On the Counting of Fully Packed Loop Configurations: Some new conjectures

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Abstract

New conjectures are proposed on the numbers of FPL configurations pertaining to certain types of link patterns. Making use of the Razumov and Stroganov Ansatz, these conjectures are based on the analysis of the ground state of the Temperley-Lieb chain, for periodic boundary conditions and so-called "identified connectivities", up to size $2n = 22$.

1. Introduction

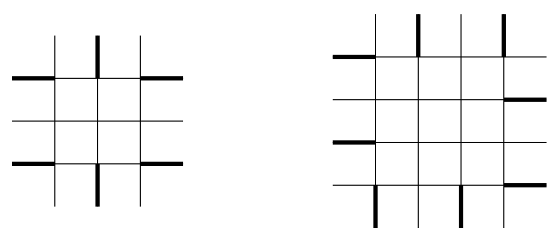


Fig. 1: The $n \times n$ grid (here $n = 3$ and $n = 4$) with $2n$ external links occupied

Consider a $n \times n$ square grid, with its $4n$ external links, see Figure 1. We are interested in *Fully Packed Loops* (FPL in short), i.e. sets of disconnected paths which pass through each of the n^2 vertices of the grid and exit through $2n$ of the external links, every second of them being occupied (see figure 2 for the case $n = 4$).

There is a simple one-to-one correspondence between such FPL and alternating-sign matrices (ASM), obtained as follows: divide the n^2 vertices into odd and even as usual, and associate $+1$ (resp. -1) to each horizontal segment of the path passing through an even (resp. odd) vertex, the opposite if the segment is vertical, and 0 if the path has a corner at that vertex.

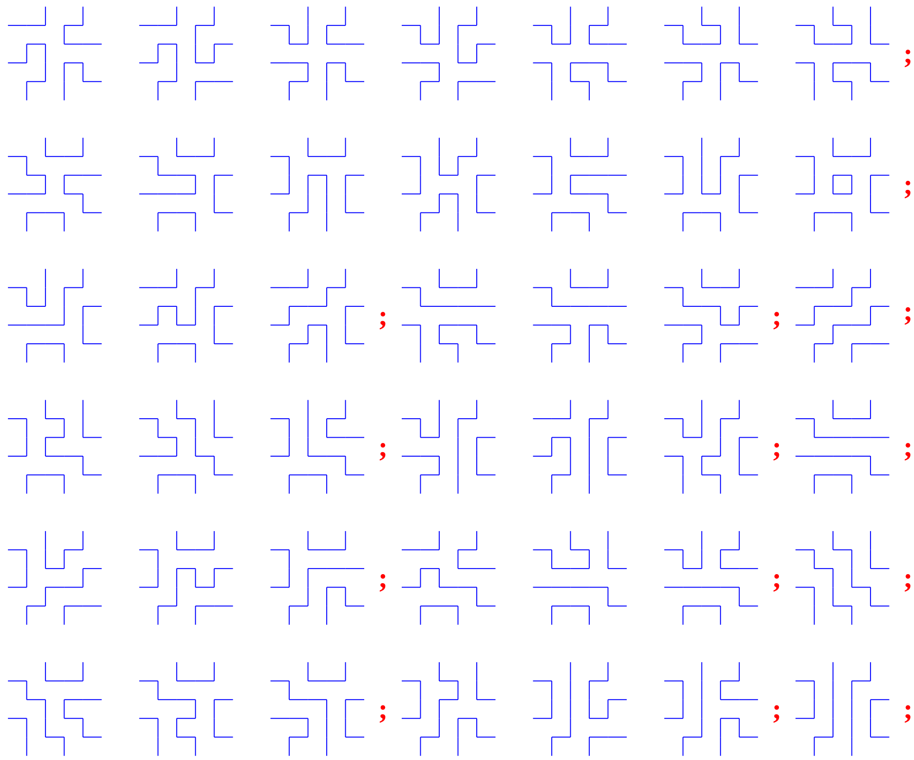


Fig. 2: The 42 FPL configurations on a 4×4 grid. Configurations corresponding to distinct link patterns are separated by semi-colons.

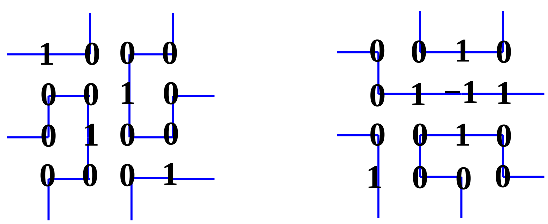


Fig. 3: FPL–ASM correspondence

This prescription associates an $n \times n$ ASM matrix to the FPL configuration in a one-to-one way. Thanks to the celebrated result on ASM's [1,2], the total number of FPL is thus known to be

$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!} \ . \tag{1.1}$$

For a review, see [3,4,5].

Considering FPL rather than ASM enables one to ask different questions, which are more natural in the path picture. Each FPL configuration defines a certain connectivity pattern, or *link pattern*, between the $2n$ occupied external links. Let $A_n(\pi)$ be the number of FPL configurations for a given link pattern π. We want to collect results and conjectures about these numbers $A_n(\pi)$. The next two sections recall well-known results and conjectures, while the following one gathers a certain number of conjectures which had not appeared in print before to the best of my knowledge. It is hoped that they will stimulate someone else's interest or suggest to an ingenious reader a connection with a different problem.

2. Counting the orbits

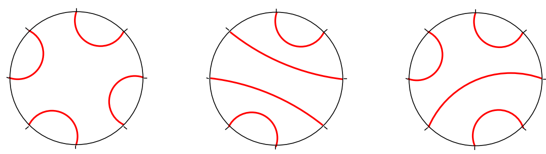


Fig. 4: The three link patterns up to rotations and reflections for $n = 4$

Although the problem of evaluating the $A_n(\pi)$ seems to admit only the symmetries of the square, it is convenient to represent the link patterns by arches connecting $2n$ points regularly distributed on a circle (see figure 4).

Wieland [6] has proved the remarkable result that $A_n(\pi)$ depends only on the equivalence class of π under the action of the dihedral group D_n generated by the rotations by $2\pi/2n$ and reflections across any diameter passing through a pair of these points. While it is easy to convince oneself that the number of link patterns equals $C_n = \frac{(2n)!}{n!(n+1)!}$ (the Catalan number), computing the number O_n of orbits under the action of D_n, i.e. of independent link patterns, is more subtle and appeals to Polya's theory of orbit counting (see for example [7]). In fact, using an alternative representation by the dual graph (see Figure 5), one realizes that these orbits are in one-to-one correspondence with the *projective planar trees* (PPT's) on $n+1$ points, whose generating function $T(x) = \sum_{n=1} O_n x^n$ has been computed by Stockmeyer [8]. We recall here his result for the convenience of the reader. Let $z_1, z_2, \cdots, z_n$ and y be $n+1$ indeterminates and define the *modified cycle index* of the dihedral group D_n as

$$Z(D_n^*; z_1, z_2, \cdots, z_n, y) = \frac{1}{2n} \sum_{i|n} \phi(i) z_i^{n/i} + \begin{cases} \frac{1}{2} y z_2^{(n-1)/2} & \text{if } n \text{ is odd} \\ \frac{1}{4} y^2 z_2^{(n-2)/2} + z_2^{n/2} & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

where $\phi(n)$ is the Euler totient function, counting the number of positive integers less than n which are relatively prime to n. Let $c(x) = \sum_{n=0} \frac{(2n)!}{n!(n+1)!} x^{n+1}$ be the generating function of the Catalan numbers and define $a(x) = x/(1 - x - c(x^2))$. The generating function $R(x)$ of the numbers of rooted planar projective trees is then given by

$$R(x) = x Z(D_n^*; c(x), c(x^2), \cdots, c(x^n), a(x)) \tag{2.2}$$

while the one of unrooted PPT's, which we want, is

$$T(x) = R(x) - Z(D_2^*; c(x), c(x^2), a(x)) + c(x^2) \ . \tag{2.3}$$

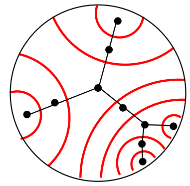


Fig. 5: The dual picture of a link pattern as a planar tree

One finds

$$T(x) = x+x^2+2x^3+3x^4+6x^5+12x^6+27x^7+65x^8+175x^9+490x^{10}+1473x^{11}+4588x^{12}+\cdots \tag{2.4}$$

In Table 1, we list the values of A_n, C_n and O_n for low values of n.

n :	1	2	3	4	5	6	7	8	9	10	11
A_n :	1	2	7	42	429	7436	218348	10850216	911835460	129534272700	31095744852375
C_n :	1	2	5	14	42	132	429	1430	4862	16796	58786
O_n :	1	1	2	3	6	12	27	65	175	490	1473

Table 1.

In the following, we use either the notation of link patterns with arches, or their dual PPT graphs, or both. The $2n$ external links are numbered from 1 to $2n$ in cyclic order. A link pattern π_a may be regarded as an involutive permutation on $\{1, \cdots, 2n\}$, with $\pi_a(i) = j$ for each arch connecting i and j.

3. The $A_n(\pi)$ as solutions of a linear problem

The work of Razumov and Stroganov [9] and Batchelor, de Gier and Nienhuis [10] contains a certain number of conjectures on the numbers $A_n(\pi)$. The most remarkable one connects them to a linear problem, as follows.

The periodic Temperley-Lieb algebra $PTL_p(\beta)$ is the algebra generated by the identity and p generators e_i, with the index i running on $\{1, \cdots, p\}$ modulo p, satisfying [(1)]

$$\begin{aligned} e_i^2 &= \beta e_i \\ e_i e_{i\pm 1} e_i &= e_i \\ e_i e_j &= e_j e_i \qquad \text{if} \quad |i-j| \bmod p > 1 \ . \end{aligned} \tag{3.1}$$

(1) Note that because we are working on a disk rather than a cylinder (more precisely we let the e's act on link patterns on a disk), we don't have to consider non-contractible loops nor to introduce additional relations between the e's: we are working in the so-called "identified connectivities" periodic sector [11].

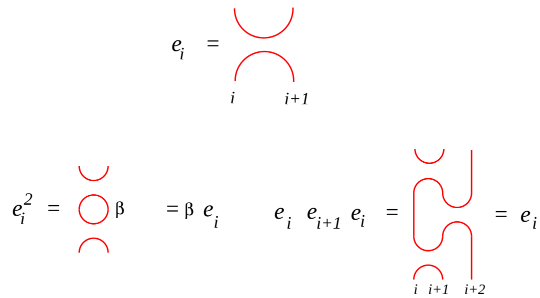


Fig. 6: The graphical representation of the Temperley-Lieb algebra $PTL_p(\beta)$, with $i = 1, \cdots, p$ mod p.

There exists a faithful graphical representation of PTL_p, see figure.

Now take $\beta = 1$ and let $PTL_{2n}(1)$ act on the link patterns π_a, $a = 1, \cdots, C_n$: using the graphical representation above, it is clear that e_i maps π_a on itself if $\pi_a(i) = i + 1$, while $\pi_b = e_i\pi_a$ connects j and k (as well as i and $i + 1$) if $\pi_a(i) = j$, $\pi_a(i + 1) = k$. Define

$$H = \sum_{i=1}^{2n} e_i \ . \tag{3.2}$$

In the basis $\{\pi_a\}$, H admits $(1, 1, \cdots, 1)$ as a left eigenvector of eigenvalue $2n$. This is its largest eigenvalue, and as the matrix H is irreducible and has non negative entries, one may use Perron-Frobenius theorem to assert that the right eigenvector for that largest eigenvalue must have non negative components. According to [9], one has

Conjecture 1. [9] *The right eigenvector of H of eigenvalue $2n$ is* $\Psi = \sum_a A_n(\pi_a)\pi_a$

$$\sum_{i=1}^{2n} e_i \sum_a A_n(\pi_a)\pi_a = 2n \sum_a A_n(\pi_a)\pi_a \ . \tag{3.3}$$

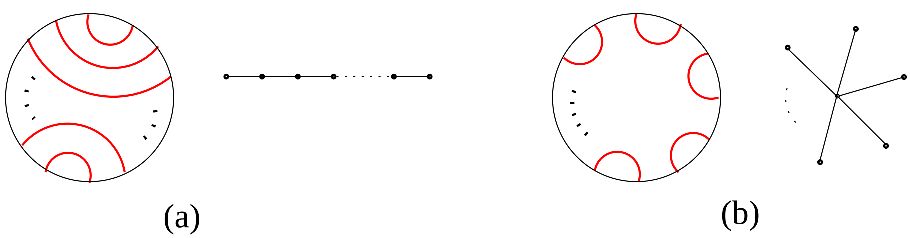


Fig. 7: the configurations of (a) smallest, (b) largest component

This assumes that the eigenvector has been normalised in such a way that its smallest component be equal to 1. This smallest component corresponds to the link patterns shown on figure 7(a), with n *nested* arches, or in the alternative dual picture, to linear trees, and it is possible to prove, independently of Conjecture 1, that there is a unique FPL configuration for each such link pattern [12].

Then, another conjecture deals with the *largest* component:

Conjecture 2. [10] *The largest component of the eigenvector occurs for link patterns of n level 1 arches, see figure 7(b), and equals A_{n-1}, i.e. the total number of FPL (or ASM) of size $n-1$.*

In the present work, we have taken Conjecture 1 for granted and used the linear problem to compute the $A_n(\pi)$ up to $n = 11$. We have found helpful to use the symmetry properties of sect. 1 to reduce the dimension of the problem. The Hamiltonian H commutes with the generators of the group D_n and the eigenvector of largest eigenvalue is expected to be completely symmetric under these symmetries, in agreement with Conjecture 1 and Wieland's theorem. One may thus determine the $A_n(\pi)$ by looking at a reduced Hamiltonian acting on orbits. As a glance at Table 1 above will convince the reader, this results in a large gain of computing time and size. In practice, we have been able to determine all the $A_n(\pi)$ up to $n = 11$ with an unsophisticated Mathematica code. The following conjectures have been extracted from the analysis and extrapolation of these data (which are available on request).

4. New results and conjectures

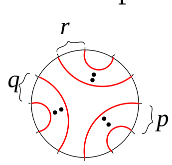
4.1. Expression of $A_n(\pi)$ for several classes of link patterns π

In view of its frequent occurrence, it is convenient to introduce a new notation for the "superfactorial"

$$m\text{¡} \;:= \prod_{r=1}^{m} r! = \prod_{j=1}^{m} (m-j+1)^j \ , \quad (-1)\text{¡} = 0\text{¡} = 1. \tag{4.1}$$

Then all the results up to $n = 11$ are consistent with

Conjecture 3.



$$= \frac{(p+q+r-1)\text{¡}\,(p-1)\text{¡}\,(q-1)\text{¡}\,(r-1)\text{¡}}{(p+q-1)\text{¡}\,(q+r-1)\text{¡}\,(r+p-1)\text{¡}} \qquad p,q,r,\geq 0 \ .$$

(4.2)

This may also be written in a simpler but less symmetric form, using the notation $n = p+q+r$

$$\frac{\binom{n-1}{p}\binom{n-2}{p}\cdots\binom{n-q}{p}}{\text{same for } n=p+q} \ . \tag{4.3}$$

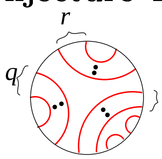
But the expert will also recognize in (4.2) MacMahon's formula for plane partitions in a box of size (p,q,r) [2], i.e.

$$\prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{k=1}^{r} \frac{i+j+k-1}{i+j+k-2} \ .$$

[2] Many thanks to S. Mitra and D. Wilson for this observation. A similar connection between FPL with different boundary conditions and a tiling problem had been observed and proved by de Gier [5].

It would be very interesting to find a bijection between FPL configurations with those link patterns and these plane partitions [(3)].

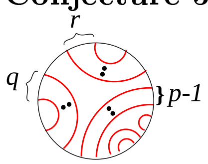
The factorized form does not persist for more complicated configurations. For example,
Conjecture 4.[(4)] For $p \geq 1, q, r, \geq 0$,



$$= \frac{(q-1)\text{¡}\,(r-1)\text{¡}}{(q+r-1)\text{¡}}\,\frac{p\text{¡}\,(p+q+r)\text{¡}}{(p+q+1)\text{¡}\,(p+r+1)\text{¡}}(p+q)!(p+r)! \;\times \tag{4.4}$$

$$\times\,[p^3 + 2p^2(q+r+1) + p(q^2+qr+r^2+3(q+r)+1) + q(q+1) + r(r+1)]$$

Conjecture 5. [(4)] For $p \geq 1, q, r, \geq 0$,



$$\begin{aligned}
&= \frac{(q-1)\text{¡}\,(r-1)\text{¡}}{2(q+r-1)\text{¡}}\,\frac{(p+1)\text{¡}\,(p+q+r+1)\text{¡}}{(p+q+3)\text{¡}\,(p+r+3)\text{¡}}\\
&\times (p+q+2)!\,(p+q+1)!\,(p+r+3)!\,(p+r)!\,(p+2)\\
&\times \Big[p^5 + p^4(7+4\,q+4\,r) + p^3(17+22\,q+6\,q^2+24\,r+10\,q\,r+6\,r^2)\\
&\quad + p^2(17+40\,q+24\,q^2+4\,q^3+46\,r+42\,q\,r+8\,q^2\,r+30\,r^2+8\,q\,r^2+4\,r^3)\\
&\quad + p(6+28\,q+29\,q^2+10\,q^3+q^4+32\,r+49\,q\,r+17\,q^2\,r+2\,q^3\,r+41\,r^2\\
&\qquad +23\,q\,r^2+3\,q^2\,r^2+16\,r^3+2\,q\,r^3+\,r^4)\\
&\quad + 6\,q+11\,q^2+6\,q^3+q^4+6\,r+13\,q\,r+3\,q^2\,r+15\,r^2+15\,q\,r^2+3\,q^2\,r^2+12\,r^3+2\,q\,r^3+3\,r^4\Big]
\end{aligned} \tag{4.5}$$

4.2. Polynomial behavior in n and asymptotic behaviour for large n

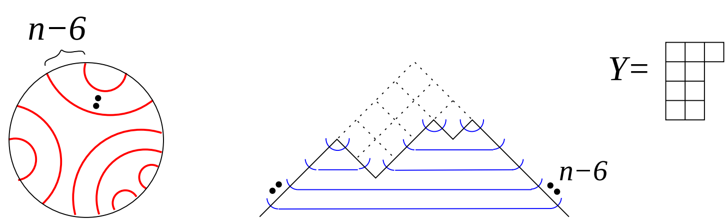


Fig. 8: Describing a configuration by a Dyck path or a Young diagram

Let us consider link patterns π made of a given set $\mathcal{S}$ of r arches plus $n-r$ nested arches as in Conjectures 3 and 4 above, and let n vary, while keeping $\mathcal{S}$ fixed. Any such link pattern is also encoded by a (Dyck) path, or by the complementary Young diagram

(3) Note added : This has now been achieved in [18], thus proving Conjecture 3.

(4) Note added: This has now been proved by Caselli and Krattenthaler [19]. Note that the proofs of Conjectures 3-5 are independent of Conjecture 1, but that the results are consistent with it.

Y, see Figure 8[5]. We denote by $|Y|$ the number of boxes of Y and by $\dim Y$ the dimension of the representation of the symmetric group $S_{|Y|}$ labelled by Y. We recall (see for example [13]) the useful expression for the ratio $\frac{\dim Y}{|Y|!} = \frac{1}{\mathrm{hl}(Y)}$, the inverse *hook length* of the diagram, i.e. the inverse product of the hook lengths of all its boxes. Finally, we denote by $F(Y)$ the set of diagrams obtained by adjonction of one box to Y according to the usual rules. Alternatively, if D_Y is the corresponding irreducible representation of $Sl(N)$, $F(Y)$ labels the set of representations appearing in the decomposition into irreducibles of $D_\square \otimes D_Y$. Then

Conjecture 6. *For* $n \geq r$

$$A_n(\pi) = \frac{1}{|Y|!} P_Y(n) \tag{4.6}$$

where $P_Y(n)$ is a polynomial of degree $|Y|$ with coefficients in $\mathbb{Z}$ and its highest degree coefficient is equal to $\dim Y$.

For example, in the case covered by equation (4.3), Y is a rectangular $p \times q$ Young diagram, $|Y| = pq$ and $(pq)! \frac{2! \cdots (q-1)!}{p!(p+1)! \cdots (p+q-1)!}$ is indeed an integer. See more examples in Appendix A.

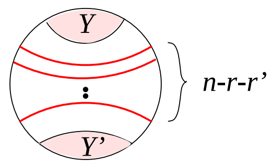


Fig. 9: Configuration described by two Young diagrams

As a corollary of Conjecture 6, the asymptotic behavior for large n is given by

$$A_n(\pi) \approx \frac{\dim Y}{|Y|!} n^{|Y|} \ . \tag{4.7}$$

Such an asymptotic behavior had been observed in the case of open boundary conditions by Di Francesco [14], who derived it as a consequence of the eigenvector equation. The action of the Temperley-Lieb generator e_i on an open link pattern associated with one Young diagram Y or on the corresponding Dyck path is described by the "raise and peel" process of [15]: the resulting Young diagram $\bar{Y}$ is either Y itself if the site i is a local peak of the path, or has one less box than Y if i is a local minimum of the path (and then $Y \in F(\bar{Y})$), or is a diagram with a larger number of boxes than Y otherwise. What changes in the case of periodic boundary conditions is the possibility of an action on the "other side" of the link pattern. In order to carry out the discussion in the periodic case, we thus have to generalize our considerations to configurations described

(5) The ambiguity between the Young diagram Y and its transpose in this definition will be immaterial in what follows.

by *two* Young diagrams Y and Y', with r and r' arches, separated by a number $n-r-r'$ of parallel arches (see Fig. 9). Then

Conjecture 7. *For* $n \geq r + r'$

$$A_n(\pi) =: A_n(Y,Y') = \frac{1}{|Y|!|Y'|!} P_{Y,Y'}(n) \tag{4.8}$$

with $P_{Y,Y'}(n)$ *a polynomial of degree* $|Y| + |Y'|$ *with coefficients in* $\mathbb{Z}$ *and its highest degree coefficient is* $\dim Y \dim Y'$.

This is exemplified on the configurations of Conjectures 4 or 5: for given q and r, one Young diagram is a $q \times r$ rectangle, the other is made of one or two boxes, and Y and Y' are separated by $p-1$ arches; then in the expressions given in Conj. 4 or 5, the first factor represents $\frac{\dim Y}{|Y|!} \frac{\dim Y'}{|Y'|!}$, the second (the ratio of superfactorials) is seen to be a polynomial in p, and the degree of the whole expression is easily computed.

Again, one derives from this conjecture the asymptotic behavior

$$A_n(Y,Y') \approx \frac{\dim Y}{|Y|!} \frac{\dim Y'}{|Y'|!} n^{|Y|+|Y'|} \tag{4.9}$$

We shall now show that this asymptotic behavior is consistent with the eigenvector equation (3.3). Let π_a be a link pattern described by a pair of Young diagrams (Y, Y'), as in Fig. 9, and e_i be a generator of the periodic Temperley-Lieb algebra. The link pattern $\pi_b = e_i \pi_a$ is described by a pair $(\bar{Y}, \bar{Y}')$. Identifying the coefficient of π_b in (3.3) and using the Ansatz (4.9), we find that for n large, the only terms to contribute are either $Y = \bar{Y}$, $Y' \in F(\bar{Y}')$ or $Y \in F(\bar{Y})$, $Y' = \bar{Y}'$

$$2nA_n(\bar{Y},\bar{Y}') = \sum_{Y \in F(\bar{Y})} A_n(Y,\bar{Y}') + \sum_{Y' \in F(\bar{Y}')} A_n(\bar{Y},Y') + O(\frac{1}{n}) \tag{4.10}$$

which is consistent with the behaviour (4.9), since

$$2\frac{\dim \bar{Y}}{|\bar{Y}|!} \frac{\dim \bar{Y}'}{|\bar{Y}'|!} = \sum_{Y \in F(\bar{Y})} \frac{\dim Y}{|Y|!} \frac{\dim \bar{Y}'}{|\bar{Y}'|!} + \sum_{Y' \in F(\bar{Y}')} \frac{\dim \bar{Y}}{|\bar{Y}|!} \frac{\dim Y'}{|Y'|!}$$

which results itself from the identity

$$\frac{\dim \bar{Y}}{|\bar{Y}|!} = \sum_{Y \in F(\bar{Y})} \frac{\dim Y}{|Y|!} \ . \tag{4.11}$$

4.3. Recursion formulae generalizing Conjecture 2

In the same way as Conjecture 2 relates the number of FPL configurations for a certain link pattern, made of n simple arches, to the *inclusive sum* of all FPL configurations

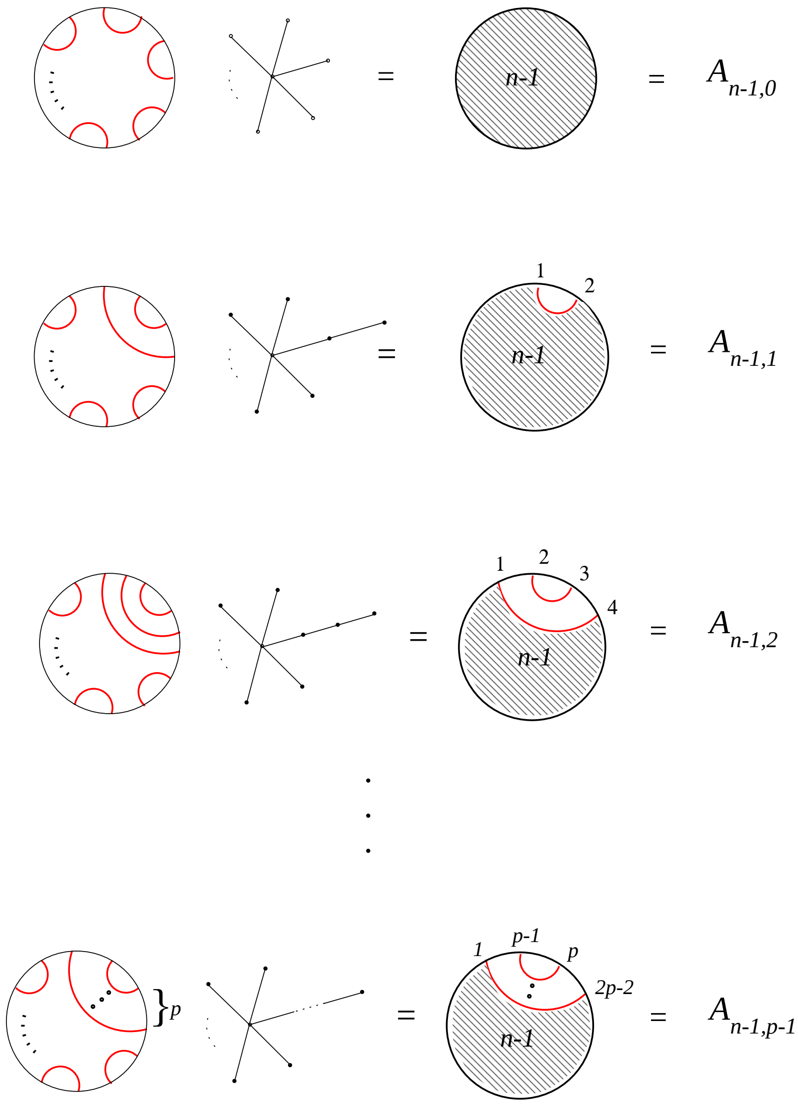


Fig. 10: Relating FPL configurations of size n with inclusive configurations of size $n-1$

of size $n-1$, one finds relations between other configuration numbers of size n and inclusive sums of size $n-1$.

Conjecture 8. (i) [6] *We have the relations depicted on Figure 10, where for example the expression $A_{n-1,1}$ on the r.h.s. is the number of FPL configurations of size $n-1$ containing an arch between external links 1 and 2.*

(ii) *The rhs of these relations*, at size n, *take respectively the values*

$$A_{n,0} = A_n\,, \quad A_{n,1} = \frac{3}{2}\frac{n^2+1}{(2n-1)(2n+1)}A_n\,,$$

$$A_{n,2} = \frac{1}{16}\frac{59n^6+299n^4+866n^2+576}{(2n-3)(2n-1)^2(2n+1)^2(2n+3)}A_n$$

and

$$A_{n,3} = \frac{3}{512}\,\frac{2579\, n^{12}+39364\, n^{10}+374412\, n^{8}+2174092\; n^{6}+6601109\, n^{4}+11674044\, n^{2}+6350400}{(2n-5)(2n-3)^{2}(2n-1)^{3}(2n+1)^{3}(2n+3)^{2}(2n+5)} A_n$$

It is easy to guess the general form of $A_{n,p} = (P_{p(p+1)}(n^2)/\prod_{\ell=1}^{p}(4n^2-(2\ell-1)^2)^{p+1-\ell})A_n$ as a ratio of two even polynomials of degree $p(p+1)$ in n, although the detailed form of the numerator remains unclear. The expressions of $A_{n,p}$, $p=1,2$ in (ii) were known to D. Wilson [16], while the one of $A_{n,3}$ seems to be new.

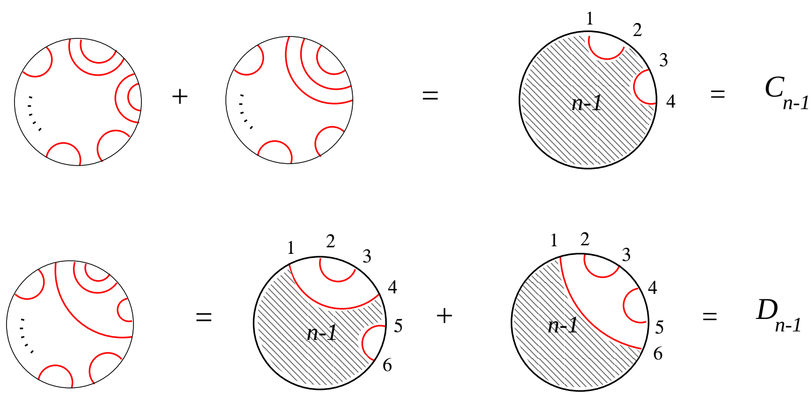


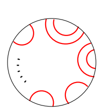
Fig. 11: Relating FPL configurations of size n with inclusive configurations of size $n-1$, cont'd

Conjecture 9. *There are equalities as shown on Figure 11 between the sum of two configuration numbers $A_n(\pi)$ and an inclusive sum C_{n-1} of size $n-1$, or vice versa, with*

$$C_n = \frac{97\, n^6 + 82\, n^4 - 107\, n^2 - 792}{8(2n-3)(2n-1)^2(2n+1)^2(2n+3)} A_n$$

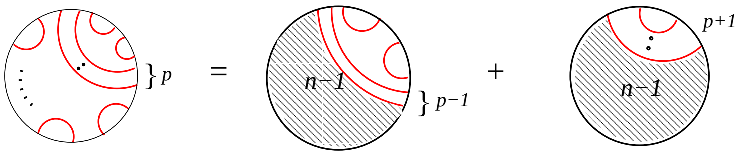
$$D_n = \frac{9}{256}\,\frac{5977\, n^{12}+16622\, n^{10}+54681\, n^{8}-216784\, n^{6}-2071808\, n^{4}-337488\, n^{2}+3456000}{(2n-5)(2n-3)^{2}(2n-1)^{3}(2n+1)^{3}(2n+3)^{2}(2n+5)} A_n \ .$$

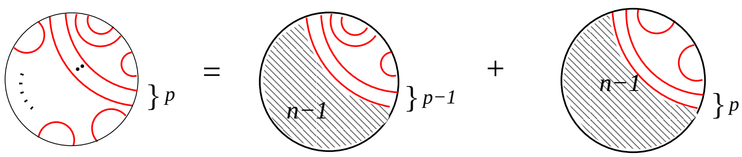
By combining the previous formulae it follows that for $n \geq 3$

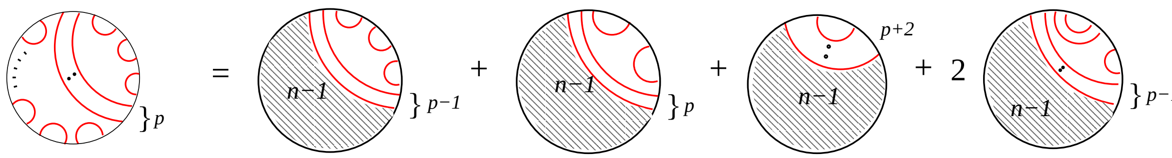


$$\Big|_{n+1} = \frac{3^3.5}{2^4}\,\frac{(n^2-4)(n^4+3n^2+4)}{(2n-3)(2n-1)^2(2n+1)^2(2n+3)} A_n \ .$$

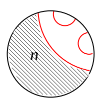
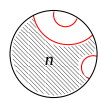
These identities are just the beginning of a host of relations, such as







but their systematics has remained elusive so far.

One may also conjecture that  and  are again both of the form

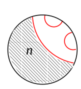
$$P_{12}(n)A_n/\left((4n^2-25)(4n^2-9)^2(4n^2-1)^3\right)$$

with the even polynomials $P_{12}(n)$ equal to respectively

$$\tfrac{3}{512}\left(12631\,n^{12}+101096\,n^{10}+586518\,n^{8}+1237988\,n^{6}-5800349\,n^{4}-19336284\,n^{2}-23976000\right)$$

and

$$\tfrac{3}{512}\left(23231\,n^{12}-1364\,n^{10}-258432\,n^{8}-2538692\,n^{6}-6630499\,n^{4}+17311356\,n^{2}+44712000\right)\ .$$

The expression of $A_{n,3}+$  was known to D. Wilson [16].

5. Discussion

This paper has presented a certain number of conjectural expressions and recursion formulae for the numbers of configurations of FPL with periodic boundary conditions. At this stage all these expressions remain empirical, and based on the actual data of the linear problem. The connection with the numbers of FPL thus relies on another conjecture (Conjecture 1). In some cases, however, the numbers given in this paper have been tested against the direct computation of FPL configurations [12]. A similar discussion is currently being carried out for the other types of boundary conditions by another group [17].

More conjectural expressions have been collected for other types of configurations (see Appendix A), but this seems a gratuitous game in the absence of a guiding principle. Observe however the simplicity of the "three-point-functions" (Conjecture 3) as compared to the cumbersomeness of the others. Could this suggest that the latter may be obtained from the former, in the same way as higher correlation functions in Conformal Field Theories, say, may be constructed from the 3-point functions ?

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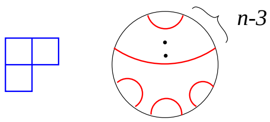
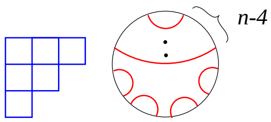
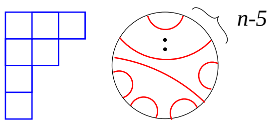
Appendix A. More configurations



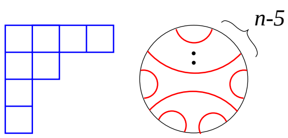
$$= \frac{(n-2)}{6}(2n^2 - 5n + 9)$$



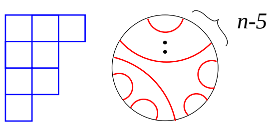
$$= \frac{(n-1)(n-3)}{180}(4n^4 - 32n^3 + 155n^2 - 394n + 540)$$



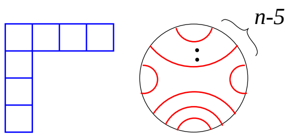
$$= \frac{(n-1)(n-3)(n-4)}{720}(5n^4 - 38n^3 + 197n^2 - 522n + 840)$$



$$= \frac{(n-1)(n-4)}{20160}(45n^6 - 635n^5 + 4639n^4 - 21865n^3 + 68924n^2 - 136740n + 146160)$$



$$= \frac{(n-1)(n-2)(n-3)(n-4)}{4!5!}(5n^4 - 46n^3 + 275n^2 - 802n + 1440)$$



$$= \frac{(n-4)}{2520}(10n^6 - 135n^5 + 853n^4 - 3378n^3 + 9343n^2 - 17403n + 18270)$$