathbf{36} : (12,36) \to \neg y$; $\mathbf{36} : (20,36) \to \neg x$, thus $[36, z]$;
$\mathbf{13} : (13,12) \to \neg y$; $\mathbf{13} : (13,36) \to \neg z$, thus $[13, x]$;
$\mathbf{14} : (9,14) \to \neg z$; $\mathbf{14} : (14,13) \to \neg x$, thus $[14, y]$;
$\mathbf{15} : (8,15) \to \neg x$; $\mathbf{15} : (15,14) \to \neg y$; $\mathbf{15} : (15,36) \to \neg z$;
and thus, we get a contradiction at 15. $\dashv$

Finally, let us also consider the subgraph $\dot{G}_+^\times$ of $G_+^\times$ defined as follows: The vertex-set of $\dot{G}_+^\times$ is again the set of positive integers, and n joined to m if for some *distinct* $x, y \in \mathbb{N}$ we have $n = x + y$ and $m = x \cdot y$.

Like for $G_+^\times$, we can show that the chromatic number of $\dot{G}_+^\times$ is at least 4, but the subgraph which provides the counterexample is much larger.

PROPOSITION 4.2. $\chi\big(\dot{G}_+^\times\big) \geq 4$.

Proof. Let $\dot{G}_{29}$ be the subgraph of $\dot{G}_+^\times$ induced by the 29 vertices 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 32, 33, 36, 40, 44, 45, 48, 72, 80, 84, 90, and 120.

Assume towards a contradiction that $\chi(\dot{G}_{29}) = 3$. So, let us colour the vertices of $\dot{G}_{29}$ with three colours x, y, and z. Without loss of generality, let us colour 14 with colour x. Since, the four numbers 13, 14, 22, and 40 form a pair of triangles sharing the edge $(13, 40)$, 22 must get the same colour as 14, so we have to colour 22 also with x. Now, since 22, 23, and 120 form a triangle, 23 and 120 must get the colours y and z. Thus, without loss of generality, let us colour 23 with z and 120 with y. Finally, because of the edges $(24,23)$ and $(14,24)$, and since 23 has colour z and 14 has colour x, 24 must get colour y. So far, we have the following colouring: $[14,x],[22,x],[23,z],[24,y],[120,y]$.

Consequently we get the following:

$\mathbf{10}:(10,24)\to\neg y$; $\mathbf{11}:(11,24)\to\neg y$; $\mathbf{13}:(14,13)\to\neg x$; $\mathbf{15}:(15,14)\to\neg x$; $\mathbf{21}:(22,21)\to\neg x$; $\mathbf{25}:(25,24)\to\neg y$; $\mathbf{26}:(26,120)\to\neg y$; $\mathbf{33}:(14,33)\to\neg x$; $\mathbf{40}:(14,40)\to\neg x$; $\mathbf{44}:(24,44)\to\neg y$; $\mathbf{45}:(14,45)\to\neg x$; $\mathbf{48}:(14,48)\to\neg x$; $\mathbf{72}:(22,72)\to\neg x$; $\mathbf{80}:(24,80)\to\neg y$; $\mathbf{90}:(23,90)\to\neg z$.

Now, let us consider the numbers 12 and 19. Each we can colour with x,y or z, thus, there are 9 possible ways to colour these two numbers. We will see that in each case, we get a contradiction.

$\boldsymbol{[12,x],[19,x]}$ **:**

$\mathbf{11}:(12,11)\to\neg x$; $\mathbf{18}:(19,18)\to\neg x$; $\mathbf{32}:(12,32)\to\neg x$; $\mathbf{90}:(19,90)\to\neg x$; $\mathbf{33}:(33,90)\to\neg y$; $\mathbf{18}:(11,18)\to\neg z$; $\mathbf{32}:(18,32)\to\neg y$; $\mathbf{33}:(33,32)\to\neg z$;
and thus, we get a contradiction at 33.

$\boldsymbol{[12,x],[19,y]}$ **:**

$\mathbf{20}:(12,20)\to\neg x$; $\mathbf{20}:(20,19)\to\neg y$; $\mathbf{48}:(19,48)\to\neg y$; $\mathbf{84}:(19,84)\to\neg y$; $\mathbf{84}:(20,84)\to\neg z$; $\mathbf{44}:(44,84)\to\neg x$; $\mathbf{15}:(15,44)\to\neg z$; $\mathbf{16}:(16,48)\to\neg z$; $\mathbf{16}:(16,15)\to\neg y$;
and thus, we get a contradiction at 16.

$\boldsymbol{[12,x],[19,z]}$ **:**

$\mathbf{18}:(19,18)\to\neg z$; $\mathbf{20}:(12,20)\to\neg x$; $\mathbf{20}:(20,19)\to\neg z$; $\mathbf{21}:(21,20)\to\neg y$; $\mathbf{80}:(21,80)\to\neg z$; $\mathbf{84}:(19,84)\to\neg z$; $\mathbf{84}:(20,84)\to\neg y$; $\mathbf{44}:(44,84)\to\neg x$; $\mathbf{45}:(45,44)\to\neg z$; $\mathbf{18}:(18,80)\to\neg x$; $\mathbf{45}:(18,45)\to\neg y$;
and thus, we get a contradiction at 45.

$\boldsymbol{[12,y],[19,x]}$ **:**

$\mathbf{20}:(12,20)\to\neg y$; $\mathbf{20}:(20,19)\to\neg x$; $\mathbf{21}:(21,20)\to\neg z$; $\mathbf{90}:(19,90)\to\neg x$; $\mathbf{90}:(21,90)\to\neg y$;
and thus, we get a contradiction at 90.

$\boldsymbol{[12,y],[19,y]}$ **:**

$\mathbf{18}:(19,18)\to\neg y$; $\mathbf{48}:(19,48)\to\neg y$; $\mathbf{26}:(26,48)\to\neg z$; $\mathbf{84}:(19,84)\to\neg y$; $\mathbf{25}:(26,25)\to\neg x$; $\mathbf{16}:(16,48)\to\neg z$; $\mathbf{84}:(25,84)\to\neg z$; $\mathbf{44}:(44,84)\to\neg x$; $\mathbf{15}:(15,44)\to\neg z$; $\mathbf{16}:(16,15)\to\neg y$; $\mathbf{17}:(17,16)\to\neg x$; $\mathbf{10}:(10,16)\to\neg x$; $\mathbf{11}:(11,10)\to\neg z$; $\mathbf{18}:(11,18)\to\neg x$; $\mathbf{17}:(18,17)\to\neg z$; $\mathbf{72}:(17,72)\to\neg y$; $\mathbf{72}:(18,72)\to\neg z$;
and thus, we get a contradiction at 72.

$\boldsymbol{[12,y],[19,z]}$ **:**

$\mathbf{18}:(19,18)\to\neg z$; $\mathbf{20}:(12,20)\to\neg y$; $\mathbf{20}:(20,19)\to\neg z$; $\mathbf{36}:(13,36)\to\neg z$; $\mathbf{36}:(20,36)\to\neg x$; $\mathbf{48}:(19,48)\to\neg z$; $\mathbf{15}:(15,36)\to\neg y$; $\mathbf{16}:(16,48)\to\neg y$;

$\mathbf{16} : (16, 15) \to \neg z\,;\quad \mathbf{17} : (17, 16) \to \neg x\,;\quad \mathbf{10} : (10, 16) \to \neg x\,;\quad \mathbf{11} : (11, 10) \to \neg z\,;$
$\mathbf{18} : (11, 18) \to \neg x\,;\quad \mathbf{17} : (18, 17) \to \neg y\,;\quad \mathbf{72} : (17, 72) \to \neg z\,;\quad \mathbf{72} : (18, 72) \to \neg y\,;$
and thus, we get a contradiction at 72.

$[\mathbf{12}, \boldsymbol{z}], [\mathbf{19}, \boldsymbol{x}]$ **:**

$\mathbf{11} : (12, 11) \to \neg z\,;\quad \mathbf{20} : (12, 20) \to \neg z\,;\quad \mathbf{20} : (20, 19) \to \neg x\,;\quad \mathbf{21} : (21, 20) \to \neg y\,;$
$\mathbf{10} : (11, 10) \to \neg x\,;\quad \mathbf{21} : (10, 21) \to \neg z\,;$
and thus, we get a contradiction at 21.

$[\mathbf{12}, \boldsymbol{z}], [\mathbf{19}, \boldsymbol{y}]$ **:**

$\mathbf{20} : (12, 20) \to \neg z\,;\quad \mathbf{20} : (20, 19) \to \neg y\,;\quad \mathbf{48} : (19, 48) \to \neg y\,;\quad \mathbf{26} : (26, 48) \to \neg z\,;$
$\mathbf{84} : (19, 84) \to \neg y\,;\quad \mathbf{84} : (20, 84) \to \neg x\,;\quad \mathbf{25} : (25, 84) \to \neg z\,;\quad \mathbf{26} : (26, 25) \to \neg x\,;$
and thus, we get a contradiction at 26.

$[\mathbf{12}, \boldsymbol{z}], [\mathbf{19}, \boldsymbol{z}]$ **:**

$\mathbf{11} : (12, 11) \to \neg z\,;\quad \mathbf{18} : (19, 18) \to \neg z\,;\quad \mathbf{48} : (19, 48) \to \neg z\,;\quad \mathbf{10} : (11, 10) \to \neg x\,;$
$\mathbf{16} : (10, 16) \to \neg z\,;\quad \mathbf{18} : (11, 18) \to \neg x\,;\quad \mathbf{17} : (18, 17) \to \neg y\,;\quad \mathbf{16} : (16, 48) \to \neg y\,;$
$\mathbf{72} : (18, 72) \to \neg y\,;\quad \mathbf{17} : (17, 16) \to \neg x\,;\quad \mathbf{72} : (17, 72) \to \neg z\,;$
and thus, we get a contradiction at 72. $\dashv$

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