When Can You Tile a Box With Translates of Two Given Rectangular Bricks?

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Abstract

When can a d-dimensional rectangular box R be tiled by translates of two given d-dimensional rectangular bricks B_1 and B_2? We prove that R can be tiled by translates of B_1 and B_2 if and only if R can be partitioned by a hyperplane into two sub-boxes R_1 and R_2 such that R_i can be tiled by translates of the brick B_i alone ($i = 1, 2$). Thus an obvious sufficient condition for a tiling is also a necessary condition. (However, there may be tilings that do not give rise to a bipartition of R.)

There is an equivalent formulation in terms of the (not necessarily integer) edge lengths of R, B_1, and B_2. Let R be of size $z_1 \times z_2 \times \cdots \times z_d$, and let B_1 and B_2 be of respective sizes $v_1 \times v_2 \times \cdots \times v_d$ and $w_1 \times w_2 \times \cdots \times w_d$. Then there is a tiling of the box R with translates of the bricks B_1 and B_2 if and only if

(a) z_i/v_i is an integer for $i = 1, 2, \ldots, d$; or

(b) z_i/w_i is an integer for $i = 1, 2, \ldots, d$; or

(c) there is an index k such that z_i/v_i and z_i/w_i are integers for all $i \neq k$, and $z_k = \alpha v_k + \beta w_k$ for some nonnegative integers α and β.

Our theorem extends some well known results (due to de Bruijn and Klarner) on tilings of rectangles by rectangles with integer edge lengths.

1 Introduction and Main Theorem

A d-dimensional *rectangular box* or *brick* of size $v_1 \times v_2 \times \cdots \times v_d$ is any translate of the set

$$\{(x_1, x_2, \ldots, x_d) \in \mathbf{R}^d : 0 \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, d\}.$$

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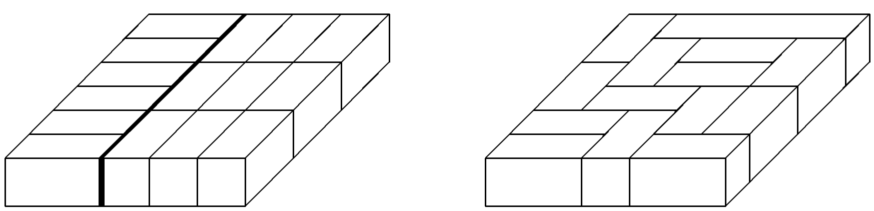
Thus a box or brick in dimension $d = 2$ is simply a rectangle with sides parallel to the coordinate axes. We study the problem of tiling a d-dimensional rectangular box with translates of two given d-dimensional rectangular bricks. We use the term *tile* in the following sense: The interiors of the bricks must be disjoint, and their union must be the entire box.

We will provide two different characterizations of the boxes that can be tiled by translates of two given bricks. One characterization is geometric. The other is arithmetic and involves the edge lengths of the bricks and the box. We do not require that the bricks and the box have integer edge lengths, although this special case is a crucial element of our analysis. Our main theorem extends several 2-dimensional tiling theorems in a pleasing manner.

Tilings of a box with translates of a single brick are readily characterized. We say that the $z_1 \times z_2 \times \cdots \times z_d$ box is a *multiple* of the $v_1 \times v_2 \times \cdots \times v_d$ brick provided z_i/v_i is an integer for $i = 1, 2, \ldots, d$. The following observation is clear.

Observation. *The d-dimensional box R can be tiled by translates of a given brick B if and only if R is a multiple of B. Moreover, any such tiling is unique.*

When we have two bricks at our disposal, the situation is more complicated. Note that a tiling of a box with translates of two bricks need not be unique. For instance, Figure 1 shows two tilings of a box R with translates of two rectangular bricks B_1 and B_2. In (a) the box R is partitioned by a plane into two sub-boxes R_1 and R_2, and the sub-box R_i is a multiple of the brick B_i for $i = 1, 2$. We refer to such a tiling as a *bipartite tiling* of R with B_1 and B_2. In (b) we exhibit a non-bipartite tiling of R with the same bricks B_1 and B_2. Because the trivial box of size $0 \times 0 \times \cdots \times 0$ is a multiple of every non-trivial d-dimensional box, either of the two sub-boxes may be trivial in a bipartite tiling of a box R; this degenerate situation occurs precisely when R is a multiple of B_1 or B_2.



(a) (b)

Figure 1: (a) A bipartite tiling (b) A non-bipartite tiling

Clearly, the existence of a bipartite tiling is sufficient for the existence of a tiling of a box with translates of two given bricks. The thrust of our main theorem is that this obvious sufficient condition is also necessary:

Theorem 1 (Geometric). *The d-dimensional box R can be tiled by translates of two given d-dimensional bricks B_1 and B_2 if and only if R can be partitioned by a hyperplane into two sub-boxes R_1 and R_2 such that R_i is a multiple of B_i for $i = 1, 2$.*

We emphasize that Theorem 1 does *not* say that every tiling must be bipartite. However, the existence of a non-bipartite tiling implies the existence of bipartite tiling.

Theorem 1 gives a satisfying and complete geometric characterization of the boxes that can be tiled by translates of two given bricks. We now provide an equivalent arithmetic characterization in terms of the edge lengths of the box and the bricks. For real numbers v and w we denote the set of all nonnegative integer linear combinations of v and w by

$$\langle v, w\rangle = \{\alpha v + \beta w : \ \alpha = 0, 1, 2, \ldots, \ \beta = 0, 1, 2, \ldots\}.$$

Theorem 1′ (Arithmetic). *Let z_i, v_i, and w_i be positive real numbers for $i = 1, 2, \ldots, d$. There is a tiling of a $z_1 \times z_2 \times \cdots \times z_d$ box with translates of $v_1 \times v_2 \times \cdots \times v_d$ and $w_1 \times w_2 \times \cdots \times w_d$ bricks if and only if*

(a) *z_i/v_i is an integer for $i = 1, 2, \ldots, d$;* *or*

(b) *z_i/w_i is an integer for $i = 1, 2, \ldots, d$;* *or*

(c) *there is an index k such that z_k is in $\langle v_k, w_k\rangle$, and the numbers z_i/v_i and z_i/w_i are integers for all $i \neq k$.*

Condition (a) or (b) holds when the box is tiled by translates of one of the bricks, while (c) holds when a tiling uses both bricks.

Example. It is possible to tile a $12 \times 12 \times 11$ box with $4 \times 4 \times 4$ and $3 \times 3 \times 3$ cubical bricks. Condition (c) of Theorem 1′ is satisfied (with $k = 3$) because $12/4$ and $12/3$ are both integers, and $11 = 2 \cdot 4 + 1 \cdot 3$. Figure 2 shows a bipartite tiling. The two sub-boxes are of sizes $12 \times 12 \times 8$ and $12 \times 12 \times 3$.

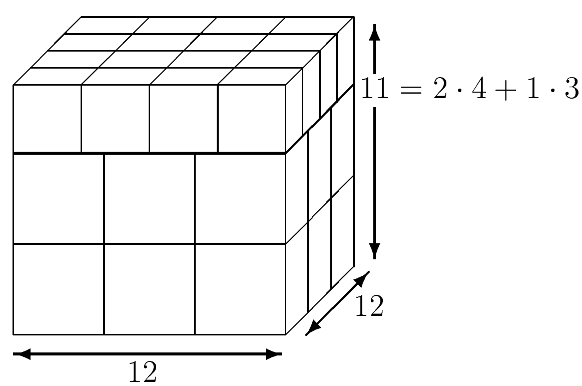


Figure 2: A tiling of a $12 \times 12 \times 11$ box with $4 \times 4 \times 4$ and $3 \times 3 \times 3$ cubical bricks

The preceding example helps reveal the equivalence of Theorem 1 and Theorem 1′. The two sub-boxes R_1 and R_2 in Theorem 1 are non-trivial and are separated by a hyperplane perpendicular to the k-th coordinate axis exactly when k is an index for which condition (c) holds in Theorem 1′. The integrality conditions imposed on z_i/v_i and z_i/w_i for $i \neq k$ guarantee that that R_1 and R_2 are multiples of the two respective bricks.

The arithmetic conditions in Theorem 1′ supply us with an algorithm to recognize when a $z_1 \times z_2 \times \cdots \times z_d$ box can be tiled by translates of $v_1 \times v_2 \times \cdots \times v_d$ and $w_1 \times w_2 \times \cdots \times w_d$

bricks. Integrality of the $2d$ real numbers z_i/v_i and z_i/w_i for $i = 1, 2, \ldots, d$ is readily checked, and z_k is in $\langle v_k, w_k\rangle$ if and only if the real number $(z_k - \alpha v_k)/w_k$ is an integer for some α in $\{0, 1, \ldots, \lfloor z_k/v_k\rfloor\}$.

Sections 2 through 6 contain preliminary results and a discussion of important special cases. The proof of Theorem $1'$ is in Section 7. Our discussion is elementary and accessible to a wide audience.

2 Re-Scaling and a Counting Lemma

We begin with two basic lemmas on tilings. The first result is easy, and we omit the proof.

Lemma 2. *Let h be a positive real number. Then there is a tiling of a $z_1 \times z_2$ rectangle with translates of $v_1 \times v_2$ and $w_1 \times w_2$ rectangular bricks if and only if there is a tiling of an $(hz_1) \times z_2$ rectangle with translates of $(hv_1) \times v_2$ and $(hw_1) \times w_2$ rectangular bricks.*

The number h represents a re-scaling factor applied to all horizontal edge lengths of the rectangles. There is a corresponding result for vertical re-scalings, as well as an extension to re-scalings in higher dimensions.

Our second basic result uses a counting argument to obtain a fundamental necessary condition for a box to be tiled by translates of two given bricks.

Lemma 3. *Suppose that there is a tiling of a $z_1 \times z_2 \times \cdots \times z_d$ box with translates of $v_1 \times v_2 \times \cdots \times v_d$ and $w_1 \times w_2 \times \cdots \times w_d$ bricks in d dimensions. Then there are nonnegative integers α_i and β_i such that $z_i = \alpha_i v_i + \beta_i w_i$ for $i = 1, 2, \ldots, d$. Moreover, if v_i is irrational and w_i is rational, then α_i and β_i are unique.*

Proof. Count the number of bricks of each type incident with an edge of length z_i (parallel to the i-th coordinate axis) of the box. If there are α_i bricks of size $v_1 \times v_2 \times \cdots \times v_d$ and β_i bricks of size $w_1 \times w_2 \times \cdots \times w_d$, then this count shows that $z_i = \alpha_i v_i + \beta_i w_i$.

Suppose that v_i is irrational and w_i is rational. Let $\alpha_i, \beta_i, \alpha_i'$, and β_i' be nonnegative integers such that $z_i = \alpha_i v_i + \beta_i w_i = \alpha_i' v_i + \beta_i' w_i$. Then $(\alpha_i - \alpha_i')\, v_i = (\beta_i' - \beta_i)\, w_i$. The expression on the right is rational, and thus $\alpha_i' = \alpha_i$. Then $\beta_i' = \beta_i$, and the uniqueness of α_i and β_i is established. ∎

The following elementary result tells us that the necessary counting condition of Lemma 3 is also sufficient for the tiling of an interval by translates of two given intervals on the real line. Thus our main theorem is true in dimension 1.

Theorem 4. *Let z, v, and w be positive real numbers. The following statements are equivalent.*

(a) *An interval of length z can be tiled by translates of intervals of length v and w.*
(b) *An interval of length z has a bipartite tiling with intervals of length v and w.*
(c) *There are nonnegative integers α and β such that $z = \alpha v + \beta w$.*

Proof. If an interval of length z is tiled by α intervals of length v and β intervals of length w, then $z = \alpha v + \beta w$, and we may place α intervals of length v followed by β intervals of length w to produce a bipartite tiling. Thus (a) implies (b) and (c). This construction also makes it clear that (c) implies (a). ∎

3 The Divisibility Lemma for Tilings

In an *integer rectangle* the length of each edge is an integer. The next result is a key step in our proof of Theorem $1'$.

Divisibility Lemma. *Let v and w be positive integers. Suppose that the integer rectangle R is tiled by integer rectangular bricks, each of which has width divisible by v or height divisible by w. Then R itself has width divisible by v or height divisible by w.*

Generalizations and variations of the divisibility lemma appear throughout the literature on tiling. For example, the divisibility lemma can be deduced from Wagon's [9] result on "semi-integer rectangles" by using a re-scaling argument. To keep our discussion self-contained we include a charge-counting proof of the divisibility lemma. The related checkerboard coloring scheme [4, 5, 8, 9] and polynomial factorizations [1, 3, 6] also work.

Proof. Let R be an $n_1 \times n_2$ rectangle with a tiling of the specified type. Partition R into $n_1 n_2$ unit cells with segments parallel to the edges. Index the rows $1, 2, \ldots, n_2$ from bottom to top. Place a unit positive charge in each of the n_1 cells in row j for $j = 1, w+1, 2w+1, \ldots$ and a unit negative charge in each of the n_1 cells in row j for $j = w, 2w, 3w, \ldots,$ as in Figure 3. All other cells in R receive charge 0.

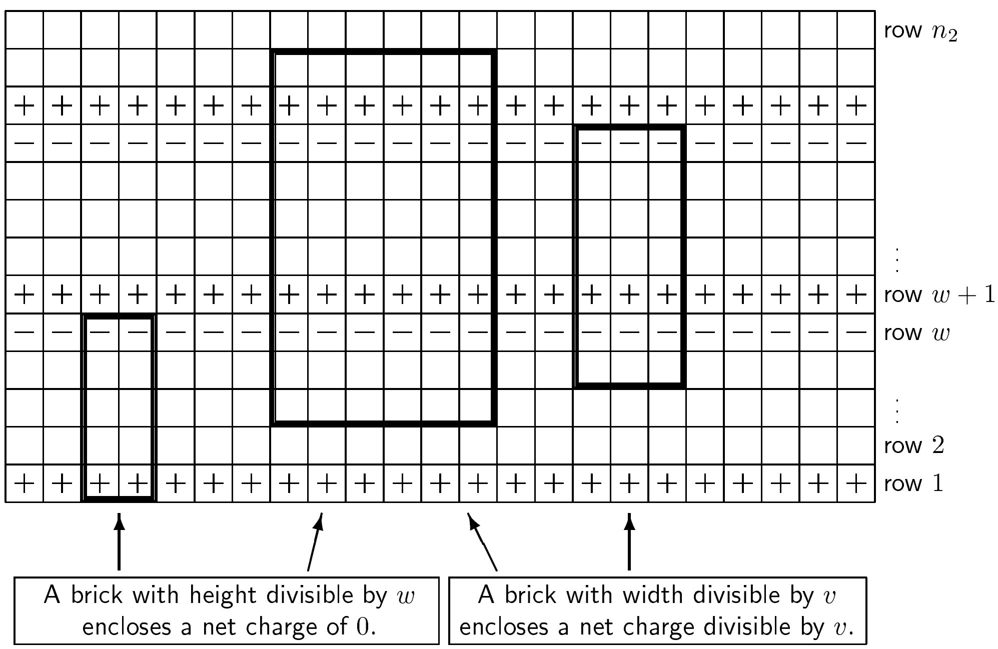


Figure 3: A charge-counting argument, illustrated for $v = 3$ and $w = 5$

Consider the net charge enclosed by R and by each rectangular brick. Observe that the net charge of the entire rectangle R equals n_1 if w does not divide n_2 and equals 0 if w does divide n_2. Now each rectangular brick with height divisible by w encloses a net charge of 0, while each rectangular brick with width divisible by v encloses a net charge that is divisible by v. It follows that if w does not divide n_2, then v must divide n_1. ∎

4 Tiling with Integer Rectangles

We now state and prove Theorem $1'$ for integer rectangles.

Theorem 5. *Let v_1, v_2, w_1, and w_2 be positive integers with* $\gcd(v_1, w_1) = \gcd(v_2, w_2) = 1$. *Then an integer rectangle of size $n_1 \times n_2$ can be tiled by translates of $v_1 \times v_2$ and $w_1 \times w_2$ rectangular bricks if and only if*

(a) *v_1 divides n_1, and v_2 divides n_2; or*
(b) *w_1 divides n_1, and w_2 divides n_2; or*
(c) *$v_1 w_1$ divides n_1, and n_2 is in $\langle v_2, w_2 \rangle$; or*
(d) *$v_2 w_2$ divides n_2, and n_1 is in $\langle v_1, w_1 \rangle$.*

Proof. Let B_1 and B_2 be $v_1 \times v_2$ and $w_1 \times w_2$ rectangular bricks, respectively, and let R be an $n_1 \times n_2$ rectangle. Suppose that R is tiled by translates of B_1 and B_2. Lemma 3 tells us that n_1 is in $\langle v_1, w_1 \rangle$ and n_2 is in $\langle v_2, w_2 \rangle$. The brick B_1 has width v_1, while B_2 has height w_2. The divisibility lemma implies that v_1 divides n_1, or w_2 divides n_2. Similarly, B_2 has width w_1, while B_1 has height v_2, and so the divisibility lemma implies that w_1 divides n_1, or v_2 divides n_2. It follows that one of the four conditions (a)–(d) must hold.

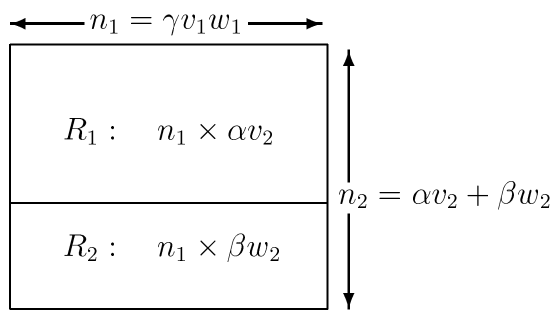


Figure 4: The proof of Theorem 5

We now show that R can be tiled by translates of B_1 and B_2 if any of (a)–(d) holds. If (a) or (b) holds, then R is a multiple of B_1 or B_2, and the desired tiling certainly exists. Suppose that (c) holds, where $n_1 = \gamma v_1 w_1$ and $n_2 = \alpha v_2 + \beta w_2$. Then a horizontal line partitions R into a rectangle R_1 of size $n_1 \times \alpha v_2$ and a rectangle R_2 of size $n_1 \times \beta w_2$, as in Figure 4. Now R_1 is a multiple of the brick B_1, while R_2 is a multiple of the brick B_2. Thus R has a bipartite tiling with B_1 and B_2. Condition (d) is treated similarly. ∎

As is clear from the proof of Theorem 5, each of the conditions (a)–(d) implies the existence of a bipartite tiling of R with B_1 and B_2.

The hypothesis that corresponding edge lengths of the bricks be relatively prime is not an obstacle in applying Theorem 5; a re-scaling argument allows us to treat the cases where this hypothesis is not met, as in the proof of Corollary 6 below.

5 Corollaries

Theorem 5 contains several important tiling results as special cases.

Corollary 6 (de Bruijn [3] and Klarner [7]). *Let v and w be positive integers. An integer rectangle of size $n_1 \times n_2$ can be tiled by $v \times w$ rectangular bricks (with both orientations allowed) if and only if*

(a) *v divides n_1 or n_2;* *and*
(b) *w divides n_1 or n_2;* *and*
(c) *n_1 is in $\langle v, w\rangle$;* *and*
(d) *n_2 is in $\langle v, w\rangle$.*

Proof. If v and w are relatively prime, then the result follows from Theorem 5 with $v_1 = w_2 = v$ and $w_1 = v_2 = w$. If v and w are not relatively prime, then we first divide n_1, n_2, v, and w by $\gcd(v, w)$. The re-scaled rectangle must be an integer rectangle for a tiling to exist, and we are in the previous situation. By Lemma 2 there is a tiling with the original rectangular bricks if and only if there is a tiling with the re-scaled bricks. ■

We also obtain the following less well known result, which appeared in 1995.

Corollary 7 (Fricke [4]). *Let v and w be relatively prime positive integers. An $n_1 \times n_2$ rectangle can be tiled by $v \times v$ and $w \times w$ squares if and only if*

(a) *v divides n_1 and n_2;* *or*
(b) *w divides n_1 and n_2;* *or*
(c) *vw divides n_1, and n_2 is in $\langle v, w\rangle$;* *or*
(d) *vw divides n_2 and n_1 is in $\langle v, w\rangle$.*

Proof. In Theorem 5 let $v_1 = v_2 = v$ and $w_1 = w_2 = w$. ■

6 Tiling Rectangles with Rectangles

We now extend Theorem 5 to obtain necessary and sufficient conditions for a rectangle to be tiled by translates of two (not necessarily integer) rectangles. In other words, we prove our main theorem in dimension 2. We will see that the 2-dimensional case is the crucial one for establishing the general theorem.

Theorem 8 (Geometric). *The rectangle R can be tiled by translates of two given rectangular bricks B_1 and B_2 if and only if R can be partitioned by a line into two sub-rectangles R_1 and R_2 such that R_i is a multiple of B_i for $i = 1, 2$.*

Proof. Clearly, if R can be partitioned into a multiple of B_1 and a multiple of B_2, then R has a tiling with translates of B_1 and B_2.

Suppose that R can be tiled by translates of the bricks B_1 and B_2. Without loss of generality B_1 is a $v \times 1$ rectangle and B_2 is a $1 \times w$ rectangle, as suitable horizontal and vertical re-scalings bring about this situation. Let R be a $z_1 \times z_2$ rectangle and consider a particular tiling of R with translates of B_1 and B_2. If this tiling uses translates of only one of the two bricks, then R is a multiple of that brick, and we have our desired (degenerate) bipartition of R. We henceforth suppose that the tiling uses translates of both B_1 and B_2.

Case 1: Suppose that v and w are both rational. Then after suitable horizontal and vertical re-scalings we may assume that B_1, B_2, and R are integer rectangles and that the corresponding edge lengths of B_1 and B_2 are relatively prime. Theorem 5 establishes the existence of the desired bipartite tiling.

Case 2: Suppose that at least one of v and w is irrational. Without loss of generality v is irrational. By Lemma 3

$$z_1 = \alpha v + \beta, \tag{1}$$

where α and β are unique nonnegative integers. Now a vertical line partitions R into the sub-rectangles R_1 and R_2 of respective sizes $(\alpha v) \times z_2$ and $\beta \times z_2$. We will show that R_i is a multiple of the brick B_i for $i = 1, 2$, which will complete the proof.

Claim 1: *The sub-rectangle R_1 of size $(\alpha v) \times z_2$ is a multiple of the brick B_1 of size $v \times 1$.* Clearly, $(\alpha v)/v = \alpha$ is an integer. We use a tile-sliding argument to show that $z_2/1 = z_2$ is an integer, which will establish the claim. First remove the translates of the brick B_2 from the tiling. Then draw a horizontal line k units from the bottom of R for $k = 1, 2, \ldots, \lfloor z_2 \rfloor$ to slice R into horizontal strips. Beginning from the lowest strip and working upward, we see that (1) implies that each strip in turn wholly contains exactly α copies of the $v \times 1$ brick B_1. We slide the bricks to the left within each successive strip to tile a portion of the sub-rectangle R_1 with translates of B_1, as shown in Figure 5. If z_2 is not an integer, then in the final step we see that the topmost horizontal strip of size $z_1 \times (z_2 - \lfloor z_2 \rfloor)$ must contain α bricks of size $v \times 1$, which is impossible because $0 < z_2 - \lfloor z_2 \rfloor < 1$. Therefore z_2 in an integer, and our claim is true.

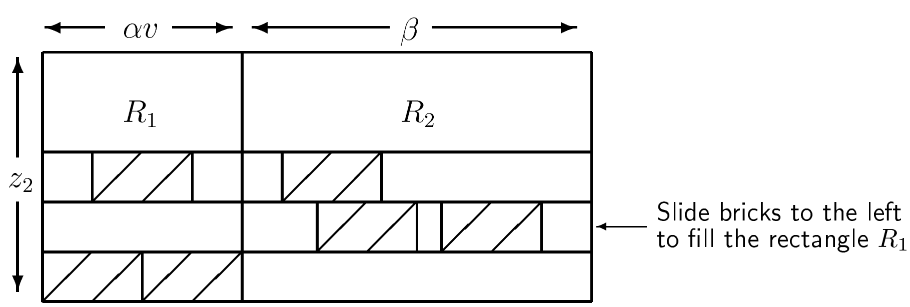


Figure 5: The proof of Claim 1

Claim 2: *The sub-rectangle R_2 of size $\beta \times z_2$ is a multiple of the brick B_2 of size $1 \times w$.* The argument is almost identical to the one given above; we remove the bricks B_1, slice R into strips with horizontal lines at height $w, 2w, \ldots, \lfloor z_2/w \rfloor w$, and slide the bricks B_2 to the right to fill R_2. ■

Here is the equivalent arithmetic formulation of Theorem 8; the equivalence is clear from our discussion following the example in Section 1.

Theorem 8′ (Arithmetic). *A $z_1 \times z_2$ rectangle can be tiled by translates of $v_1 \times v_2$ and $w_1 \times w_2$ rectangles if and only if*

(a) z_1/v_1 *and* z_2/v_2 *are integers;* *or*
(b) z_2/w_1 *and* z_2/w_2 *are integers;* *or*
(c) z_1 *is in* $\langle v_1, w_1 \rangle$, *and the numbers* z_2/v_2 *and* z_2/w_2 *are integers;* *or*
(d) z_2 *is in* $\langle v_2, w_2 \rangle$, *and the numbers* z_1/v_1 *and* z_1/w_1 *are integers.*

7 Proof of Theorem 1′

We prove our main theorem in its arithmetic formulation. We have seen that Theorem 1′ is true in dimension 1 (Theorem 4) and dimension 2 (Theorem 8′). We henceforth suppose that $d \geq 3$. If either (a) or (b) is true, then the box can be tiled by translates of one brick, while if (c) is true, then there is a bipartite tiling.

Conversely, suppose that a $z_1 \times z_2 \times \cdots \times z_d$ box R is tiled by translates of bricks of size $v_1 \times v_2 \times \cdots \times v_d$ and $w_1 \times w_2 \times \cdots \times w_d$. Also, suppose that neither (a) nor (b) holds. We will show that condition (c) must hold, which will complete the proof. Because (a) and (b) fail, there are indices j and k such that neither z_j/v_j nor z_k/w_k is an integer. We claim that j must equal k. For if $j \neq k$, then an inspection of a suitable 2-dimensional face of R reveals a tiling of a $z_j \times z_k$ rectangle with translates of $v_j \times v_k$ and $w_j \times w_k$ rectangular bricks. However, each of the conditions in Theorem 8′ fails, and therefore no such tiling exists. Therefore $j = k$. Of course, z_k is in $\langle v_k, w_k \rangle$ by Lemma 3. We have shown that (c) holds. ■

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