Bijections and Congruences for Generalizations of Partition Identities of Euler and Guy

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Abstract

In 1958, Richard Guy proved that the number of partitions of n into odd parts greater than one equals the number of partitions of n into distinct parts with no powers of 2 allowed, which is closely related to Euler's famous theorem that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. We consider extensions of Guy's result, which naturally lead to a new algorithm for producing bijections between various equivalent partition ideals of order 1, as well as to two new infinite families of parity results which follow from Euler's Pentagonal Number Theorem and a well-known series-product identity of Jacobi.

1 Introduction

A *partition* λ of the integer n is a representation of n as a sum of positive integers wherein the order of the summands is considered irrelevant. Accordingly, the summands can be rearranged in any order that seems convenient. Often in the literature, the summands are placed in nonincreasing order, but as we shall see, other canonical representations of partitions are useful in various contexts. A summand in a partition is called a *part* of the partition.

One of the most elegant partition identities known was discovered by Euler:

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Euler's Partition Identity. *The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Proofs of this result abound [3, p. 5], [9, p. 277].

In a brief note published in 1958, Richard Guy [8] gave a variety of proofs for a similar result:

Guy's Partition Identity. *The number of partitions of n into odd parts greater than one equals the number of partitions of n into distinct parts which are not powers of 2.*

In 1971, M.V. Subbarao generalized the results of Euler and Guy as follows:

Subbarao's Partition Identity. *For any sets of positive integers S_1 and S_2, the number of partitions of n into parts taken from S_1 equals the number of partitions of n into parts taken from S_2 with no part repeated more than $m-1$ times if and only if $mS_2 \subseteq S_2$ and $S_1 = S_2 - mS_2$.*

Our goal in this paper is to naturally extend the results of Guy and Subbarao via a 2-parameter generalization, which will in turn motivate some more general results. A bijective proof of an extension of Guy's result (Theorem 3.1), combined with P. A. MacMahon's "partitions of infinity", will naturally suggest bijective proofs for an infinite class of equivalent partition ideals of order 1, as defined by Andrews [2]. This rather general result is given as Theorem 3.6. Note that the bijections produced are those of Remmel [24], but the algorithm given for producing them here is direct and straight forward, whereas Remmel's method is based on the involution principle of Garsia and Milne [5], and is therefore quite arduous. Gordon [6] and O'Hara [20] have supplied accelerated algorithms for producing the Remmel bijections, but these still do not remove the mystery of why Glaisher-type bijections arise from their application. We shall discuss this further in Section 3.4. We will also prove a number of parity results for two special cases of our 2-parameter generalization in Section 4. We shall begin by reviewing the necessary background material in Section 2.

2 Background Material

2.1 Andrews' Partition Ideals of Order 1

In [2], and again in [3, Chapter 8], Andrews demonstrated how to place a lattice theoretic structure on certain sets of partitions, and derived some rather general results which explain a large class of partition identities. In order to keep this present work relatively self-contained, we shall informally review the material necessary for our present purposes. The interested reader is strongly encouraged to consult [2] or [3] directly.

Many famous partition identities involve restricted sets of partitions C in which the conditions on the parts are such that if one or more parts of a partition $\lambda \in C$ are removed, the resulting partition λ' is also in C. For example, in Euler's partition identity, a partition which has only odd parts will still be a partition into only odd parts if one or more of its parts are removed. The same property holds for partitions into distinct parts and to both classes of partitions mentioned in Guy's partition identity. Such sets of partitions are called *partition ideals*.

Any partition λ may be written in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \cdots ,$$

or more briefly, as

$$\{f_1, f_2, f_3, f_4, \dots\},$$

where f_i represents the number of times the positive integer i occurs as a part in the partition.

For example, the partition

$$\begin{aligned} &6+6+6+6+4+4+3+2+2+2+2+1+1 \\ &= 2\cdot 1 + 4\cdot 2 + 1\cdot 3 + 2\cdot 4 + 0\cdot 5 + 4\cdot 6 + 0\cdot 7 + 0\cdot 8 + 0\cdot 9 + \cdots \end{aligned}$$

may be represented by the frequency sequence

$$\{2, 4, 1, 2, 0, 4, 0, 0, 0, 0, 0, 0, \dots\}.$$

Thus each sequence $\{f_i\}_{i=1}^{\infty}$, where each f_i is a nonnegative integer and only finitely many of the f_i are nonzero, represents a partition of the integer $\sum_{i=1}^{\infty} f_i \cdot i$. Let $\mathcal{S}$ denote the set of all such sequences $\{f_i\}_{i=1}^{\infty}$.

Andrews goes on to define the *order* of a partition ideal [2, p. 19, Definition 8]. Informally, the order of a partition ideal is the "width of the lens" necessary to determine whether or not a given partition is a member of a given partition ideal. Each of the partition ideals in the partition identities of Euler and Guy is of order 1 since one never has to look at more than one f_i at a time to determine whether a given partition is in the partition ideal in question. For example, the partition ideal $\mathcal{O}$ of partitions into odd parts contains all partitions whose frequency sequence representation has $f_i = 0$ whenever i is even. The partition ideal $\mathcal{D}$ of partitions into distinct parts contains all partitions whose frequency sequence representation has $f_i \leq 1$ for all positive integers i.

In contrast, the partition ideal $\mathcal{R}$ of partitions into distinct, nonconsecutive parts is a partition ideal of order greater than 1 because consideration of individual terms of $\{f_i\}_{i=1}^{\infty}$ is not always sufficient to determine whether or not a given partition is in $\mathcal{R}$.

To formally define a partition ideal of order 1, we need to first define the auxilliary sequence $\lambda_k^{(1)}$. If λ is a partition (thought of in terms of its frequency sequence representation $\{f_i\}_{i=1}^{\infty}$), then define $\lambda_k^{(1)}$ to be the sequence obtained by changing every term of $\{f_i\}_{i=1}^{\infty}$ to 0 except for the kth term, which is left unchanged. Thus,

$$\lambda_k^{(1)} := \{0, 0, 0, 0, 0, \dots, 0, 0, f_k, 0, 0, 0, 0, \dots\},$$

where f_k is the kth term of the sequence. A partition ideal C is of order 1 if and only if for any partition λ which is *not* in C, there exists a corresponding $\lambda_k^{(1)}$ such that $\lambda_k^{(1)}$ is also not in C.

Andrews [1, p. 124, Theorem 8.1] proved that a partition ideal C is of order 1 if and only if there exists a sequence $\{d_j\}_{j=1}^{\infty}$ such that

$$C = \left\{ \{f_i\}_{i=1}^{\infty} \in \mathcal{S} \,\middle|\, f_i \leq d_i \text{ for all } i \right\}$$

where each d_i is a nonnegative integer or $+\infty$. Since, for each positive integer j, it suffices to take $d_j = \sup_{\{f_i\}_{i=1}^{\infty}} f_j$, let us make the following definition:

Definition 2.1. For any partition ideal C of order 1, define its *minimal bounding sequence*, denoted by $\{d_j^C\}_{j=1}^{\infty}$, by

$$d_j^C = \sup_{\{f_i\}_{i=1}^{\infty} \in C} f_j,$$

for $j = 1, 2, 3, \ldots$.

Thus, using the examples mentioned above, $\{d_j^{\mathcal{O}}\}_{j=1}^{\infty} = \{\infty, 0, \infty, 0, \infty, 0, \ldots\}$ and $\{d_j^{\mathcal{D}}\}_{j=1}^{\infty} = \{1, 1, 1, 1, 1, 1, \ldots\}$.

Let $p(C, n)$ denote the number of partitions of an integer n in the partition ideal C. We say that two partition ideals C_1 and C_2 are equivalent, and write $C_1 \sim C_2$, if $p(C_1, n) = p(C_2, n)$ for all integers n. (It is easily verified that $\sim$ is an equivalence relation.) Thus Euler's partition identity is merely the assertion that $\mathcal{O} \sim \mathcal{D}$. Similarly, Guy's partition theorem asserts the equivalence of two partition ideals.

Finally, we make the following definition:

Definition 2.2. For any partition ideal C, define *the multiset associated with* C, $M(C)$, as follows:

$$M(C) := \{ j(d_j^C + 1) \,\big|\, j \in \mathbb{N} \text{ and } d_j^C < \infty \},$$

where $\mathbb{N}$ denotes the positive integers.

Andrews [2, p. 22, Theorem 3] classified all partition ideals of order 1 by proving that $C_1 \sim C_2$ if and only if $M(C_1) = M(C_2)$.

2.2 MacMahon's Partitions of Infinity

In a footnote [17, p. 119], and again in [18, p. 642], MacMahon defined a *partition of infinity* to be a formal expression of the form

$$(g_1 - 1) \cdot 1 + (g_2 - 1) \cdot g_1 + (g_3 - 1) \cdot (g_1 g_2) + (g_4 - 1) \cdot (g_1 g_2 g_3) + \cdots$$

where each g_i is an integer larger than 1, *or* for some fixed K, $g_1, g_2, g_3, \ldots, g_{K-1}$ are each integers greater than 1, $g_K = \infty$, and $g_i = 1$ if $i > K$.

Note that a partition of infinity may be thought of as the partition ideal of order 1 with minimal bounding sequence given by

$$d_i = \begin{cases} g_1 - 1, & \text{if } i = 1 \\ g_{k+1} - 1, & \text{if } i = \prod_{j=1}^{k} g_j \text{ for some } k \in \mathbb{Z}_+ \\ 0, & \text{otherwise.} \end{cases}$$

An important special case of a partition of infinity occurs when $g_i = m$ for all i, where m is a fixed integer greater than 1. This leads to the "base m expansion" of any integer.

All partitions of infinity have generating function $1/(1-q)$, and thus contain a unique partition of each nonnegative integer.

3 Extending Guy's Partition Identity

3.1 A first step

Fix an integer $m > 1$ and an integer $j > 0$ such that $m \nmid j$. Let $p_1(n; m, j)$ be the number of partitions of n with no parts divisible by m and no parts equal to j. Let $p_2(n; m, j)$ be the number of partitions of n with no parts of the form $m^k j$, $k \geq 0$, and at most $m-1$ copies of each part present. The following theorem then generalizes Guy's result (which is the case $m = 2$, $j = 1$ of Theorem 3.1).

Theorem 3.1. *For all n, $p_1(n; m, j) = p_2(n; m, j)$.*

Remark 3.2. We note that Theorem 3.1 is a simple corollary of Andrews' classification theorem for partition ideals of order 1 [2, p. 22, Theorem 3], which was proved via generating functions. We provide a proof via a bijection between the two sets of partitions in question. The idea behind this bijection appears to have been used first by Glaisher [7] in a slightly different setting. We present it here to motivate several natural generalizations, which will lead to bijections between a large number of pairs of equivalent partition ideals of order 1 in Sections 3.2 and 3.3.

Proof of Theorem 3.1. Consider a partition λ of n which is counted by $p_1(n; m, j)$. Then we know that no part λ_i of λ is equal to j and no part λ_i is divisible by m. Thus,

$$n = f_{\lambda_1}\lambda_1 + f_{\lambda_2}\lambda_2 + \cdots + f_{\lambda_s}\lambda_s,$$

for some frequency values $f_{\lambda_1}, f_{\lambda_2}, \ldots, f_{\lambda_s}$. To obtain a partition λ' which is counted by $p_2(n; m, j)$, we write each coefficient f_{λ_i} in base m notation. That is,

$$f_{\lambda_i} = a_{\lambda_i,k}m^k + a_{\lambda_i,k-1}m^{k-1} + \cdots + a_{\lambda_i,1}m + a_{\lambda_i,0}$$

with $0 \leq a_{i,j} \leq m-1$ for each i and j. Then we know

$$\begin{aligned} n \quad = \quad & \ (a_{\lambda_1,0} + a_{\lambda_1,1}m + a_{\lambda_1,2}m^2 + \cdots + a_{\lambda_1,k}m^k)\lambda_1 \\ & + (a_{\lambda_2,0} + a_{\lambda_2,1}m + a_{\lambda_2,2}m^2 + \cdots + a_{\lambda_2,k}m^k)\lambda_2 \\ & \vdots \\ & + (a_{\lambda_s,0} + a_{\lambda_s,1}m + a_{\lambda_s,2}m^2 + \cdots + a_{\lambda_s,k}m^k)\lambda_s. \end{aligned}$$

Distributing each of the λ_i over the base m expansion of f_{λ_i}, we obtain

$$\begin{aligned} n \;=\; & \quad a_{\lambda_1,0}(\lambda_1) + a_{\lambda_1,1}(\lambda_1 m) + a_{\lambda_1,2}(\lambda_1 m^2) + \cdots + a_{\lambda_1,k}(\lambda_1 m^k) \\ & + a_{2,0}(\lambda_2) + a_{\lambda_2,1}(\lambda_2 m) + a_{\lambda_2,2}(\lambda_2 m^2) + \cdots + a_{\lambda_2,k}(\lambda_2 m^k) \\ & \vdots \\ & + a_{\lambda_s,0}(\lambda_s) + a_{\lambda_s,1}(\lambda_s m) + a_{\lambda_s,2}(\lambda_s m^2) + \cdots + a_{\lambda_s,k}(\lambda_s m^k). \end{aligned}$$

Since $\lambda_i \neq j$ and $m \nmid \lambda_i$ for each i, $1 \leq i \leq s$, we know that none of the parenthesized terms of the form $(m^t \lambda_i)$ in the above are of the form $(m^l j)$ for some integer l. Thus, we know that the above is a partition counted by $p_2(n; m, j)$ since each coefficient $a_{i,j} \leq m-1$ for each value of i and j.

The inverse of the map is more straightforward to write. We begin with a partition $\lambda' = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ counted by $p_2(n; m, j)$. For each i, $1 \leq i \leq r$, we write $\lambda_i = m^{k_i}\alpha_i$ with $k_i \geq 0$ and $(m, \alpha_i) = 1$. Then the corresponding partition λ counted by $p_1(n; m, j)$ is

$$n = m^{k_1}\alpha_1 + m^{k_2}\alpha_2 + \cdots + m^{k_r}\alpha_r.$$

Note that $m \nmid \alpha_i$ for each i and that $\alpha_i \neq j$ for any i (since no part λ_i is of the form $m^k j$ for some k). Therefore, λ is indeed a partition counted by $p_1(n; m, j)$. $\square$

An example may prove beneficial at this time.

Example 3.3. Consider the partition

$$\begin{aligned} &8+8+8+8+8+7+5+5+5+4+4+1+1+1+1+1+1+1+1+1+1+1+1+1 \\ &= (13)1 + (2)4 + (3)5 + (1)7 + (5)8 \end{aligned}$$

counted by $p_1(83; 3, 2)$. Writing the base 3 expansion of each of the parenthesized frequencies, we have

$$\begin{aligned} 83 \;=\; & \quad (1\cdot 1 \; + 1\cdot 3 + 1\cdot 3^2)\,1 \\ & + (2\cdot 1)\,4 \\ & + (\quad\;\; + 1\cdot 3)\,5 \\ & + (1\cdot 1)\,7 \\ & + (2\cdot 1 \; + 1\cdot 3)\,8, \end{aligned}$$

and by distributing the λ_i over the expanded frequencies, we obtain

$$\begin{aligned} 83 \;=\; & \quad 1(1) \; + 1(3) + 1(9) \\ & + 2(4) \\ & + \qquad\; 1(15) \\ & + 1(7) \\ & + 2(8) \; + 1(24). \end{aligned}$$

This provides the partition

$$24 + 15 + 9 + 8 + 8 + 7 + 4 + 4 + 3 + 1$$

counted by $p_2(83;3,2)$.

Similarly, we can write the parts of the partition

$$24+15+9+8+8+7+4+4+3+1$$

counted by $p_2(83;3,2)$ as

$$3^1\cdot 8+3^1\cdot 5+3^2\cdot 1+3^0\cdot 8+3^0\cdot 8+3^0\cdot 7+3^0\cdot 4+3^0\cdot 4+3^1\cdot 1+3^0\cdot 1$$

which yields the partition

$$8+8+8+8+8+7+5+5+5+4+4+1+1+1+1+1+1+1+1+1+1+1+1+1$$

rather quickly.

A comment is in order regarding another proof technique for Theorem 3.1. In 1969, Andrews [1] defined an *Euler pair* as a pair of sets (S_1, S_2) such that, for all natural numbers n, the number of partitions of n into parts taken from S_1 equals the number of partitions of n into *distinct* parts taken from S_2. Two years later, M. V. Subbarao [25] generalized Andrews' idea by defining an Euler pair (S_1, S_2) of order m where the number of partitions of n into parts taken from S_1 equals the number of partitions of n into parts taken from S_2 with no part repeated more than $m-1$ times in any partition. He then proved that (S_1, S_2) is an Euler pair of order m if and only if $mS_2 \subseteq S_2$ and $S_1 = S_2 - mS_2$. We close this section by noting that Theorem 3.1 can be proved using Subbarao's result. The set of parts allowable in those partitions counted by $p_1(n;m,j)$ is $S_1 = \mathbb{N} - (\{j\} \cup \{mi \mid i \in \mathbb{N}\})$, while the set of parts allowable in those partitions counted by $p_2(n;m,j)$ is $S_2 = \mathbb{N} - \{m^k j \mid k = 0,1,2,\dots\}$. It is a straightforward exercise to prove that $mS_2 \subseteq S_2$ and $S_1 = S_2 - mS_2$, yielding infinitely many (previously unpublished) Euler pairs of order m.

3.2 Further generalization

Let us now define $p_1(n;m,J)$ where n is an integer, m is a positive integer and J is a *set* of positive integers, none of which is a multiple of m, to be the number of partitions of n into nonmultiples of m where no element of the set J appears as a part. A bijective map between the partitions enumerated by $p_1(n;m,J)$ and another class of partitions can be found by a generalization of the method given in the proof of Theorem 3.1.

The idea is as follows: For any valid m and J, the partitions enumerated by $p_1(n;m,J)$ come from a partition ideal of order 1 whose minimal bounding sequence contains only zeros and infinities. In Theorem 3.1, the bijection was created by taking the base m expansion of each f_i. But, the base m expansion is simply a special case of MacMahon's partitions of infinity. So by expanding the f_i by various different partitions of infinity, we can map the partitions enumerated by $p_1(n;m,J)$ to *any other partition ideal of order 1 which is in the same equivalence class.*

Note also that the set of equivalence classes which contain a partition ideal whose minimial bounding sequence consists only of zeros and infinities is precisely the equivalence class of order 1 partition ideals C whose associated multiset $M(C)$ contains no repeated elements.

Also, we acknowledge that while the use of two parameters m and J provides a convenient way to generalize the Euler and Guy identities, the m is actually superfluous in the following sense: once we allow J to be a *set* of forbidden parts, we have the freedom to allow J to contain all multiples of some fixed m, or not, as we see fit.

Let now us consider the following example.

Example 3.4. We will give a bijective proof of the fact that the number of partitions of n into parts not equal to 2, 5, 6, 9, 10, 12, 18 or 20 is equal to the number of partitions of n where 5 and 6 do not appear as parts, 1, 9, and 10 may appear at most once, 3 and 4 may appear at most twice, 2 may appear at most four times, and all other positive integers may appear without restriction. The former set of partitions C is a partition ideal of order 1 whose minimal bounding sequence has a zero for terms 2, 5, 6, 9, 10, 12, 18, and 20, and ∞ for all other terms. The latter set of partitions C' has minimal bounding sequence

$$\{d_j^{C'}\}_{j=1}^{\infty} = \{1, 4, 2, 2, 0, 0, \infty, \infty, 1, 1, \infty, \infty, \infty, \infty, \infty, \infty, \dots\}.$$

Any partition of n in C can be written in the form

$$n = f_1 \cdot 1 + f_3 \cdot 3 + f_4 \cdot 4 + f_7 \cdot 7 + f_8 \cdot 8 + f_{11} \cdot 11 + \sum_{i=13}^{17} f_i \cdot i + f_{19} \cdot 19 + \sum_{i=21}^{\infty} f_i \cdot i,$$

where each f_i is a nonnegative integer.

- Expand f_1 by the partition of infinity defined by $g_{1,1} = 2$, $g_{1,2} = 5$, $g_{1,3} = 2$, $g_{1,4} = \infty$, $g_{1,k} = 1$ if $k > 4$.
- Expand f_3 by the partition of infinity defined by $g_{3,1} = 3$, $g_{3,2} = 2$, $g_{3,3} = \infty$, $g_{3,k} = 1$ if $k > 3$.
- Expand f_4 by the partition of infinity defined by $g_{4,1} = 3$, $g_{4,2} = \infty$, $g_{4,k} = 1$ if $k > 2$.

Thus we have

$$\begin{aligned} n \;=\; & \big(a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(2\cdot 5) + a_{1,4}(2\cdot 5\cdot 2)\big)1 \\ &+ \big(a_{3,0}(1) + a_{3,1}(3) + a_{3,2}(3\cdot 2)\big)3 \\ &+ \big(a_{4,0}(1) + a_{4,1}(3)\big)4 \\ &+ \big(a_{7,0}(1)\big)7 + \big(a_{8,0}(1)\big)8 + \big(a_{11,0}(1)\big)11 + \big(a_{13,0}(1)\big)13 + \cdots \end{aligned}$$

where $0 \le a_{j,k} \le g_{j,k+1} - 1 = d^{C'}_{(j)(g_{j,1})(g_{j,2})\dots(g_{j,k})}$. (Recall that every partition of infinity contains a unique partition of each nonnegative integer n, so the preceeding decomposition is uniquely determined.)

Apply the distributive property to obtain

$$\begin{aligned} n \;=\; & \quad a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(10) + a_{1,4}(20) \\ & + a_{2,0}(3) + a_{2,1}(9) + a_{2,2}(18) \\ & + a_{4,0}(4) + a_{4,1}(12) \\ & + a_{7,0}(7) + a_{8,0}(8) + a_{11,0}(11) + a_{13,0}(13) + \cdots \end{aligned}$$

It should be clear that the proposed partitions of infinity in the preceeding example allow the construction of the desired bijection. But how can one *find* the appropriate partitions of infinity necessary to map C to C'? We propose the following algorithm for generating the necessary partitions of infinity which give rise to a bijective map $\pi : C \to C'$, where C is a partition ideal of order 1 whose minimal bounding sequence consists of all zeros and infinities, and C' is any partition ideal of order 1 which is equivalent to C.

Algorithm 3.5. *Input:* The respective minimal bounding sequences $\{d_j^C\}_{j=1}^{\infty}$ and $\{d_j^{C'}\}_{j=1}^{\infty}$, for two equivalent partition ideals C and C' where for all j, $d_j^C = 0$ or ∞.
Output: For each j in $\{d_j^C\}_{j=1}^{\infty}$ such that $d_j^C = \infty$, a corresponding partition of infinity as indicated by the sequence $\{g_{j,k}\}_{k=1}^{\infty}$.

1. Set $j := 1$.
2. If $d_j^C = \infty$, then go to step 5, otherwise go to step 3.
3. Increment j.
4. Go to step 2.
5. $g_{j,1} := d_j^{C'} + 1$.
6. If $g_{j,1} = \infty$, then go to step 7, otherwise go to step 10.
7. $g_{j,k} := 1$ for all $k > 1$.
8. Increment j.
9. Go to step 2.
10. Set $k = 2$.
11. $g_{j,k} := d^{C'}_{(j)(g_{j,1})(g_{j,2})(g_{j,3})\dots(g_{j,k-1})} + 1$.
12. If $g_{j,k} = \infty$, then go to step 13, otherwise go to step 16.
13. $g_{i,L} := 1$ for all $L > k$.
14. Increment j.

15. Go to step 2.

16. Increment k.

17. Go to step 11.

Proof of the correctness of Algorithm 3.5. Let $S = \{j | d_j^C \neq 0\}$ and $S' = \{j | d_j^{C'} \neq 0\}$.

1. *Claim: No element of S' is generated more than once by the algorithm.*

   Proof of First Claim. Suppose not. Then there exists i, I, l and L with $i \neq l$ such that $ig_{i,1}g_{i,2}\dots g_{i,I} = lg_{l,1}g_{l,2}\dots g_{l,L}$. Without loss of generality, suppose that $i < l$ and that of all such coincident products, this is the least.

   - If $I = L = 0$, then $i = l$, a contradiction.
   - If $I = 0$ and $L > 0$, then $i > l$, a contradiction.
   - If $I > 0$ and $L = 0$, then $ig_{i,1}g_{i_2}\dots g_{i,I} = l$. Since $d_l^C = \infty$, it must be the case that $l \notin M(C)$. Also, $d^{C'}_{(i)(g_{i,1})(g_{i_2})\dots(g_{i,I-1})} = g_{i,I} - 1$. So the value of $j(d_j^{C'} + 1)$, when $j = (i)(g_{i,1})(g_{i,2})\dots(g_{i,I-1})$, is $(ig_{i,1}g_{i_2}\dots g_{i,I-1})(g_{i,I}) = l$. Thus $l \in M(C')$. But we already saw that $l \notin M(C)$. Thus $C \not\sim C'$, a contradiction.
   - If $I > 0$ and $L > 0$, then

     $$ig_{i,1}g_{i,2}\dots g_{i,I} = lg_{l,1}g_{l,2}\dots g_{l,L}. \tag{1}$$

     Note that $g_{i,I} - 1 = d^{C'}_{(i)(g_{i,1})(g_{i,2})\dots(g_{i,I-1})}$ and $g_{l,L} - 1 = d^{C'}_{(l)(g_{l,1})(g_{l,2})\dots(g_{l,L-1})}$. Consider two values of $j(d_j^{C'}+1)$. First, if $j = ig_{i,1}g_{i,2}\dots g_{i,I-1}$, then $j(d_j^{C'}+1) = ig_{i,1}g_{i,2}\dots g_{i,I}$. Next, if $j = lg_{l,1}g_{l,2}\dots g_{l,L-1}$, then $j(d_j^{C'}+1) = lg_{l,1}g_{l,2}\dots g_{l,L}$. (Note that $ig_{i,1}g_{i,2}\dots g_{i,I-1} \neq lg_{l,1}g_{l,2}\dots g_{l,L-1}$, because this would contradict the assumption that the coincident products in (1) are minimal.) Therefore, by (1), two distinct $j \in \mathbb{Z}_+$ give rise to the same contribution to $M(C')$. Thus $M(C')$ has a repeated element. But, since $\{d_j^C\}_{j=1}^{\infty}$ contains only zeros and infinities, $M(C)$ has no repeated elements. Thus $M(C) \neq M(C')$. Therefore, $C \not\sim C'$, a contradiction.

   □

2. *Claim: Every element of S' arises as the result of multiplying some $i \in S$ by a term of some partition of infinity as determined by the algorithm.* That is, if $\sigma \in S'$, then for some $i \in S$, and some nonnegative integer I, $\sigma = i\prod_{l=1}^{I} g_{i,l}$, where $\prod_{l=1}^{0} g_{i,l}$ is the empty product, which is taken to be 1.

   Proof of Second Claim. We consider the two cases.

- If $\sigma \in S$, then $d_\sigma^C = \infty$. Expanding this ∞ by any partition of infinity yields

$$\big((g_{\sigma,1} - 1) \cdot 1 + \dots\big)\sigma,$$

thus σ is generated by $i = \sigma$, $I = 0$, and we are done.

- Next, consider the case where $\sigma \in S' \setminus S$.

We will show that the value of any finite-valued expression $(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})$ with $I \geq 1$ is an element of $S' \setminus S$. Suppose that $d^{C'}_{(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})} = 0$. By step 11 of Algorithm 3.5, it follows that $g_{i,I+1} = 1$. But then $g_{i,K} = \infty$ for some $K \leq I$, which contradicts the finiteness of the expression $(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})$. Thus,

$$d^{C'}_{(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})} \neq 0,$$

and so

$$(j)(g_{j,1})(g_{j,2})\dots(g_{j,I}) \in S'.$$

Also, since $I \geq 1$, $(j)(g_{j,1})(g_{j,2})\dots(g_{j,I}) \notin S$, so the set of all finite valued expressions $(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})$ where $I > 1$ is contained in $S \setminus S'$.

We will now show that there is a one-to-one correspondence between the elements of $S' \setminus S$ and these $(j)(g_{j,1})(g_{j,2})\dots(g_{j,I})$.

There is a one-to-one correspondence between each $g_{j,k}$ for which Algorithm 3.5 assigns a finite value of at least two and each $d^{C'}_{(j)(g_{j,1})(g_{j,2})\dots(g_{j,k-1})}$ whose value is neither zero nor infinity (by steps 5 and 11 of Algorithm 3.5). Now the set of all such d's is, by definition, the set $S' \setminus S$. Map each $g_{j,k}$ which is assigned a finite value of at least two to the value $(j)(g_{j,1}g_{j,2}\dots g_{j,k})$. By the first claim, each such $(j)(g_{j,1}\dots g_{j,k})$ is unique, so it corresponds to some value σ in $S' \setminus S$.

□

The proof of the above two claims together comprises the proof of Algorithm 3.5. □

A Maple implementation of Algorthm 3.5 is available for free download from the second author's web site `http://www.math.rutgers.edu/~asills`.

With Algorithm 3.5 in hand, we may now state the desired bijection:

Theorem 3.6. *Given two equivalent partition ideals C and C', both of order 1, where d_j^C is either zero or infinity for all j, the following map $\pi : C \to C'$ is a bijection: Let $\lambda = \sum_{i=1}^\infty f_i \cdot i$ be a partition of n in the partition ideal C. Let $S = \{j | d_j^C \neq 0\}$ and $S' = \{j | d_j^{C'} \neq 0\}$.*

$$\pi(\lambda) = \sum_{i \in S} \left((a_{i,0})(i) + \sum_{k=1}^\infty a_{i,k} \left(i \prod_{l=1}^k g_{i,l} \right) \right),$$

where $\{g_{i,l}\}_{i \in S,\ l \in \mathbb{N}}$ is determined by Algorithm 3.5.

Proof. We may write

$$\begin{aligned} n &= \sum_{i=1}^{\infty} f_i \cdot i, \text{ for some frequencies } f_i \\ &= \sum_{i \in S} f_i \cdot i \\ &= \sum_{i \in S} \left(a_{i,0} + \sum_{k=1}^{\infty} a_{i,k} \left(\prod_{l=1}^{k} g_{i,l} \right) \right) \cdot i, \end{aligned}$$

where $a_{i,0} + \sum_{k=1}^{\infty} a_{i,k} \left(\prod_{l=1}^{k} g_{i,l} \right)$ is the unique representation of the number f_i in the partition of infinity determined by $\{g_{i,k}\}_{k=1}^{\infty}$. Thus,

$$\begin{aligned} n &= \sum_{i \in S} \left(a_{i,0}(i) + \sum_{k=1}^{\infty} a_{i,k} \left(i \prod_{l=1}^{k} g_{i,l} \right) \right) \\ &= \sum_{\sigma \in S'} a_{i,k}\sigma, \text{ where } \sigma = i \textstyle\prod_{l=1}^{k} g_{i,l} \end{aligned}$$

for some $i \in S$ and $k \geq 0$ as determined by Algorithm 3.5.

The last line represents a partition in C'. That the map π is well-defined and bijective follows from the proof of the correctness of Algorithm 3.5. ☐

3.3 More general bijections

Above we provide an algorthm for finding a bijective map $\pi_1 : C \to C'$ where C is a partition ideal of order 1 whose minimal bouding sequence contains only zeros and infinities, $C' \sim C$, and C' is also of order 1. If π_2 is the analogous bijective map from C to C''', (where $C \sim C''$ and C'' is also of order 1), we can compose the maps $\pi_1^{-1} \circ \pi_2$ to obtain a bijection between any two partition ideals of order 1 which lie in the same equivalence class as C. This provides bijections between all partition ideals C of order 1 whose associated multiset $M(C)$ contains no repeated elements, an infinite subclass of all partition ideals of order 1.

3.4 A comparison with Remmel's bijections

In [24], Remmel provides a method for finding bijections between many equivalent partition ideals, including *any* two equivalent partition ideals of order one. However, Remmel [24, p. 279] admits that "the actual algorithm is torturously inefficient." For example, even in the simple case of Euler's partition identity (page 1), to find the image of the partition $1+1+3+3+5+5$ of 18 (into odd parts), which is the partition $2+6+10$ (with distinct parts), Remmel's algorithm requires *fourteen* iterations. [24, p. 279, Table 1]. Furthermore, Remmel [24, p. 278] states that it "is remarkable that θ [the Glaisher bijection] is exactly the bijection given by our general bijection of Theorem 2." However, our methods here, inspired by the MacMahon partitions of infinity generalization of the Glaisher-type bijection, provide the bijection in a direct and transparent manner, and are arguably as elegant as Glaisher's original bijection [7]. One referee noted that it is possible to see that our bijections in fact recover those of Remmel via essentially the same argument as Remmel used to show that his bijections recovered those of Glaisher [24, p. 278, Thm. 4].

It must be noted that Basil Gordon [6] and Kathy O'Hara [20] each published algorithms for recovering the Remmel bijections in a more efficient manner. In [23], Peter Paule demonstrated that the Garsia-Milne Involution Principle is in fact a direct consequence of the Linkage Lemma of Ingleton and Piff [15, Lemma 3]. However, with these as with Remmel's algorithm, the fact that Glaisher-type bijections are found can be *proved* but is far from *obvious* that this will be the case. With Algorithm 3.5, it is *transparent* that the Glaisher bijections arise where applicable, and the nature of how they fit into a more general setting (i.e. that a "base m" expansion of a number is a special type of partition of infinity) is also clear.

An excellent summary of the bijective work of Garsia-Milne, Gordon, and O'Hara is presented by Wilf in [28, pp. 19–28].

Regarding Remmel's application of their involution principle, Garsia and Milne [5, p. 329] remark:

> In point of fact Remmel shows that a number of "ad hoc" bijections occuring in the literature can be derived in a systematic way from the involution principle... On the basis of what happens in the case of Euler's theorem we should be tempted to suspect that, forbidding as our bijection may look on the surface, there may be a more direct underlying "number theoretical" description for it in the style of the Glaisher bijection. Clearly this may be worth further investigation.

We believe that our algorithm may be an important step in this further investigation.

4 Parity Results for $p_1(n; 2, J)$ and $p_1(n; 4, J)$

A wide variety of parity results involving a number of different partition functions exist in the literature. The interested reader is encouraged to see [4, 11, 12, 13, 14, 16, 19, 21, 26, 27]. Our goal in this section is to prove a set of parity results for $p_1(n; 2, J)$ and $p_1(n; 4, J)$.

Lemma 4.1. *Fix a prime $p > 3$ and an integer j. If r is an integer strictly between 0 and p such that $24(r-j)+1$ is a quadratic nonresidue modulo p, then $pn + r - j \neq \frac{3}{2}m^2 - \frac{1}{2}m$ for any integers m and n.*

Proof. Suppose that $pn + r = \frac{3}{2}m^2 - \frac{1}{2}m + j$ for some integers m and n. Then,

$$\begin{array}{rrcll} & pn + r - j & = & \frac{3}{2}m^2 - \frac{1}{2}m & \\ \Longrightarrow & r - j & \equiv & \frac{3}{2}m^2 - \frac{1}{2}m & \pmod{p} \\ \Longrightarrow & 24(r-j) + 1 & \equiv & 36m^2 - 12m + 1 & \pmod{p} \\ & & \equiv & (6m-1)^2 & \pmod{p}. \end{array}$$

Thus $24(r-j)+1$ is a quadratic residue mod p, a contradiction. □

Lemma 4.2. *Fix a prime $p > 3$ and an integer j. If r is an integer strictly between 0 and p such that $8(r-j)+1$ is a quadratic nonresidue modulo p, then $pn + r - j \neq \frac{1}{2}m^2 + \frac{1}{2}m$ for any integers m and n.*

Proof. Suppose that $pn + r = \frac{1}{2}m^2 + \frac{1}{2}m + j$ for some integers m and n. Then,

$$\begin{array}{rrcll} & pn + r - j & = & \frac{1}{2}m^2 + \frac{1}{2}m & \\ \Longrightarrow & r - j & \equiv & \frac{1}{2}m^2 + \frac{1}{2}m & \pmod{p} \\ \Longrightarrow & 8(r-j) + 1 & \equiv & 4m^2 + 4m + 1 & \pmod{p} \\ & & \equiv & (2m+1)^2 & \pmod{p}. \end{array}$$

Thus $8(r-j)+1$ is a quadratic residue mod p, a contradiction. □

We will also require two classical results:

Euler's Pentagonal Number Theorem. *If $|q| < 1$, then*

$$\prod_{k \geq 1}(1 - q^k) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{3}{2}m^2 - \frac{1}{2}m}. \tag{2}$$

Jacobi's Identity. *If $|q| < 1$, then*

$$\prod_{k \geq 1}(1 - q^k)^3 = \sum_{m \geq 0} (-1)^m (2m+1) q^{\frac{1}{2}m^2 + \frac{1}{2}m}. \tag{3}$$

Proposition 4.3. *Fix a prime $p > 3$, an odd positive integer j, and an integer r strictly between 0 and p such that $24(r-j)+1$ is a quadratic nonresidue modulo p. Then the coefficient of q^{pn+r} in the expansion of*

$$q^j \prod_{k \geq 1} \frac{1}{1 - q^{2k-1}}$$

is even.

Proof.

$$\begin{aligned}
q^j \prod_{k\geq 1} \frac{1}{1-q^{2k-1}} &= q^j \prod_{k\geq 1} \frac{1-q^{2k}}{1-q^k} \\
&\equiv q^j \prod_{k\geq 1} \frac{(1-q^k)^2}{(1-q^k)} \pmod 2 \\
&= q^j \prod_{k\geq 1} (1-q^k) \\
&\equiv q^j \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}m^2-\frac{1}{2}m} \pmod 2 \quad \text{by (2)} \\
&= \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}m^2-\frac{1}{2}m+j}.
\end{aligned}$$

Thus, we see that

$$q^j \prod_{k\geq 1} \frac{1}{1-q^{2k-1}} \equiv \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}m^2-\frac{1}{2}m+j} \pmod 2. \tag{4}$$

Now we compare the coefficients of q^{pn+r} on either side of (4). By Lemma 4.1, $pn+r$ can never be represented as $\frac{3}{2}m^2 - \frac{1}{2}m + j$. So the contribution to the coefficient of q^{pn+r} on the right-hand side of (4) must be zero. Thus, the coefficient of q^{pn+r} in $q^j \prod_{k\geq 1} \frac{1}{1-q^{2k-1}}$ is even for all integers n. □

Definition 4.4. Let T_J be the set of all exponents which arise in the expansion of

$$\prod_{j\in J}(1-q^j),$$

thus

$$T_J := \left\{ \sum_{e\in E} e : E \subseteq J \right\},$$

where the sum over the empty set is taken to be zero.

Corollary 4.5. *Fix a prime $p > 3$, an integer r strictly between 0 and p, and a set W of odd positive integers. If $24(r-t)+1$ is a quadratic nonresidue mod p for every $t \in T_W$, then*

$$p_1(pn+r, 2; W) \equiv 0 \pmod 2$$

for all integers n.

Proof.

$$\sum_{n\geq 0} p_1(n;2,W)q^n = \prod_{\omega\in W}(1-q^\omega) \prod_{k\geq 1} \frac{1}{(1-q^{2k-1})} = \sum_{t\in T_W} q^t \prod_{k\geq 1} \frac{1}{(1-q^{2k-1})}.$$

Thus, by $|T_W|$ applications of Proposition 4.3,

$$p_1(pn+r;2,W) \equiv 0 \pmod 2$$

for all integers n. $\square$

Example 4.6. Take $p=5$, $W=\{1\}$, $r=4$. Then $T_W=\{0,1\}$. Note that $0<r=4<5=p$. Since $24(4-0)+1=97\equiv 2 \pmod 5$ and $24(4-1)+1=73\equiv 3 \pmod 5$, and 2 and 3 are quadratic nonresidues modulo 5, we know $p_1(5n+4;2,\{1\})\equiv 0 \pmod 2$ for all n.

Example 4.7. Take $p=17$, $W=\{1,3,17\}$, $r=14$. Then $T_W=\{0,1,3,4,17,18,20,21\}$. Note that for each $t\in T_W$, $24(14-t)+1$ is congruent to 3, 7, 10, or 14 (mod 17), and that 3, 7, 10, and 14 are all quadratic nonresidues modulo 17. Thus we know that

$$p_1(17n+14;2,\{1,3,17\})\equiv 0 \pmod 2$$

for all integers n.

Proposition 4.8. *Fix a prime $p>3$, a postive integer j which is not a multiple of 4, and an integer r strictly between 0 and p such that $8(r-j)+1$ is a quadratic nonresidue modulo p. Then the coefficient of q^{pn+r} in the expansion of*

$$q^j\prod_{k\geq 1}\frac{1-q^{4k}}{1-q^k}$$

is even.

Proof.

$$\begin{aligned}
q^j\prod_{k\geq 1}\frac{1-q^{4k}}{1-q^k} &\equiv q^j\prod_{k\geq 1}\frac{(1-q^k)^4}{(1-q^k)} \pmod 2\\
&= q^j\prod_{k\geq 1}(1-q^k)^3\\
&\equiv q^j\sum_{m\geq 0} q^{\frac{1}{2}m^2+\frac{1}{2}m} \pmod 2 \quad \text{by (3)}\\
&= \sum_{m\geq 0} q^{\frac{1}{2}m^2+\frac{1}{2}m+j}.
\end{aligned}$$

Thus, we see that

$$q^j\prod_{k\geq 1}\frac{1-q^{4k}}{1-q^k}\equiv\sum_{m\geq 0} q^{\frac{1}{2}m^2+\frac{1}{2}m+j} \pmod 2. \tag{5}$$

Now we compare the coefficients of q^{pn+r} on either side of (5). By Lemma 4.2, $pn+r$ can never be represented as $\frac{1}{2}m^2+\frac{1}{2}m+j$. So the contribution to the coefficient of q^{pn+r} on the right-hand side of (5) must be zero. Thus, the coefficient of q^{pn+r} in $q^j\prod_{k\geq 1}\frac{1-q^{4k}}{1-q^k}$ is even for all integers n. $\square$

Corollary 4.9. *Fix a prime $p > 3$, an integer r strictly between 0 and p, and a set $\mathcal{F}$ of positive integers which are not multiples of 4. If $8(r-t)+1$ is a quadratic nonresidue mod p for every $t \in T_{\mathcal{F}}$, then*

$$p_1(pn+r, 4; \mathcal{F}) \equiv 0 \pmod{2}$$

for all integers n.

Proof.

$$\sum_{n \geq 0} p_1(n; 4, \mathcal{F}) q^n = \prod_{f \in \mathcal{F}} (1-q^f) \prod_{k \geq 1} \frac{1-q^{4k}}{1-q^k} = \sum_{t \in T_{\mathcal{F}}} q^t \prod_{k \geq 1} \frac{1-q^{4k}}{1-q^k}.$$

Thus, by $|T_{\mathcal{F}}|$ applications of Proposition 4.8,

$$p_1(pn+r; 4, \mathcal{F}) \equiv 0 \pmod{2}$$

for all integers n. □

5 Concluding Remarks

In this paper, we show how to efficiently construct generalized Glaisher-type bijections between all partition ideals C of order 1 whose associated multiset $M(C)$ contains no repeated elements, an infinite subclass of all partition ideals of order 1. There are, however, uncountably many partition ideals $\mathcal{C}$ of order 1 whose associated multiset $M(\mathcal{C})$ *does* contain one or more repeated elements. We hope that this paper is a first step towards the loftier goal of finding a straightforward bijection between *any* two equivalent partition ideals of order 1. Since Remmel's application of the Garsia-Milne involution principle automatically produces Glaisher's bijections in the instances where Glaisher's original methods apply, it seems plausible that Remmel's bijections may be the "natural" (i.e. generalized Glaisher-type) ones, even in the instances where the methods of this paper are not applicable. Accordingly, it may be that some natural extension of the methods of this paper could be used to obtain Remmel's bijections via a straightforward bijection. This appears to merit further investigation.

The parity results stated in Section 4 use the fact that Euler's Pentagonal Number Theorem and Jacobi's Identity have nice series representations for the infinite product $\prod_{k \geq 1}(1-q^k)^{m-1}$ for $m = 2$ and $m = 4$ respectively. These in turn generate parity results for $p_1(n; m, J)$. Using series expansions of $\prod_{k \geq 1}(1-q^k)^{m-1}$ for values of m other than 2 and 4 could yield analogous results.

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