# Explicit Ramsey graphs and orthonormal labelings

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**Abstract**

We describe an explicit construction of triangle-free graphs with no independent sets of size $m$ and with $\Omega(m^{3/2})$ vertices, improving a sequence of previous constructions by various authors. As a byproduct we show that the maximum possible value of the Lovász $\theta$-function of a graph on $n$ vertices with no independent set of size 3 is $\Theta(n^{1/3})$, slightly improving a result of Kashin and Konyagin who showed that this maximum is at least $\Omega(n^{1/3}/\log n)$ and at most $O(n^{1/3})$. Our results imply that the maximum possible Euclidean norm of a sum of $n$ unit vectors in $R^n$, so that among any three of them some two are orthogonal, is $\Theta(n^{2/3})$.

## 1 Introduction

Let $R(3,m)$ denote the maximum number of vertices of a triangle-free graph whose independence number is at most $m$. The problem of determining or estimating R(3,m) is a well studied Ramsey type problem. Ajtai, Komlós and Szemerédi proved in [1] that $R(3,m) \leq O(m^2/\log m)$, (see also [17] for an estimate with a better constant). Improving a result of Erdös , who showed in [7] that $R(3,m) \geq \Omega((m/\log m)^2)$, (see also [18], [13] or [4] for a simpler proof), Kim [10] proved, very recently, that the upper bound is tight, up to a constant factor, that is: $R(3,m) = \Theta(m^2/\log m)$. His proof, as well as that of Erdös, is probabilistic, and does not supply any explicit construction of such a graph. The problem of finding an explicit construction of triangle-free graphs of independence number $m$ and many vertices has also received a considerable amount of attention. Erdös [8] gave an explicit construction of such graphs with

$$\Omega(m^{(2\log 2)/3(\log 3-\log 2)}) = \Omega(m^{1.13})$$

vertices. This has been improved by Cleve and Dagum [6], and improved further by Chung, Cleve and Dagum in [5], where the authors present a construction with

$$\Omega(m^{\log 6/\log 4}) = \Omega(m^{1.29})$$

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vertices. The best known explicit construction is given in [2], where the number of vertices is $\Omega(m^{4/3})$.

Here we improve this bound and describe an explicit construction of triangle free graphs with independence numbers $m$ and $\Omega(m^{3/2})$ vertices. Our graphs are Cayley graphs and their construction is based on some of the properties of certain Dual BCH error-correcting codes. The bound on their independence numbers follows from an estimate of their Lovász $\theta$-function. This fascinating function, introduced by Lovász in [14], can be defined as follows. If $G = (V, E)$ is a graph, an *orthonormal labeling* of $G$ is a family $(b_v)_{v \in V}$ of unit vectors in an Euclidean space so that if $u$ and $v$ are distinct non-adjacent vertices, then $b_u^t b_v = 0$, that is, $b_u$ and $b_v$ are orthogonal. The $\theta$-number $\theta(G)$ is the minimum, over all orthonormal labelings $b_v$ of $G$ and over all unit vectors $c$, of

$$max_{v \in V} \frac{1}{(c^t b_v)^2}.$$

It is known (and easy; see [14]) that the independence number of $G$ does not exceed $\theta(G)$. The graphs $G_n$ we construct here are triangle free graphs on $n$ vertices satisfying $\theta(G_n) = \Theta(n^{2/3})$, and hence the independence number of $G_n$ is at most $O(n^{2/3})$.

The construction and the properties of the $\theta$-function settle a geometric problem posed by Lovász and partially solved by Kashin and Konyagin [12], [9]. Let $\Delta_n$ denote the maximum possible value of the Euclidean norm $||\sum_{i=1}^n u_i||$ of the sum of $n$ unit vectors $u_1, \ldots, u_n$ in $R^n$, so that among any three of them some two are orthogonal. Motivated by the study of the $\theta$-function, Lovász raised the problem of determining the order of magnitude of $\Delta_n$. In [12] it is shown that $\Delta_n \leq O(n^{2/3})$ and in [9] it is proved that this is nearly tight, namely that $\Delta_n \geq \Omega(n^{2/3}/(\log n)^{1/2})$. Here we show that the upper bound is tight up to a constant factor, that is:

$$\Delta_n = \Theta(n^{2/3}).$$

The rest of this note is organized as follows. In Section 2 we construct our graphs and estimate their $\theta$-numbers and their independence numbers. The resulting lower bound for $\Delta_n$ is described in Section 3. Our method in these sections combines the ideas of [9] with those in [2]. The final Section 4 contains some concluding remarks.

## 2 The graphs

For a positive integer $k$, let $F_k = GF(2^k)$ denote the finite field with $2^k$ elements. The elements of $F_k$ are represented, as usual, by binary vectors of length $k$. If $a, b$ and $c$ are three such vectors, let $(a, b, c)$ denote their concatenation, i.e., the binary vector of length $3k$ whose coordinates are those of $a$, followed by those of $b$ and those of $c$. Suppose $k$ is not divisible by 3 and put $n = 2^{3k}$. Let $W_0$ be the set of all nonzero elements $\alpha \in F_k$ so that the leftmost bit in the binary representation of $\alpha^7$ is 0, and let $W_1$ be the set of all nonzero elements $\alpha \in F_k$ for which the leftmost bit of $\alpha^7$ is

1. Since 3 does not divide $k$, 7 does not divide $2^k-1$ and hence $|W_0| = 2^{k-1}-1$ and $|W_1| = 2^{k-1}$, as when $\alpha$ ranges over all nonzero elements of $F_k$ so does $\alpha^7$.

Let $G_n$ be the graph whose vertices are all $n = 2^{3k}$ binary vectors of length $3k$, where two vectors $u$ and $v$ are adjacent if and only if there exist $w_0 \in W_0$ and $w_1 \in W_1$ so that $u + v = (w_0, w_0^3, w_0^5) + (w_1, w_1^3, w_1^5)$, where here the powers are computed in the field $F_k$ and the addition is addition modulo 2. Note that $G_n$ is the Cayley graph of the additive group $(Z_2)^{3k}$ with respect to the generating set $S = U_0 + U_1 = \{u_0 + u_1 : u_0 \in U_0, u_1 \in U_1\}$, where $U_0 = \{(w_0, w_0^3, w_0^5) : w_0 \in W_0\}$, and $U_1$ is defined similarly. The following theorem summerizes some of the properties of the graphs $G_n$.

**Theorem 2.1** *If $k$ is not divisible by 3 and $n = 2^{3k}$ then $G_n$ is a $d_n = 2^{k-1}(2^{k-1}-1)$-regular graph on $n = 2^{3k}$ vertices with the following properties.*

1. *$G_n$ is triangle-free.*
2. *Every eigenvalue $\mu$ of $G_n$, besides the largest, satisfies*
$$-9 \cdot 2^k - 3 \cdot 2^{k/2} - 1/4 \leq \mu \leq 4 \cdot 2^k + 2 \cdot 2^{k/2} + 1/4.$$
3. *The $\theta$-function of $G_n$ satisfies*
$$\theta(G_n) \leq n \frac{36 \cdot 2^k + 12 \cdot 2^{k/2} + 1}{2^k(2^k-2) + 36 \cdot 2^k + 12 \cdot 2^{k/2} + 1} \leq (36 + o(1))n^{2/3},$$
*where here the $o(1)$ term tends to 0 as $n$ tends to infinity.*

**Proof.** The graph $G_n$ is the Cayley graph of $Z_2^{3k}$ with respect to the generating set $S = S_n = U_0 + U_1$, where $U_i$ are defined as above.

Let $A_0$ be the $3k$ by $2^{k-1}-1$ binary matrix whose columns are all vectors of $U_0$, and let $A_1$ be the $3k$ by $2^{k-1}$ matrix whose columns are all vectors of $U_1$. Let $A = [A_0, A_1]$ be the $3k$ by $2^k - 1$ matrix whose columns are all those of $A_0$ and those of $A_1$. This matrix is the parity check matrix of a binary BCH-code of designed distance 7 (see, e.g., [16], Chapter 9), and hence every set of six columns of $A$ is linearly independent over $GF(2)$. In particular, all the sums $(u_0 + u_1)_{u_0 \in U_0, u_1 \in U_1}$ are distinct and hence $|S_n| = |U_0||U_1|$. It follows that $G_n$ has $2^{3k}$ vertices and it is $|S_n| = 2^{k-1}(2^{k-1}-1)$ regular.

The fact that $G_n$ is triangle-free is equivalent to the fact that the sum (modulo 2) of any set of 3 elements of $S_n$ is not the zero-vector. Let $u_0 + u_1$, $u_0' + u_1'$ and $u"_0 + u"_1$ be three distinct elements of $S_n$, where $u_0, u_0', u"_0 \in U_0$ and $u_1, u_1', u"_1 \in U_1$. If the sum (modulo 2) of these six vectors is zero then, since every six columns of $A$ are linearly independent, every vector must appear an even number of times in the sequence $(u_0, u_0', u"_0, u_1, u_1', u"_1)$. However, since $U_0$ and $U_1$ are disjoint this implies that every vector must appear an even number of times in the sequence $(u_0, u_0', u"_0)$ and this is clearly impossible. This proves part 1 of the theorem.

In order to prove part 2 we argue as follows. Recall that the eigenvalues of Cayley graphs of abelian groups can be computed easily in terms of the characters of the group. This result, decsribed in, e.g., [15], implies that the eigenvalues of the graph $G_n$ are all the numbers

$$\sum_{s \in S_n} \chi(s),$$

where $\chi$ is a character of $Z_2^{3k}$. By the definition of $S_n$, these eigenvalues are precisely all the numbers

$$(\sum_{u_0 \in U_0} \chi(u_0))(\sum_{u_1 \in U_1} \chi(u_1)).$$

It follows that these eigenvalues can be expressed in terms of the Hamming weights of the linear combinations (over $GF(2)$) of the rows of the matrices $A_0$ and $A_1$ as follows. Each linear combination of the rows of $A$ of Hamming weight $x+y$, where the Hamming weight of its projection on the columns of $A_0$ is $x$ and the weight of its projection on the columns of $A_1$ is $y$, corresponds to the eigenvalue

$$(2^{k-1} - 1 - 2x)(2^{k-1} - 2y).$$

Our objective is thus to bound these quantities.

The linear combinations of the rows of $A$ are simply all words of the code whose generating matrix is $A$, which is the dual of the BCH-code whose parity-check matrix is $A$. It is known (see [16], pages 280-281) that the Carlitz-Uchiyama bound implies that the Hamming weight $x+y$ of each non-zero codeword of this dual code satisfies

$$2^{k-1} - 2^{1+k/2} \leq x + y \leq 2^{k-1} + 2^{1+k/2}. \tag{1}$$

Let $p$ denote the characteristic vector of $W_1$, that is, the binary vector indexed by the non-zero elements of $F_k$ which has a 1 in each coordinate indexed by a member of $W_1$ and a 0 in each coordinate indexed by a member of $W_0$. Note that the sum (modulo 2) of $p$ and any linear combination of the rows of $A$ is a non-zero codeword in the dual of the BCH-code with designed distance 9. Therefore, by the Carlitz-Uchiyama bound, the Hamming weight of the sum of $p$ with the linear combination considered above, which is $x + (2^{k-1} - y)$, satisfies

$$2^{k-1} - 3 \cdot 2^{k/2} \leq x + 2^{k-1} - y \leq 2^{k-1} + 3 \cdot 2^{k/2}. \tag{2}$$

Since for any two reals $a$ and $b$,

$$-(\frac{a-b}{2})^2 \leq ab \leq (\frac{a+b}{2})^2$$

we conclude from (1) that

$$(2^{k-1} - 1 - 2x)(2^{k-1} - 2y) \leq \frac{(2^k - 1 - 2(x+y))^2}{4} \leq 4 \cdot 2^k + 2 \cdot 2^{k/2} + 1/4.$$

Similarly, (2) implies that

$$(2^{k-1}-1-2x)(2^{k-1}-2y) \geq -\frac{(1+2(x-y))^2}{4} \geq -9\cdot 2^k - 3\cdot 2^{k/2} - 1/4.$$

This completes the proof of part 2 of the theorem.

Part 3 follows from part 2 together with Theorem 9 of [14] which asserts that for $d$-regular graphs $G$ with eigenvalues $d=\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$,

$$\theta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n}.$$

It is worth noting that the fact that the right hand side in the last inequality bounds the independence number of $G$ is due to A. J. Hoffman. □

Since the independence number of each graph $G$ does not exceed $\theta(G)$ the following result follows.

**Corollary 2.2** *If $k$ is not divisible by 3 and $n=2^{3k}$, then the graph $G_n$ is a triangle-free graph with independence number at most $(36+o(1))n^{2/3}$.* □

Let $G_n$ be one of the graphs above and let $\overline{G}_n$ denote its complement. Since $G_n$ is a Cayley graph, Theorem 8 in [14] implies that $\theta(\overline{G}_n)\theta(G_n) = n$ and hence, by Theorem 2.1, $\theta(\overline{G}_n) \geq (1+o(1))\frac{1}{36}n^{1/3}$.

In [9] it is proved (in a somewhat disguised form), that for any graph $H$ with $n$ vertices and no independent set of size 3, $\theta(H) \leq 2^{2/3}n^{1/3}$. (See also [3] for an extension). Since $\overline{G}_n$ has no independent set of size 3 and since for every graph $H$, $\theta(H)\theta(\overline{H}) \geq n$ (see Corollary 2 of [14]) the following result follows.

**Corollary 2.3** *If $k$ is not divisible by 3 and $n=2^{3k}$, then $\theta(G_n)=\Theta(n^{2/3})$ and $\theta(\overline{G}_n)=\Theta(n^{1/3})$. Therefore, the minimum possible value of the $\theta$-number of a triangle-free graph on $n$ vertices is $\Theta(n^{2/3})$ and the maximum possible value of the $\theta$-number of an $n$-vertex graph with no independent set of size 3 is $\Theta(n^{1/3})$.*

## 3 Nearly orthogonal systems of vectors

A system of $n$ unit vectors $u_1,\ldots,u_n$ in $R^n$ is called *nearly orthogonal* if any set of three vectors of the system contains an orthogonal pair. Let $\Delta_n$ denote the maximum possible value of the Euclidean norm $||\sum_{i=1}^n u_i||$, where the maximum is taken over all systems $u_1,\ldots,u_n$ of nearly orthogonal vectors. Lovász raised the problem of determining the order of magnitude of $\Delta_n$. Konyagin showed in [12] that $\Delta_n \leq O(n^{2/3})$ and that

$$\Delta_n \geq \Omega(n^{4/3-\log 3/2\log 2}) \geq \Omega(n^{0.54}).$$

The lower bound was improved by Kashin and Konyagin in [9], where it is shown that

$$\Delta_n \geq \Omega(n^{2/3}/(\log n)^{1/2})$$

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The following theorem asserts that the upper bound is tight up to a constant factor.

**Theorem 3.1** *There exists an absolute positive constant $a$ so that for every $n$*

$$\Delta_n \geq an^{2/3}.$$

*Thus,* $\Delta_n = \Theta(n^{2/3})$.

**Proof.** It clearly suffices to prove the lower bound for values of $n$ of the form $n = 2^{3k}$, where $k$ is an integer and 3 does not divide $k$. Fix such an $n$, let $G = G_n = (V, E)$ be the graph constructed in the previous section and define $\theta = \theta(G)$. By Theorem 2.1, $\theta \leq (36 + o(1))n^{2/3}$. By the definition of $\theta$ there exists an orthonormal labeling $(b_v)_{v \in V}$ of $G$ and a unit vector $c$ so that $(c^t b_v)^2 \geq 1/\theta$ for every $v \in V$. Therefore, the norm of the projection of each $b_v$ on $c$ is at least $1/\sqrt{\theta}$ and by assigning appropriate signs to the vectors $b_v$ we can ensure that all these projections are in the same direction. With this choice of signs, the norm of the projection of $\sum_{v \in V} b_v$ on $c$ is at least $n/\sqrt{\theta}$, implying that

$$||\sum_{v \in V} b_v|| \geq n/\sqrt{\theta} \geq (\frac{1}{6} - o(1))n^{2/3}.$$

Note that since the vectors $b_v$ form an orthonormal labeling of $G$, which is triangle-free, among any three of them there are some two which are orthogonal. This implies that $(b_v)_{v \in V}$ is a nearly orthogonal system and shows that for every $n = 2^{3k}$ as above

$$\Delta_n \geq (\frac{1}{6} - o(1))n^{2/3},$$

completing the proof of the theorem. □

# 4 Concluding remarks

The method applied here for explicut constructions of triangle-free graphs with small independence numbers cannot yield asymptotically better constructions. This is because the independence number is bounded here by bounding the $\theta$-number which, by Corollary 2.3, cannot be smaller than $\Theta(n^{2/3})$ for any triangle-free graph on $n$ vertices.

Some of the results of [9] can be extended. In a forthcoming paper with N. Kahale [3] we show that for every $k \geq 3$ and every graph $H$ on $n$ vertices with no independent set of size $k$,

$$\theta(H) \leq Mn^{1-2/k}, \tag{3}$$

for some absolute positive constant $M$. It is not known if this is tight for $k > 3$. Combining this with some of the properties of the $\theta$-function, this can be used to show that for every $k \geq 3$ and any system of $n$ unit vectors $u_1, \ldots, u_n$ in $R^n$ so that among any $k$ of them some two are orthogonal, the inequality

$$||\sum_{i=1}^{n} u_i|| \leq O(n^{1-1/k})$$

holds. This is also not known to be tight for $k > 3$. Lovász (cf. [11]) conjectured that there exists an absolute constant $c$ so that for every graph $H$ on $n$ vertices and no independent set of size $k$,

$$\theta(H) \leq ck\sqrt{n}.$$

Note that this conjecture, if true, would imply that the estimate (3) above is *not* tight for all fixed $k > 4$.

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