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Rings of Global Sections in Two-dimensional Schemes

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Abstract. In this paper we study the ring of global sections $\Gamma(U, \mathcal{O})$ of an open subset $U = D(I) \subseteq \operatorname{Spec} A$, where A is a two-dimensional noetherian ring. The main concern is to give a geometric criterion when these rings are finitely generated, in order to correct an invalid statement of Schenzel in [7].

1. Introduction

Let A be a noetherian ring with an ideal $I \subseteq A$ and $U = D(I) \subseteq \operatorname{Spec} A$ the corresponding open subset. If U is an affine scheme, then the ring of global sections $B = \Gamma(U, \mathcal{O}_X)$ – which is also called the ideal-transform $T(I)$ – is of finite type over A. The converse is by no means true, in dimension two however we have the following result due to Eakin et. al. ([4], Theorem 3.2): Suppose A is a local excellent[1] Cohen-Macaulay domain of dimension two, and let I be an ideal of height one. Then (among other characterizations) $D(I)$ is affine if and only if B is noetherian if and only if B is of finite type over A.

Schenzel states in [7], Theorem 4.1 and 4.2, that this holds more general for two-dimensional excellent local domains. However, this is not true, as the following example shows.[2]

[1]In fact the result was stated under the somewhat weaker conditions that the normalization is finite and the local rings of the normalization are analytically irreducible, instead of excellent.

[2]The mistake in [7] is at the end of the proof of Theorem 4.1, where the statement $T \subseteq T_N$ is wrong.

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Example. Let $X = \operatorname{Spec} A$ be an affine excellent irreducible surface which is regular outside one single closed point P and such that in the normalization two points P_1, P_2 lie over P. Outside these points the normalization mapping $\tilde{X} \longrightarrow X$ is isomorphic.

Let $Y = V(I)$ be the image of an irreducible curve Y' passing through P_1, but not through P_2. Then $U = X - Y$ is not affine, since the preimage of Y consists of the curve Y' and of the isolated point P_2. On the other hand, $U = X - Y$ is normal and isomorphic to $\tilde{X} - Y' - P_2$, so the rings of global sections are identical. Since $\tilde{X}$ is normal, this ring equals also the ring of global sections of $\tilde{X} - Y'$. $\tilde{X}$ is a normal excellent affine surface, thus the complement of a curve is affine, and B is finitely generated. For an explicit example see below.

In this paper we give a criterion for two-dimensional local rings to decide the finiteness of the ring of global sections of $U = D(I)$, I an ideal of height one. The criterion is based on the combinatoric of the components in the completion $\hat{A}$ of A. It says that in case U is not affine the ring of global sections of U is not finitely generated if and only if there exists an irreducible component of $\operatorname{Spec} \hat{A}$ where U is affine and a component where U is not affine such that their intersection is one-dimensional.

The criterion is (due to the connectedness theorem of Hartshorne) seen to be fulfilled in case A is Cohen-Macaulay, thus we recover the result of Eakin et. al. as a corollary (Cor. 2.4). Another consequence is that if $D(I)$ is non-affine and connected, then $T(I)$ is not noetherian (Cor. 2.3).

In the third section we extend the result to the non-complete case and describe the conditions used in the criterion in terms of the normalization.

2. The complete case

Let $X = \operatorname{Spec} A$ be the spectrum of a local complete noetherian ring A of dimension 2, and let P denote the closed point. Let $X_j = V(\mathbf{p}_j) = \operatorname{Spec} A/\mathbf{p}_j$ be the irreducible components of X corresponding to the minimal primes $\mathbf{p}_j$, $j \in J$.

Let I be an ideal in A, $Y = V(I)$ and $U = D(I)$. U is affine if and only if $U_j = U \cap X_j$ is affine on every component, and this is due to the theorem of Lichtenbaum-Hartshorne (see [3], 8.2.1) the case if and only if $\operatorname{ht} I(A/\mathbf{p}_j) \leq 1$ for every $j \in J$. Thus U is not affine if and only if there exists a two-dimensional component X_j where $Y_j = Y \cap X_j$ consists just of the single point P.

We want to know for an ideal I of height one whether the ring of global sections of $D(I)$ is finitely generated. If $D(I)$ is affine, this is the case, so we suppose furtheron that $D(I)$ is not affine. We divide $J = J_0 \cup J_1$ in such a way, that for $j \in J_1$ the open subsets $U_j \subseteq X_j$ are affine and for $j \in J_0$ not. Thus the X_j, $j \in J_0$, are the two-dimensional components of X where Y_j is just the closed point. The affineness of U is equivalent with $J_0 = \emptyset$.

Put $\mathbf{a}_0 = \bigcap_{j \in J_0} \mathbf{p}_j$ and $\mathbf{a}_1 = \bigcap_{j \in J_1} \mathbf{p}_j$ and $X_0 = \operatorname{Spec} A/\mathbf{a}_0$, $X_1 = \operatorname{Spec} A/\mathbf{a}_1$. We denote the structure sheaves on these closed subschemes of X with $\mathcal{O}_i$, $i = 0, 1$.

Furthermore we put $U_i = U \cap X_i$, $i = 0, 1$, considered as an open subset in X_i with the induced scheme structure, put $B_i = \Gamma(U_i, \mathcal{O}_i)$. $U_1 = \operatorname{Spec} B_1 \subset X_1$ is affine, U_0 is not affine. The closed embedding $X_1 \hookrightarrow X$ yields a (closed) restriction map $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U_1, \mathcal{O}_1) = B_1$.

Finally, let $\mathbf{b} = \mathbf{a}_0 + \mathbf{a}_1 \subseteq \mathbf{m}$ and $R = A/\mathbf{b}$. R is a zero- or one-dimensional local complete noetherian ring, let $Z = \operatorname{Spec} R$ und $Z^\times = D(\mathbf{m}) \subset Z$. The dimension of $Z = V(\mathbf{b}) = X_0 \cap X_1$ is the crucial point for $\Gamma(U, \mathcal{O}_X)$ to be noetherian or not.

For our proof we have to put on A the condition S_1 of Serre, meaning that every associated prime of A is minimal, equivalently that every zero-divisor lies in a minimal prime or that every ideal of height one contains a non-zero-divisor. This is fulfilled for example if A is reduced.

Theorem 2.1. *Let A be a two-dimensional complete local noetherian ring, fulfilling the condition S_1. Let I be an ideal of height one and suppose that $U = D(I)$ is not affine. Then the following are equivalent.*

(1) $\Gamma(U, \mathcal{O}_X)$ *is not of finite type.*

(2) $\Gamma(U, \mathcal{O}_X)$ *is not noetherian.*

(3) *The image of* $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U_1, \mathcal{O}_1)$ *is not noetherian.*

(4) *The intersection Z of X_0 and X_1 is one-dimensional.*

Proof. The implications (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are clear. (1) $\Rightarrow$ (4). Suppose $Z = \{P\}$ is only the closed point. Then U is the disjoint union of U_0 and U_1 (both closed hence open in U). Thus we have

$$\Gamma(U, \mathcal{O}_X) = \Gamma(U_0, \mathcal{O}_0) \oplus \Gamma(U_1, \mathcal{O}_1)\,.$$

Since U_1 is affine, the second component is of finite type. Since $U_0 = X_0 - \{P\}$, the mapping $A/\mathbf{a}_0 \longrightarrow \Gamma(U_0, \mathcal{O}_X)$ is also of finite type, see Lemma 2.2 (1).

So we have to show (4) $\Rightarrow$ (3). We denote the image of $\Gamma(U, \mathcal{O}_X) \longrightarrow \Gamma(U_1, \mathcal{O}_1)$ by C.

Let $h \in A$ be an element such that in $Z = \operatorname{Spec} R$ we have $V(h) = V(\mathbf{m}) = \{P\}$. Thus $1/h$ is a function defined on $Z^\times = Z - \{P\} = D(h)$. Since $Z^\times \hookrightarrow U_1$ is a closed embedding and since U_1 is affine, there exists a function $q \in B_1 = \Gamma(U_1, \mathcal{O}_1)$ with $q\,|_{Z^\times} = 1/h$.

$$\begin{array}{ccccc} X_0 & \hookleftarrow & Z & \hookrightarrow & X_1 \\ \cup & & \cup & & \cup \\ U_0 & \hookleftarrow & Z^\times & \hookrightarrow & U_1 \\ & \searrow & \cap & \swarrow & \\ & & U & & \end{array}$$

Let $a \in \mathbf{b} \subset A$ be a regular element (i.e. a non-zero-divisor) inside the describing ideal of Z. The functions aq^n are defined on U_1 and the restrictions to $Z^\times$ are zero, thus they are extendible to Z. Since $Z \hookrightarrow X_0$ is closed and X_0 is affine, these functions are also extendible to X_0 and in particular to U_0. So we may assume that these functions are defined on U and we see that they lie in C.

Consider in C the ideal $(a, aq, aq^2, aq^3, ...)$ spanned by this functions, and suppose that it is finitely generated. Then we have an equation

$$aq^{n+1} = a_n aq^n + ... + a_1 aq + a_0 a$$

with $a_i \in C \subset B_1$. We may assume that $a_i \in \Gamma(U, \mathcal{O}_X)$. Since a is regular in A, it is also a regular in $A/\mathbf{a}_i$. (For if $ax \in \mathbf{a}_i = \bigcap_{j \in J_i} \mathbf{p}_j$, we have $ax \in \mathbf{p}_j$ for all $j \in J_i$ and thus $x \in \mathbf{p}_j$ for all $j \in J_i$, so $x = 0 \mod \mathbf{a}_i$.) Since the restriction $A/\mathbf{a}_1 = \Gamma(X_1, \mathcal{O}_1) \longrightarrow \Gamma(U_1, \mathcal{O}_1)$ is injective, a is also a regular element in B_1.

This yields in B_1 (on U_1) the equation $q^{n+1} = a_n q^n + ... + a_1 q + a_0$. This equation restricted to $Z^\times \subseteq U_1$ yields an integral equation for $q = 1/h$ over $R[a_i'] \subseteq R_h$, where the a_i' denote the restrictions of a_i on $R_h = \Gamma(Z^\times, \mathcal{O}_Z)$.

We claim that the a_i' are integral over R: Consider the elements $a_i \in \Gamma(U, \mathcal{O}_X)$ as functions on U_0 – as elements of B_0. Since $U_0 = X_0 - \{P\}$, the $a_i \in B_0$ are integral over $A/\mathbf{a}_0 = \Gamma(X_0, \mathcal{O}_0)$, see Lemma 2.2. The closed embeddings $(Z^\times \subset Z) \hookrightarrow (U_0 \subset X_0)$ show that the a_i' are integral over $R = \Gamma(Z, \mathcal{O}_Z)$. It follows that $q \mid_Z = 1/h$ would be integral over R, but this is not possible. □

Lemma 2.2. *Let A be a local noetherian ring of dimension two fulfilling S_1. Let $\mathbf{m}$ be the maximal ideal and $B = \Gamma(D(\mathbf{m}), \mathcal{O})$ the ring of global sections. Then the following hold.*

(1) *$A \longrightarrow B$ is of finite type.*

(2) *If furthermore all components of* $\operatorname{Spec} A$ *have dimension two, B is even finite over A.*

Proof. We first prove the second part, using [6], 5.11.4 (or [2], 2.5.). A point $x \in \operatorname{Ass} \mathcal{O}_X$ has height zero, for every ideal of bigger height contains a regular element. The closure $\bar{x}$ is a two-dimensional component and therefore the point P has codimension two on it.

The first part follows from the second part. The one-dimensional components of X meet the other components only in the closed point, thus the punctured curves are connected components of $W = D(\mathbf{m})$. These are affine and of finite type. □

We deduce from the theorem two corollaries.

Corollary 2.3. *Let A be a local complete noetherian ring of dimension two fulfilling S_1. Let I be an ideal of height one. If $U = D(I)$ is connected and $\Gamma(U, \mathcal{O}_X)$ is of finite type, then U is affine.*

Proof. Suppose U is not affine, then in the partition described above U_0 is not empty, and U_1 is not empty since I is of height one. Put $Z = X_0 \cap X_1$. Since U is connected, U_0 and U_1 are not disjoint, thus Z does not consist only of the closed point, it must be a curve. Then due to the theorem the ring of global sections can not be noetherian. □

We recover the result of Eakin et. al. in the Cohen-Macaulay case.

Corollary 2.4. *Let A be a local complete noetherian Cohen-Macaulay ring of dimension two. Let I be an ideal of height one. Then $U = D(I)$ is affine if and only if its ring of global sections is of finite type (or noetherian).*

Proof. Again, suppose U to be not affine, put $X = X_0 \cup X_1$ as before with the describing ideals $\mathbf{a}_0$ and $\mathbf{a}_1$. Then $\mathbf{a}_0 \cap \mathbf{a}_1$ is nilpotent, thus due to the connectedness theorem of Hartshorne (see [5], Theorem 18.12) the ideal $\mathbf{a}_0 + \mathbf{a}_1$ has height one. Since it describes the intersection, $Z = X_0 \cap X_1$ is one-dimensional and $\Gamma(U, \mathcal{O}_X)$ is not noetherian. □

Example. Of course, $U = D(I)$ can be affine without being connected. $A = K[[x, y, z]]/(xy)$ is Cohen-Macaulay (K a field), the complement of the common axis $V(x, y)$ is affine, but not connected.

Remark. We may associate to a complete local ring of dimension two a graph Γ in such a way, that for each irreducible two-dimensional component we associate a point, and two points are connected by an edge if and only if the intersection of the corresponding components is one-dimensional. Then an open subset as above yields a partition $\Gamma = \Gamma_0 \cup \Gamma_1$, and the ring of global sections is noetherian if and only if there is no edge between points of Γ_0 and of Γ_1.

3. Interpretation in the normalization

We want to extend the result from the complete case to the general case. Suppose we are given a curve $V(I) \subseteq \operatorname{Spec} A$ where A is a two-dimensional noetherian domain. Then $\Gamma(D(I), \mathcal{O})$ is of finite type if this is true in every (closed) point $x \in \operatorname{Spec} A$, see [1]. Furthermore, we have the following lemma.

Lemma 3.1. *Let $A \longrightarrow A'$ be faithfully flat and let $U \subseteq \operatorname{Spec} A$ denote an open subset with preimage U'. Then $B = \Gamma(U, \mathcal{O})$ is of finite type over A if and only if $B' = \Gamma(U', \mathcal{O}')$ is of finite type over A'.*

Proof. We have $B' = B \otimes_A A'$ due to flatness. This yields the first implication. If B' is of finite type, we may assume that it is generated by finitely many elements of B, thus there is a surjection $A'[T_1, \ldots, T_n] \longrightarrow B' = B \otimes_A A'$ induced by $A[T_1, \ldots, T_n] \longrightarrow B$. Due to faithfulness, this must also be surjective. □

Therefore the condition in the theorem that $\Gamma(U, \mathcal{O})$ is of finite type is preserved by passing to the completion, and we may skip in Cor. 2.4 the assumption of completeness.

So we take a look at the condition that the intersection of two components in the completion is one-dimensional, and we want to describe it in terms of the normalization of A. For this we recall some correspondences between normalization and completion, see [6], 7.6.1 and 7.6.2. Let X be the spectrum of a local excellent domain A with completion $\hat{X}$ and normalization $\tilde{X}$. Then the normalization of $\hat{X}$ equals the completion of $\tilde{X}$ (semilocal), and this consists of connected components being the normalizations of the irreducible components of $\hat{X}$ and the completion of the localizations of $\tilde{X}$ as well. In particular, there is a correspondence between the irreducible components of $\hat{X}$ and the closed points of $\tilde{X}$.

For a closed subset $C \subseteq X$ the completion of C equals the preimage of C in $\hat{X}$ yielding a canonical inclusion $\hat{C} \subseteq \hat{X}$. The irreducible components of $\hat{C}$ correspond again to closed points of $\tilde{C}$, but this is of course not the preimage of C in the normalization $\tilde{X}$.

Lemma 3.2.. *Let A be an excellent local domain of dimension two, $P_0 \in \tilde{X}$ the closed point on $\tilde{X}$ corresponding to the irreducible component X_0 of the completion $\hat{X}$. Let $C \subset X$ be an irreducible curve and let $D \subset \tilde{X}$ be the preimage of C without the isolated points.*

(1) *There exists an irreducible component of $\hat{C}$ on X_0 if and only if P_0 is not an isolated point on $\varphi^{-1}(C)$ ($\varphi : \tilde{X} \longrightarrow X$ normalization map).*

(2) *The irreducible component C_0 of $\hat{C}$ lies on X_0 if and only if there exists a point $R \in \tilde{D}$ over P_0 mapping to the point $Q_0 \in \tilde{C}$ corresponding to C_0.*

(3) *The component C_0 of $\hat{C}$ connects the irreducible components X_1 and X_2 of $\hat{X}$ if and only if the corresponding point $Q_0 \in \tilde{C}$ is reached by points $R_1, R_2 \in \tilde{D}$ lying over P_1 and P_2.*

Proof. (1) We consider the mapping (completion) $\tilde{X}_0 \longrightarrow \tilde{X}_{P_0}$, where $\tilde{X}_{P_0}$ means the localization at P_0. The preimage of $C \subset X$ in $\tilde{X}_{P_0}$ is just the closed point if and only if this is true in $\tilde{X}_0$, and this is the case if and only if $\hat{C}$ is zero-dimensional on X_0.

(2) The preimage of $\hat{C}$ in $\tilde{\hat{X}}$ without the isolated points equals $\hat{D}$, being the preimage of D. The statement $C_0 \subset X_0$ is equivalent to the statement that there exists an irreducible component $D_0 \subseteq \hat{D} \subset \tilde{\hat{X}}$ over C_0 lying on $\tilde{X}_0$. Let R be the point on $\tilde{D}$ corresponding to the component $D_0 \subseteq \hat{D}$. Suitable diagrams show that D_0 dominates C_0 is equivalent with R maps to Q_0 and that $D_0 \subseteq \tilde{X}_0$ is equivalent with R maps to P_0.

(3) follows from (2). □

This motivates the following definition.

Definition. *Let X denote a reduced irreducible noetherian scheme, $\varphi : \tilde{X} \longrightarrow X$ its normalization, $P \in X$ a closed point and $P_1, P_2 \in \tilde{X}$, $\varphi(P_1) = \varphi(P_2) = P$. We call an irreducible curve $C \subset X$ a melting curve for the points P_1 and P_2 if and only if P_1, P_2 are not isolated on $\varphi^{-1}(C)$ and there exist points $R_1, R_2 \in \tilde{D}$ (D as in Lemma 3.2) over P_1, P_2 mapping to one common point $Q \in \tilde{C}$.*

Theorem 3.3. *Let $X = \operatorname{Spec} A$, where A is an excellent local domain of dimension two. Then the intersection of the components X_1 and X_2 on $\hat{X}$ is one-dimensional if and only if there exists a melting curve for $P_1, P_2 \in \tilde{X}$.*

Proof. If C is a melting curve for P_1 and P_2 with common point Q as in the definition, then the previous proposition says that the corresponding component C_0 lies on X_1 and X_2, thus the intersection is one-dimensional.

For the converse, let C_0 be an irreducible curve on $X_1 \cap X_2$ with prime ideal $\mathbf{q} \subset \hat{A}$ of height one. Then $\mathbf{p} = \mathbf{q} \cap A$ is also of height one. For $\mathbf{q}$ is not a normal point of $\hat{A}$, since on the normalization there are at least two points above it. Then also $\mathbf{p}$ is not a normal point, because the normal locus commutes with completion under the condition of excellence (see [6], 7.8.3.1.) Thus $\operatorname{ht} \mathbf{p} = 1$, $C = V(\mathbf{p})$ is a curve, C_0 a component of its completion and we may apply the previous proposition. □

Proposition 3.4. *Let P_1, P_2 be two closed points in the normalization $\tilde{X}$ over $P \in \operatorname{Spec} A$, where A is a two-dimensional noetherian domain. Then the following hold.*

(1) *If there exist two different irreducible curves C_1, C_2 with $P_i \in C_i = V(\mathbf{q}_i)$ on $\tilde{X}$ such that $\mathbf{q}_1 \cap A = \mathbf{q}_2 \cap A = \mathbf{p}$, then $C = V(\mathbf{p})$ is a melting curve for P_1, P_2.*

(2) *If C is normal (or analytically irreducible) and P_1 and P_2 are not isolated on $\varphi^{-1}(C)$, then C is a melting curve.*

(3) *If $P_1, P_2 \in C'$ is irreducible and $\varphi(C') = C$ is a melting curve, then $\varphi|_{C'} : C' \longrightarrow C$ is not birational. A melting curve lies in the non-normal locus.*

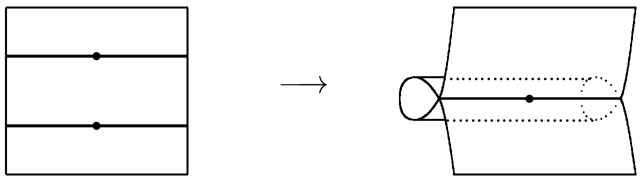
Proof. (1) Both mappings $C_1 \longrightarrow C$ and $C_2 \longrightarrow C$ are surjective, and this is then also true for the normalizations. Thus for any closed point $Q \in \tilde{C}$ there are points on $\tilde{C}_i$ over P_i mapping to Q.

(2) If C is analytically irreducible, then any closed point of $\tilde{D}$ maps to the only closed point of $\tilde{C}$.

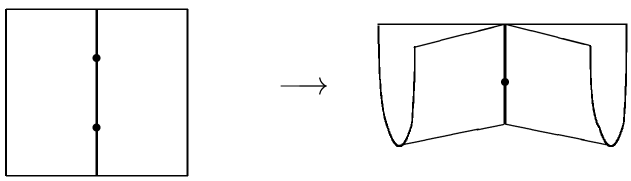
(3) Suppose $C' \longrightarrow C$ is birational. Then the normalizations of these curves are the same, and different points cannot be identified. If the generic point of a curve C is normal, then D consists just of one irreducible component, and $D \longrightarrow C$ is birational. □

Examples. We give some typical examples of (non-)melting curves to illustrate the cases the previous proposition is talking about. They are given by mappings $\mathbf{A}^2_K \longrightarrow \mathbf{A}^n_K$ such that the affine plane is the normalization of the image (K is a field).

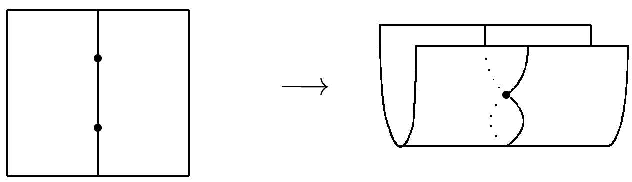
(1) $(x, y) \longmapsto (x, y^3 - y, y^2 - 1)$. This identifies the two different curves $V(y-1)$ and $V(y+1)$. The common image curve C is a melting curve.



(2) $(x, y) \longmapsto (x, y^2, xy)$. The line $V(x)$ is melted with itself, identifying the points $(0, 1)$ and $(0, -1)$. $V(x) \longrightarrow V(r, t)$ is not birational, C is a melting curve.



(3) $(x, y) \longmapsto (x, y^2, y((y-1)^2 + x^2)((y+1)^2 + x^2), xy)$. This identifies only the two points. $V(x)$ is birational with its image C, thus C is not a melting curve.



(4) Consider the mapping $(x, y) \longmapsto (x, y^2, y(x^2 - y^2(y+1))$ followed by the identification of the points $(0,0,0)$ and $(-1,0,0)$. Then $D = V(x^2 - y^2(y+1)) \longmapsto C$ is not birational, but C (= the image of D) is not a melting curve for their common point. Thus the necessary condition in Prop. 3.4 (3) is not sufficient.

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