On the chromatic number of intersection graphs of convex sets in the plane*

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Abstract

Let G be the intersection graph of a finite family of convex sets obtained by translations of a fixed convex set in the plane. We show that every such graph with clique number k is $(3k-3)$-degenerate. This bound is sharp. As a consequence, we derive that G is $(3k-2)$-colorable. We show also that the chromatic number of every intersection graph H of a family of homothetic copies of a fixed convex set in the plane with clique number k is at most $6k-6$.

1 Introduction

The *intersection graph* G of a family $\mathcal{F}$ of sets is the graph with vertex set $\mathcal{F}$ where two members of $\mathcal{F}$ are adjacent if and only if they have common elements. Asplund and Grünbaum [3] and Gyárfás and Lehel [11, 9] started studying many interesting problems

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on the chromatic number of intersection graphs of convex figures in the plane. Many problems of this type can be stated as follows. For a class $\mathcal{G}$ of intersection graphs and for a positive integer k, find or bound $f(\mathcal{G}, k)$ - the maximum chromatic number of a graph in $\mathcal{G}$ with the clique number at most k. A number of results on the topic can be found in [5, 9, 11, 13].

Recently, several papers on intersection graphs of translations of a plane figure appeared. Akiyama, Hosono, and Urabe [2] considered $f(\mathcal{C}, k)$, where $\mathcal{C}$ is the family of intersection graphs of unit squares on the plane with sides parallel to the axes. They proved that $f(\mathcal{C}, 2) = 3$ and asked about $f(\mathcal{C}, k)$ and, more generally, about chromatic number of intersection graphs of unit cubes in $\mathbb{R}^d$. In connection with channel assignment problem in broadcast networks, Clark, Colbourn, and Johnson [4] and Gräf, Stumpf, and Weißenfels [6] considered colorings of graphs in the class $\mathcal{U}$ of intersection graphs of unit disks in the plane. They proved that finding chromatic number of graphs in $\mathcal{U}$ is an NP-complete problem. In [6, 18], and [17] polynomial algorithms are given implying that $f(\mathcal{U}, k) \leq 3k - 2$. Perepelitsa [18] also considered the more general family $\mathcal{T}$ of intersection graphs of translations of a fixed compact convex figure in the plane. She proved that every graph in $\mathcal{T}$ is $(8k - 8)$-degenerate, which implies that $f(\mathcal{T}, k) \leq 8k - 7$. She also considered intersection graphs of translations of triangles and boxes in the plane.

Recall that a graph G is called *m-degenerate* if every subgraph H of G has a vertex v of degree at most m in H. It is well known that every m-degenerate graph is $(m+1)$-colorable. In fact, the property of being m-degenerate is sufficiently stronger that being $(m+1)$-colorable. In particular, every m-degenerate is also $(m+1)$-list-colorable.

Our main result strengthens Perepelitsa's bound as follows.

Theorem 1 *Let G be the intersection graph of translations of a fixed compact convex set in the plane with clique number $\omega(G) = k$. Then G is $(3k - 3)$-degenerate. In particular, the chromatic number and the list chromatic number of G do not exceed $3k - 2$.*

The bound on degeneracy in Theorem 1 is sharp. In Section 5, for every $k \geq 2$ we present the intersection graph G of a family of unit circles in the plane with $\omega(G) = k$ that is not $(3k - 4)$-degenerate.

The idea of the proof of Theorem 1 allows us to estimate the maximum degree of the intersection graph.

Theorem 2 *Let G be the intersection graph of translations of a fixed compact convex set in the plane with $\omega(G) = k$, $k \geq 2$. Then the maximum degree of G is at most $6k - 7$.*

This bound is also sharp.

Then we consider a more general setting: shrinking and blowing of the figures are now allowed.

Theorem 3 *Let H be the intersection graph of a family F of homothetic copies of a fixed convex compact set D in the plane. If $\omega(H) = k$, $k \geq 2$, then H is $(6k - 7)$-degenerate. In particular, the chromatic number and the list chromatic number of H do not exceed $6k - 6$.*

There is no upper bound on the maximum degree for intersection graphs of homothetic copies of a fixed convex set in the plane analogous to Theorem 2, since every star is a graph of this type.

The results above yield some Ramsey-type bounds for geometric intersection graphs. For a positive integer n and a family $\mathcal{F}$ of graphs, let $r(\mathcal{F}, n)$ denote the maximum r such that for every $G \in \mathcal{F}$ on n vertices, either the *clique number*, $\omega(G)$, or the independence number, $\alpha(G)$, is at least r. One can read Ramsey Theorem for graphs as the statement that for the family $\mathcal{G}$ of all graphs, $r(\mathcal{G}, n) \sim 0.5 \log_2 n$. Larman, Matousek, Pach, and Torocsik [15] proved that for the family $\mathcal{P}$ of intersection graphs of compact convex sets in the plane, $r(\mathcal{P}, n) \geq n^{0.2}$. Since $\chi(H) \geq \frac{n}{\alpha(H)}$ for every n-vertex graph H, Theorem 3 yields that for every n-vertex intersection graph H of a family of homothetic copies of a fixed convex compact set D in the plane, we have $\alpha(H)(6\omega(H) - 7) \geq n$. It follows that $r(\mathcal{D}, n) \geq \sqrt{n/6}$ for the family $\mathcal{D}$ of intersection graphs of homothetic copies of a fixed convex compact set in the plane. Similarly, Theorem 1 yields that $r(\mathcal{T}, n) \geq \sqrt{n/3}$.

The structure of the paper is as follows. In the next section we introduce our tools. Theorems 1 and 2 are proved in Section 3. In Section 4 we prove Theorem 3. Section 5 is devoted to construction of extremal graphs.

2 Preliminaries

Given sets A and B of vectors and a real α, the set $\alpha(A+B)$ is defined as $\{\alpha(a+b) \mid a \in A,\ b \in B\}$. When $B = \{b\}$, we sometimes write $A + b$ instead of $A + \{b\}$. Our first tool is the following lemma.

Lemma 4 *Let A be a convex figure and $A'' = A + s$ where s is a vector. Let P be a convex figure that intersects both A and A''. If $A' = A + \alpha s$, $0 \leq \alpha \leq 1$, then P intersects A'.*

Proof. Let $u \in A \cap P$, $u'' = u + s$, $v'' \in A'' \cap P$, $v = v'' - s$. So $v'' \in A''$, $u \in A$, and the interval $\overline{uv''}$ is in P. Let $A' = A + \alpha s$, $u' = u + \alpha s$, and $v' = v + \alpha s$. Then the interval $\overline{u'v'}$ is in A' and must intersect the interval $\overline{uv''}$ in P. $\square$

Our second tool is an old result of Minkowski [16].

Lemma 5 (Minkowski [16]) *Let K be a convex set in the plane. Then $(x+K) \cap (y+K) \neq \emptyset$ if and only if $(x + \frac{1}{2}[K + (-K)]) \cap (y + \frac{1}{2}[K + (-K)]) \neq \emptyset$.*

A proof can be also found in [12]. Note that the set $\frac{1}{2}[K+(-K)]$ is centrally symmetric for every K. Hence without loss of generality, it is enough to prove Theorems 1, 2, and 3 for centrally symmetric convex sets. For handling these sets, the notion of Minkowski norm is quite useful.

Let K be a compact convex set on the plane, centrally symmetric about the origin. For every point x on the plane, we define the *Minkowski norm*

$$\|x\|_K = \inf_{\lambda \geq 0} \{\lambda \in \mathbb{R} : x \in \lambda K\}.$$

Note that $\{x : \|x\|_K = 1\}$ is the boundary of K. It is easily checked that $u + K$ and $v + K$ intersect if and only if $\|u - v\|_K \leq 2$. The two lemmas below appear in [8]. We present their proofs, since they are very short.

Lemma 6 (Grünbaum [8]) *Let x, y, z be different points belonging to the boundary of K, such that the origin O does not belong to the open half-plane determined by x and y that contains z. Then $\|x - z\|_K \leq \|x - y\|_K$.*

Proof. If $x + y = 0$, then $\|x - y\|_K = \|2x\|_K = 2$, since x is on the boundary of K. On the other hand, $\|x - z\|_K \leq \|x\|_K + \|z\|_K = 2$. Hence $\|x - z\|_K \leq \|x - y\|_K$.

If $x + y \neq 0$, find another triple of points x^*, y^*, z^* such that $x^* + y^* = 0$ and the triangle with vertices $\{x^*, y^*, z^*\}$ is similar to $\{x, y, z\}$. We can check that z^* is inside K, hence $\|x^* - z^*\|_K \leq \|x^* - y^*\|_K$. Therefore $\|x - z\|_K \leq \|x - y\|_K$. □

Lemma 7 (Grünbaum [8]) *Let x, y, z, u be different points belonging to the boundary of K, such that z and u belong to an open half-plane determined by x and y, while O belongs to its complement. Then $\|z - u\|_K \leq \|x - y\|_K$.*

Proof. We may assume that the points are located in order x, u, z, y counterclockwise. From Lemma 6, $\|z - u\|_K \leq \|x - z\|_K \leq \|x - y\|_K$. □

3 Proofs of Theorems 1 and 2

It will be convenient to prove the following slightly refined version of Theorem 1 for centrally symmetric sets.

Theorem 8 *Let $\mathcal{M} = \{M_i\}$ be a set of translates of a centrally symmetric convex set in the plane with given axes. If the clique number of the intersection graph $G(\mathcal{M})$ of $\mathcal{M}$ is k, then every highest member A of $\mathcal{M}$ intersects at most $3k - 3$ other members.*

Proof. For an arbitrary set S, define $\mathcal{M}(S) = \{M_i \in \mathcal{M} \mid S \cap M_i \neq \emptyset\}$. Let A be a highest member of $\mathcal{M}$. For convenience, we assume that the center of A is the origin $O = (0, 0)$. Let z be the rightmost point on the X-axis that belongs to A. If $z = (0, 0)$, then A is an interval with the center O and G is an interval graph. So, we assume $z \neq (0, 0)$. Let $B = A - 2z$ and $C = A + 2z$. Since A is convex and centrally symmetric, B and C touch A but have no common interior points with A. Note that B and C may or may not belong to $\mathcal{M}$.

The following three claims are crucial for our proof.

Claim 3.1 *Let $\mathcal{M}_1(A) = \mathcal{M}(A) \cap \mathcal{M}(B)$. Then every two members of $\mathcal{M}_1(A)$ intersect.*

Claim 3.2 *Let $\mathcal{M}_2(A) = \mathcal{M}(A) \cap \mathcal{M}(C)$. Then every two members of $\mathcal{M}_2(A)$ intersect.*

Claim 3.3 *Let $\mathcal{M}_3(A) = \mathcal{M}(A) - \mathcal{M}(B) - \mathcal{M}(C)$. Then every two members of $\mathcal{M}_3(A)$ intersect.*

Indeed, $\mathcal{M}(A) = \mathcal{M}_1(A) \cup \mathcal{M}_2(A) \cup \mathcal{M}_3(A)$. If Claims 3.1, 3.2, and 3.3 hold, then, since $\omega(G) = k$, $|\mathcal{M}_i(A)| \le k - 1$ for $i = 1, 2, 3$, and hence A intersects at most $3k - 3$ members of $\mathcal{M}$. Therefore, we need only to prove the claims.

Let L be a *supporting line for A at* $(-z, 0)$, i.e. a line passing through $(-z, 0)$ and having no common points with the interior of A. Such a line exists, since A is convex. If $(-z, 0)$ is a corner of A, then L is not unique. Furthermore, since A is centrally symmetric, L is also a supporting line for B. Below, we will use the (not necessary orthogonal) coordinate system with the same origin and the X-axis as we used above, but whose Y-axis is parallel to L. We scale the new Y-axis so that the new y-coordinate of every point is the same as the old one.

Proof of Claim 3.1. Let $L_A = L + z$ and $L_B = L - z$ be the straight lines that are parallel to L and pass through the center $(0, 0)$ of A and the center $-2z$ of B, respectively. Note that L_A is the new y-axis. Let S be the strip between L_A and L_B on the plane.

Let U and V be in $\mathcal{M}_1(A)$, $U = u + A$ and $V = v + A$, $u = (x_u, y_u)$, $v = (x_v, y_v)$. Without loss of generality, we may assume that $x_u \ge x_v$. Note that u and v are in the strip S.

Case 1. $y_u \ge y_v$. Let $L_u = L_A + u$. This line passes through u and is parallel to L. Similarly, $L_v = L_A + v$ passes through v and is parallel to L. Let u' (respectively, v') be the intersection point of the x-axis and L_u (respectively, L_v) and $U' = A + u'$ (respectively, $V' = A + v'$). Since U' and V' are between A and B, by Lemma 4, U' and V' intersect V and each other. Let u'' be the point on L_u with the y-coordinate equal to y_v and $U'' = A + u''$. Since U' intersects V', U'' intersects V. But U is located between U' and U'' (or coincides with U'' if $y_u = y_v$). Therefore, Lemma 4 implies that U intersects V.

Case 2. $y_u < y_v$. Repeating the proof of Case 1 with roles of A and B switched yields this case. □

The proof of Claim 3.2 is the same (with C in place of B).

Proof of Claim 3.3. Let $A^* = 2A$, $B^* = A^* - 2z$, and $C^* = A^* + 2z$. Let $s = (x_s, y_s)$ be a lowest intersection point of A^* and B^*. Since $A^* = B^* + 2z$ and $C^* = A^* + 2z$, the point $w = s - 2z$ belongs to B^* and the point $t = s + 2z$ is an intersection point of A^* and C^*. Furthermore, $\|s - t\|_A = \|s - w\|_A = 2$.

Let W denote the figure bounded by the straight line segment from O to s, the arc of the boundary of A^* from s to t, denoted by R_2, and the straight line segment t to O (see Fig. 2). We will prove now that W contains all the points of $A^* - (B^* \cup C^*)$ with non-positive y-coordinates. Indeed, suppose that $A^* - (B^* \cup C^*)$ contains a point $u = (x_u, y_u)$ with $y_u \le 0$ on the left of the line l_s passing through O and s. Then, by the definition of B^*, the point $u' = u - 2z$ belongs to B^*.

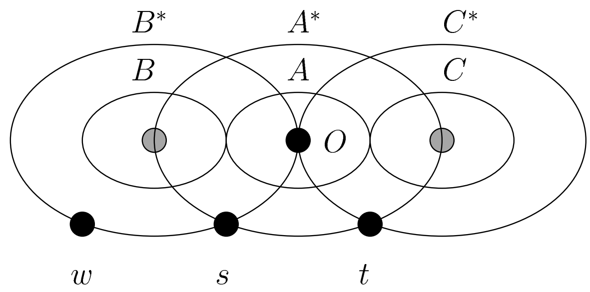

Figure 1: The intersection of boundaries of A^* and B^*.

CASE 1. $y_u < y_s$. Then the straight line segment I_1 from u' to s is contained in B^* and the straight line segment I_2 from u to $O - 2z$ is contained in A^*. Moreover, I_2 crosses I_1, and their crossing point, u^*, has the y-coordinate less than s (since it belongs to I_1). But $u^* \in A^* \cap B^*$, a contradiction to the choice of s.

CASE 2. $y_s \leq y_u \leq 0$. Let u'' be the intersection point of l_s and the line $y = y_u$. Since u'' is between O and s on l_s, it belongs to B^*. Therefore, all points on the interval between u' and u'' belong to B^*. In particular, $u \in B^*$, a contradiction to $u \in A^* - (B^* \cup C^*)$.

Similarly, $A^* - (B^* \cup C^*)$ cannot contain points with non-positive y-coordinates on the right of the line l_t passing through O and t.

Let $U, V \in \mathcal{M}_3(A)$, $U = u + A$, $V = v + A$, $u = (x_u, y_u)$, $v = (x_v, y_v)$. Then by definition, $y_u \leq 0$, and $y_v \leq 0$, and by the above, $u, v \in W$. As it was pointed out in Section 2, proving that U and V intersect is equivalent to proving that $\|u - v\|_A \leq 2$.

Let $u, v \in W$ and let l_u (respectively, l_v) be the straight lines passing through O and u (respectively, v). Since B^* and C^* are convex, the lines l_u and l_v must pass between the straight line l_s and the straight line l_t (see Fig. 2). Since R_2 connects s with t, we conclude that lines l_u and l_v intersect R_2. Let u' (respectively, v') be the intersection point of l_u (respectively, l_v) and R_2. By Lemmas 6 and 7, $\|u' - v'\|_A \leq \|s - t\|_A = \|2z\|_A = 2$. Hence $u' + A$ and $v' + A$ intersect (and both intersect A). Since v is between O and v' on l_v, Lemma 4 yields that $u' + A$ intersects $v + A$. Now, since u is between O and u' on l_u, the same lemma yields that $v + A$ intersects $u + A$. This proves the claim and thus the theorem. □

Clearly, Theorem 8 implies Theorem 1. Now we also derive Theorem 2.

Proof of Theorem 2. Let A be a member of a set $\mathcal{M} = \{M_i\}$ of translates of a centrally symmetric convex set in the plane such that the clique number of the intersection graph $H(\mathcal{M})$ of $\mathcal{M}$ is k. We want to prove that A intersects at most $6k-7$ other members of $\mathcal{M}$. Let $B \in \mathcal{M}$ intersect A. Choose a coordinate system on the plane so that the center of A is the origin and the center of B lies on the x-axis. Let $\mathcal{M}^+$ (respectively, $\mathcal{M}^-$) be the family of members of $\mathcal{M}$ with a nonnegative (respectively, non-positive) y-coordinate. Then A is a highest member of $\mathcal{M}^-$ and a lowest member of $\mathcal{M}^+$. By Theorem 8, A has at most $3k-3$ neighbors in each of $\mathcal{M}^-$ and $\mathcal{M}^+$. Moreover, B was counted in both sets. This proves the theorem. □

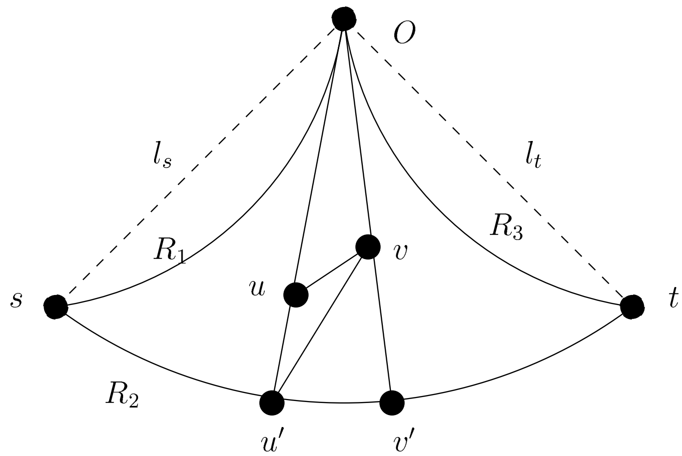

Figure 2: $\mathcal{M}_3(A) = \mathcal{M}(A) - \mathcal{M}(B) - \mathcal{M}(C)$

4 Intersection graphs of convex sets with different sizes

In this section, we study a more general case. Two convex sets K, D on the plane are called *homothetic* if $K = x + \lambda D$ for a point x on the plane and some $\lambda > 0$. We consider intersection graphs of families of homothetic copies of a fixed compact convex set. Any intersection graph of a family of different sized circles is a special example. Note that Lemma 5 does not need to hold in this more general case.

The following easy observation is quite useful for our purposes.

Lemma 9 *Let U be a convex set containing the origin. For each $v \in U$ and $0 \le \lambda \le 1$, the set $W(U, v, \lambda) = (1 - \lambda)v + \lambda U$ is contained in U and contains v.*

Proof. By the definition, $W(U, v, \lambda) = \{v + \lambda \cdot (u - v) \mid u \in U\}$.

Let $u \in U$. Since $v+0\cdot(u-v) = v \in U$, $v+1\cdot(u-v) = u \in U$, and U is convex, we have $v+\lambda\cdot(u-v) \in U$ for every $0 \le \lambda \le 1$. On the other hand, $v = v+\lambda\cdot(v-v) \in W(U, v, \lambda)$ for every $0 \le \lambda \le 1$. $\square$

Proof of Theorem 3. Let Z be a smallest homothetic copy of D in F. Let $F(Z)$ be the set of members of F intersecting Z. For every $U \in F(Z)$, let $\lambda(U)$ be the positive real such that $Z = u+\lambda(U)U$ for some u. For every $U \in F(Z)$, choose a point $z(U) \in Z\cap U$ and denote $U^* = W(U, z(U), \lambda(U)) = (1 - \lambda(U))z(U) + \lambda(U)U$. Note that U^* is a translate of Z. By Lemma 9, the intersection graph G of the family $F^*(Z) = \{Z\} \cup \{U^* \mid U \in F(Z)\}$ is a subgraph of H. In particular, the clique number of G is at most k. Moreover, because of the choice of $z(U)$, $\deg_G(Z) = \deg_H(Z)$. Since $F^*(Z)$ consists of translates of Z, Theorem 2 implies that $\deg_G(Z) \le 6k - 7$. $\square$

Remark. It is known that the maximum degree of any intersection graph of translations of a box in the plane with clique number k is at most $4k - 4$. Repeating the proof of Theorem 3 for this special case, we obtain that every intersection graph of homothetic copies of a box in the plane with clique number k is $(4k-4)$-degenerate.

5 Constructions

Our first example shows that the bound on the maximum degree in Theorem 2 is sharp.

Example 1. Let K be the unit circle whose center is the origin in the plane. Let K_2 be the circle of radius 2 whose center is the origin. For $0 \le i \le 6k-8$, let v_i be the point on the boundary of K_2 with the polar coordinates $(2, i\frac{2\pi}{6k-7})$. Let $A_i = K + v_i$ for $0 \le i \le 6k-8$. Then K intersects A_i for all i. Observe that A_i intersects A_j if and only if $|i-j| \le k-2 \pmod{6k-7}$. It follows that the clique number of the intersection graph G of the family $\{K\} \cup \{A_i : 0 \le i \le 6k-8\}$ is k and the degree of K in G is $6k-7$.

Example 2. Fix a positive real R. For a positive integer m, let $F'_m = \{(R, \frac{i}{m-1/2}) : i = 0, \pm 1, \pm 2, \ldots\}$, $F''_m = \{(R-\sqrt{3}, \frac{i}{m-1/2}) : i = 0, \pm 1, \pm 2, \ldots\}$, and $F_m = F'_m \cup F''_m$. In other words, we choose an infinite number of points on the vertical lines $x = R$ and $x = R-\sqrt{3}$. A part of $F'_m \cup F''_m$ is drawn on Figure 3 (left). Let $\mathcal{C}_m$ be the family of unit circles in the plane with the set of centers $F'_m \cup F''_m$ and let G_m be the intersection graph of $\mathcal{C}_m$. It is convenient to view G_m as the graph with the vertex set $F'_m \cup F''_m$ such that two points u and v are adjacent if and only if the (Euclidean) distance $\rho(u, v)$ is at most 2. We derive some properties of G_m in a series of claims.

The first claim is evident.

Claim 5.1 $\rho((R, y_1), (R-\sqrt{3}, y_2)) \le 2$ *if and only if* $|y_1 - y_2| \le 1$.

This simple fact and the definition of F_m imply the next claim.

Claim 5.2 *If* $(R, y_1) \in F'_m$, *then the maximum (respectively, minimum)* y_2 *such that* $(R-\sqrt{3}, y_2) \in F''_m$ *and* $\rho((R, y_1), (R-\sqrt{3}, y_2)) \le 2$ *is* $y_2 = y_1 + 1 - \frac{1}{2m-1}$ *(respectively,* $y_2 = y_1 - 1 + \frac{1}{2m-1}$*).*

It follows that every $u \in F_m$ has $2(2m-1)$ neighbors on the same vertical line and $2m-1$ neighbors on the other vertical line. Thus, we have

Claim 5.3 *For every* $u \in F_m$, $\deg_{G_m}(u) = 6m-3$.

Let $Q \subset F_m$ be a maximum clique in G_m, $Q_1 = Q \cap F'_m$, $Q_2 = Q \cap F''_m$. Suppose that the lowest point v_i in Q_i, $i = 1, 2$, has the y-coordinate y_i, and the highest point u_i in Q_i has the y-coordinate $y_i + \frac{s_i}{m-1/2}$. Then $|Q| = s_1 + s_2 + 2$. On the other hand, by Claim 5.2, we have

$$y_2 \ge y_1 + \frac{s_1}{m-1/2} - 1 + \frac{1}{2m-1} \quad \text{and} \quad y_1 \ge y_2 + \frac{s_2}{m-1/2} - 1 + \frac{1}{2m-1}.$$

Summing the last two inequalities we get

$$0 \geq \frac{s_1 + s_2 + 1}{m - 1/2} - 2,$$

i.e., $s_1 + s_2 + 1 \leq 2m - 1$. Hence, we have

Claim 5.4 $\omega(G_m) \leq 2m$.

Thus, for every even k, the graph $G_{k/2}$ is a $(3k-3)$-regular intersection graph of unit circles with clique number k. A bad side of $G_{k/2}$ is that it is an infinite graph. In order to obtain a finite graph with properties of $G_{k/2}$, we first add one more observation on G_m.

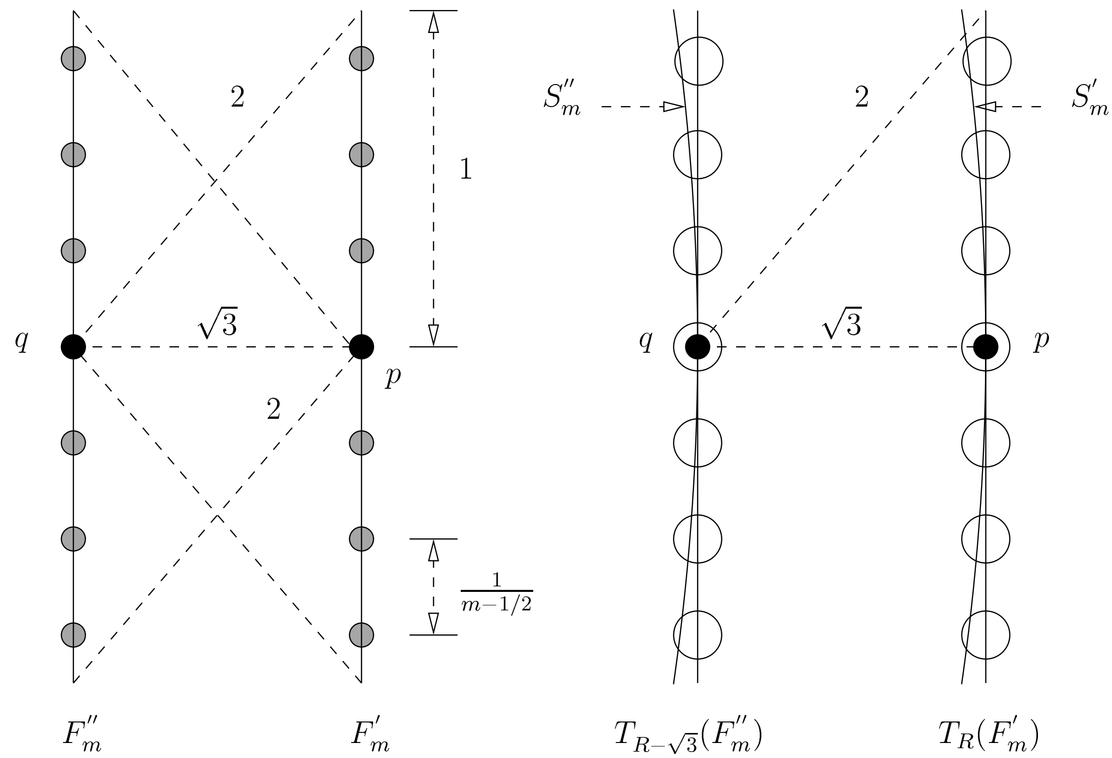

Figure 3: A fragment of F_4 (left) and S_4 (right)

Claim 5.5 *Let $u \in F'_m$, $v \in F''_m$. If $\rho(u,v) \leq 2$, then $\rho(u,v) < 2 - \frac{1}{8m}$. If $\rho(u,v) > 2$, then $\rho(u,v) \geq 2 + \frac{1}{8m}$ for $m \geq 2$.*

Proof. Assume that $\rho(u,v) \leq 2$. Then by Claim 5.2,

$$2 - \rho(u,v) \;\geq\; 2 - \sqrt{3 + \left(1 - \frac{1}{2m-1}\right)^2} = \frac{4 - 3 - \left(1 - \frac{1}{2m-1}\right)^2}{2 + \sqrt{3 + \left(1 - \frac{1}{2m-1}\right)^2}} \tag{1}$$

$$\geq \frac{1}{4}\left(\frac{2}{2m-1} - \frac{1}{(2m-1)^2}\right) \geq \frac{1}{4}\left(\frac{1}{2m-1}\right) > \frac{1}{8m}. \tag{2}$$

The calculations for the second inequality are very similar. $\square$

Now, let N be a big positive integer (say, $N = 10^6$) and $R = \frac{(2m-1)N}{\pi}$. Consider the transformation T of the plane moving every point with Cartesian coordinates (x, y) into the point with polar coordinates $(|x|, \frac{y}{R})$. For every positive x_0, the function $T_{x_0}(y) = T(x_0, y)$ is a periodic function with period $\pi R = (2m-1)N$ mapping the line $x = x_0$ onto the circle $x^2 + y^2 = x_0^2$. Let $S'_m = T(F'_m) = T_R(F'_m)$ and $S''_m = T(F''_m) = T_{R-\sqrt{3}}(F''_m)$. Then

$$S'_m = \left\{ \left(R\cos\frac{2\pi j}{(2m-1)^2N}, R\sin\frac{2\pi j}{(2m-1)^2N} \right) : j = 0, 1, \ldots, (2m-1)^2N \right\} \text{ and}$$
$$S''_m = \left\{ \left((R-\sqrt{3})\cos\frac{2\pi j}{(2m-1)^2N}, (R-\sqrt{3})\sin\frac{2\pi j}{(2m-1)^2N} \right) : j = 0, \ldots, (2m-1)^2N \right\}.$$

We claim that the intersection graph H_m of unit circles with centers in $S_m = S'_m \cup S''_m$ is also $(6m-3)$-regular and has clique number $2m$. The reason for this is that if two points in F_m are 'far' (i.e., on distance more than 2) and the corresponding points in S_m do not coincide, then these corresponding points also are 'far' apart, and that if two points in F_m are 'close', then the distance between them in S_m is almost the same. It is enough to consider situations with points $p = (R, 0)$ and $q = (R-\sqrt{3}, 0)$ (see Fig.3 (right)). Recall that $T(p) = p$ and $T(q) = q$.

Let B be the box $\{(x, y) : R-\sqrt{3} \le x \le R;\ -3 \le y \le 3\}$. We want to prove that for every point $u \in B \cap F_m$, the distance from u to p (respectively, q) is at most 2 if and only if the distance from $T(u)$ to p (respectively, q) is at most 2. Let $s = (x_0, y_0)$ be a point in B. Then $T(s) = (x_0\cos\frac{y_0}{R}, x_0\sin\frac{y_0}{R})$. Observe that

$$x_0 - x_0\cos\frac{y_0}{R} = 2x_0\sin^2\frac{y_0}{2R} \le 2R\left(\frac{3}{2R}\right)^2 \le \frac{9}{2R} < \frac{1}{20m}.$$

Similarly, $y_0 - x_0\sin\frac{y_0}{R} = (y_0 - x_0\frac{y_0}{R}) + x_0(\frac{y_0}{R} - \sin\frac{y_0}{R})$,

$$|y_0 - x_0\frac{y_0}{R}| \le \frac{|y_0|(R-R+2)}{R} \le \frac{6}{R} < \frac{1}{40m} \quad \text{and}$$
$$\left|x_0(\frac{y_0}{R} - \sin\frac{y_0}{R})\right| \le x_0\left|\left(\frac{y_0}{R}\right)^3 \Big/ 6\right| \le R\frac{27}{6R^3} < \frac{1}{40m}.$$

Therefore, for every $b \in B$, the distance between b and $T(b)$ is less than $\frac{1}{10m}$.

Let $q = (R-\sqrt{3}, 0)$. For each $b \in B \cap F_m - p$, the distance from q to $T(b)$ is less than the distance from q to b. Hence the degree of q in H_m is at least as big as in G_m. On the other hand, since the distance between b and $T(b)$ is less than $\frac{1}{10m}$, Claim 5.5 yields that q gets in H_m no new neighbor from $B \cap F_m$.

The case for $p = (R, 0)$ is very similar. T moves the points in $B \cap F''_m$ slightly away from p, but Claim 5.5 helps us again.

For every even k, this gives a finite graph $H_{k/2}$ that is a $(3k-3)$-regular intersection graph of unit circles with clique number k.

Example 3. Fix a positive real R. For a positive integer m, let $M'_m = \{(R, \frac{i+1/2}{m}) : i = 0, \pm 1, \pm 2, \ldots\}$, $M''_m = \{(R - \sqrt{3}, \frac{i}{m}) : i = 0, \pm 1, \pm 2, \ldots\}$, and $M_m = M'_m \cup M''_m$. This family is similar to F_m in Example 2, but the denominator for the y-coordinates of points is different and points in M'_m are shifted by $\frac{1}{2m}$ with respect to points in M''_m. Essentially repeating the argument of Example 2, we can see that the clique number of the intersection graph G'_m of unit circles with centers in M_m is $2m + 1$ and that G'_m is $6m$-regular. Then exactly as in Example 2, we obtain from G'_m a finite $6m$-regular intersection graph of unit circles in the plane with clique number $2m+1$. This shows that the bound of Theorem 1 is tight.

Remark. We don't know whether the bound of Theorem 3 is tight or not.

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