A Lower Bound for Schur Numbers and Multicolor Ramsey Numbers of K_3

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Abstract

For $k \geq 5$, we establish new lower bounds on the Schur numbers $S(k)$ and on the k-color Ramsey numbers of K_3.

For integers m and n, let $[m,n]$ denote the set $\{i \,|\, m \leq i \leq n\}$. A set S of integers is called *sum-free* if $i, j \in S$ implies $i + j \notin S$, where we allow $i = j$. The Schur function S(k) is defined for all positive integers as the maximum n such that $[1,n]$ can be partitioned into k sum-free sets.

The *k-color Ramsey number* of the complete graph K_n, often denoted $R_k(n)$, is defined to be the smallest integer t, such that in any k-coloring of the edges of K_t, there is a complete subgraph K_n all of whose edges have the same color. A sum-free partition of $[1,s]$ gives rise to a K_3-free edge k-coloring of K_{s+1} by identifying the vertex set of K_{s+1} with $[0,s]$ and by coloring the edge uv according to the set membership of $|u-v|$. Hence $R_k(3) \geq S(k) + 2$.

It is known that $S(1) = 1$, $S(2) = 4$, $S(3) = 13$, and $S(4) = 44$. The first three values are easy to verify; the last one is due to L. D. Baumert [1]. The best previously published bounds for $S(5)$ are $157 \leq S(5) \leq 321$, the lower bound was proved in [4] and the upper bound in [6]. For Ramsey numbers we know $R_2(3) = 6$ and $R_3(3) = 17$; the current bounds on $R_4(3)$ are 51 and 65 [5].

Below we list the five sets of a sum-free partition of $[1,160]$, Since the partition is symmetric (i and $161 - i$ always belong to the same set), only the integers from 1 to 80 are listed.

Set 1: 4 5 15 16 22 28 29 39 40 41 42 48 49 59

Set 2: 2 3 8 14 19 20 24 25 36 46 47 51 62 73

Set 3: 7 9 11 12 13 17 27 31 32 33 35 37 53 56 57 61 79

Set 4: 1 6 10 18 21 23 26 30 34 38 43 45 50 54 65 74

Set 5: 44 52 55 58 60 63 64 66 67 68 69 70 71 72 75 76 77 78 80

This proves that $S(5) \geq 160$. It follows that $R_5(3) \geq 162$. From [1] and [2] we have $S(k) \geq c(321)^{k/5} > c(3.17176)^k$ for some positive constant c. From the recurrence $R_k(3) \geq 3\,R_{k-1}(3) + R_{k-3}(3) - 3$ proved in [3] we also have $R_6(3) \geq 500$.

This construction was found using heuristic techniques that will be described below. We have found approximately 10,000 different partitions of $[1, 160]$; of these, four are symmetric. These 10,000 partitions are all "close" to each other. In other words, one can begin with one of the partitions, move an integer from one set to another, and obtain a new partition. This can be contrasted with the situation for partitions of $[1, 159]$ where we found over 100,000 partitions, most of which were not close in this sense. It is tempting to conclude that there are far fewer sum-free partitions of $[1, 160]$ than of $[1, 159]$.

Two different partitions may generate isomorphic Ramsey colorings. With this is mind, we used some simple criteria to try to determine how many non-isomorphic Ramsey colorings the 10,000 partitions gave us, and found that at least 1500 were represented.

To construct these partitions one can use any of the well known approximation heuristics for combinatorial optimization problems (simulated annealing, genetic algorithms, etc.). The key is to choose the right objective function. Of those we studied, one family of functions seems to work particularly well. To define our function, let P be a partition of $[1, n]$, and let $P = \{S_1, \ldots, S_k\}$. For $1 \leq t \leq n$, let P_t denote the partition of $[1, t]$ induced by P (i.e., $P_t = \{S_1 \cap [1, t], \ldots, S_k \cap [1, t]\}$). For integers i and j, it is convenient to define

$$g_n(i, j) = \begin{cases} 0 & \text{if } i + j \leq n, \\ 2n - i - j & \text{otherwise.} \end{cases}$$

Then define:

$$f_1(P) = max\{t | P_t \ is\ sum\text{-}free\}$$

$$f_2(P) = \sum_{i=1}^{k} \sum_{s,t \in S_i} g_n(s, t)$$

and finally

$$f(P) = c_1 f_1(P) + c_2 f_2(P).$$

For appropriate positive constants c_1 and c_2, $f(P)$ is the objective function for our maximization problem. In practice, we make c_1 relatively large and c_2 relatively small so that the f_1 term is the more important term in the function. For the case $n = 160$ and $k = 5$ we obtained best results when c_1/c_2 was allowed to randomly vary in the range 2^{12} to 2^{18}. Finally we note the somewhat surprising fact that the more obvious objective function, the number of "monochromatic" sums in a partition, seems to be far less effective.

At one point we modified the objective function so as to prefer partitions having one large set. The idea was to find a partition with a set large enough to improve the lower bound for $R_4(3)$. However, the largest set found in any sum-free partition had 44 elements. Many such partitions of $[1, n]$, $157 \leq n \leq 160$, were found, but those with sets containing 45 or more elements seem to be rare, if they exist.

References

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