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Tensor Product Surfaces of a Euclidean Space Curve and a Euclidean Plane Curve*

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Abstract. B.Y. Chen initiated the study of the tensor product immersion of two immersions of a given Riemannian manifold (see [3]). Inspired by Chen's definition, F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken (in [4]) studied the tensor product of two immersions of, in general, different manifolds; under certain conditions, this realizes an immersion of the product manifold. In [6] tensor product surfaces of Euclidean plane curves were investigated.
In the present paper, we deal with tensor product surfaces of a Euclidean space curve and a Euclidean plane curve. We classify the minimal, totally real and slant such surfaces, respectively.

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1. Tensor product immersions

Recall definitions and results of [3]. Let M and N be two differentiable manifolds and

$$f : M \to \mathbb{E}^m,$$

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$$h : N \to \mathbb{E}^n,$$

two immersions. The direct sum and tensor product maps

$$f \oplus h \colon M \times N \to \mathbb{E}^{m+n},$$
$$f \otimes h \colon M \times N \to \mathbb{E}^{mn}$$

are defined by

$$(f \oplus h)(p, q) = (f(p), h(q)),$$
$$(f \otimes h)(p, q) = f(p) \otimes h(q).$$

Necessary and sufficient conditions for $f \otimes h$ to be an immersion were obtained in [4] . It is also proved there that the pairing $(\oplus, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some subsemirings were studied in [5] by F. Decruyenaere, F. Dillen, L. Verstraelen and one of the present authors.

For many immersions f, h which are not transversal, the tensor product $f \otimes h$ is still worthwhile to be investigated and in many cases still produces an immersion. As such, in the following sections, we will consider the tensor product immersions, actually surfaces in $\mathbb{E}^6$, which are obtained from a Euclidean space curve and a Euclidean plane curve.

2. Minimal tensor product surfaces

Let $c_1 : \mathbb{R} \to \mathbb{E}^3$ and $c_2 : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean curves. Put $c_1(t) = (\alpha(t), \beta(t), \gamma(t))$ and $c_2(s) = (a(s), b(s))$. Then their tensor product is given by

$$f = c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}^6$$
$$f(t, s) = (\alpha(t)a(s), \alpha(t)b(s), \beta(t)a(s), \beta(t)b(s), \gamma(t)a(s), \gamma(t)b(s)).$$

We have

$$\frac{\partial f}{\partial t} = (\dot{\alpha}(t)a(s), \dot{\alpha}(t)b(s), \dot{\beta}(t)a(s), \dot{\beta}(t)b(s), \dot{\gamma}(t)a(s), \dot{\gamma}(t)b(s)),$$
$$\frac{\partial f}{\partial s} = (\alpha(t)\dot{a}(s), \alpha(t)\dot{b}(s), \beta(t)\dot{a}(s), \beta(t)\dot{b}(s), \gamma(t)\dot{a}(s), \gamma(t)\dot{b}(s)),$$

where $\dot{a}$ means the derivative of a.

The coefficients of the Riemannian metric g induced on Imf by the Euclidean metric of $\mathbb{E}^6$ are

$$g_{11} = \|\dot{c}_1\|^2 \|c_2\|^2,$$
$$g_{12} = <c_1, \dot{c}_1><c_2, \dot{c}_2>,$$
$$g_{22} = \|c_1\|^2 \|\dot{c}_2\|^2.$$

An orthonormal basis on $Im\, c_1 \otimes c_2$ is given by

$$e_1 = \frac{1}{\|\dot{c}_1\| \|c_2\|} \frac{\partial f}{\partial t},$$

$$e_2 = \frac{1}{\|\dot{c}_1\|\|c_2\|\sqrt{\|\dot{c}_1\|^2\|c_1\|^2\|\dot{c}_2\|^2\|c_2\|^2 - <c_1,\dot{c}_1>^2<c_2,\dot{c}_2>^2}} \times$$
$$[\|\dot{c}_1\|^2\|c_2\|^2\frac{\partial f}{\partial s} - <c_1,\dot{c}_1><c_2,\dot{c}_2>\frac{\partial f}{\partial t}].$$

The normal space is spanned by

$$\begin{aligned}
n_1 &= (-\beta(t)b(s), \beta(t)a(s), \alpha(t)b(s), -\alpha(t)a(s), 0, 0),\\
n_2 &= (0, 0 - \gamma(t)b(s), \gamma(t)a(s), \beta(t)b(s), -\beta(t)a(s)),\\
n_3 &= (-\dot{\beta}(t)\dot{b}(s), \dot{\beta}(t)\dot{a}(s), \dot{\alpha}(t)\dot{b}(s), -\dot{\alpha}(t)\dot{a}(s), 0, 0),\\
n_4 &= (0, 0 - \dot{\gamma}(t)\dot{b}(s), \dot{\gamma}(t)\dot{a}(s), \dot{\beta}(t)\dot{b}(s), -\dot{\beta}(t)\dot{a}(s)),
\end{aligned}$$

Recall that a submanifold of a Riemannian manifold is said to be *minimal* if its mean curvature vector H vanishes identically (see, for instance, Chen [1]).

In the case under consideration, Imf is minimal if and only if

$$h(e_1, e_1) + h(e_2, e_2) = 0,$$

where h denotes the second fundamental form of f, or equivalently

$$<h(e_1, e_1) + h(e_2, e_2), n_i> = 0, \quad i \in \{1, 2, 3, 4\}.$$

A straightforward calculation leads to

$$<g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2} - 2g_{12}\frac{\partial^2 f}{\partial t \partial s}, n_i> = 0, \quad i \in \{1, 2, 3, 4\}. \tag{1}$$

We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial t^2} &= (\ddot{\alpha}(t)a(s), \ddot{\alpha}(t)b(s), \ddot{\beta}(t)a(s), \ddot{\beta}(t)b(s), \ddot{\gamma}(t)a(s), \ddot{\gamma}(t)b(s)),\\
\frac{\partial^2 f}{\partial s^2} &= (\alpha(t)\ddot{a}(s), \alpha(t)\ddot{b}(s), \beta(t)\ddot{a}(s), \beta(t)\ddot{b}(s), \gamma(t)\ddot{a}(s), \gamma(t)\ddot{b}(s)),\\
\frac{\partial^2 f}{\partial t \partial s} &= (\dot{\alpha}(t)\dot{a}(s), \dot{\alpha}(t)\dot{b}(s), \dot{\beta}(t)\dot{a}(s), \dot{\beta}(t)\dot{b}(s), \dot{\gamma}(t)\dot{a}(s), \dot{\gamma}(t)\dot{b}(s)).
\end{aligned}$$

Since $<\frac{\partial^2 f}{\partial t^2}, n_i> = <\frac{\partial^2 f}{\partial s^2}, n_i> = 0$, $i = 1, 2$, (1) implies

$$g_{12}<\frac{\partial^2 f}{\partial t \partial s}, n_1> = g_{12}<\frac{\partial^2 f}{\partial t \partial s}, n_2> = 0.$$

We distinguish two cases:

a) $g_{12} = 0$; in this case c_1 is spherical or c_2 is a circle centered at the origin.

b) $<\frac{\partial^2 f}{\partial t \partial s}, n_1> = <\frac{\partial^2 f}{\partial t \partial s}, n_2> = 0$, which is equivalent to

$$\begin{aligned}
(\dot{a}(s)b(s) - a(s)\dot{b}(s))(\dot{\beta}(t)\alpha(t) - \dot{\alpha}(t)\beta(t)) &= 0,\\
(\dot{a}(s)b(s) - a(s)\dot{b}(s))(\dot{\gamma}(t)\beta(t) - \gamma(t)\dot{\beta}(t)) &= 0.
\end{aligned}$$

We have two subcases:

b$_1$) $\dot{a}(s)b(s) - a(s)\dot{b}(s) = 0$, i.e. c_2 is a portion of a straight line passing through the origin;

b$_2$) $\dot{\beta}(t)\alpha(t) - \dot{\alpha}(t)\beta(t) = 0$ and $\dot{\gamma}(t)\beta(t) - \gamma(t)\dot{\beta}(t) = 0$, i.e. c_1 is a portion of a straight line passing through the origin.

Also the case a) has two subcases:

a$_1$) c_2 is a circle centered at the origin. Then $c_2(s) = (\cos s, \sin s)$. Using equation (1) for $i = 3, 4$, we get

$$<g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_3> = <g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_4> = 0,$$

or equivalently

$$\|c_1\|^2 <\frac{\partial^2 f}{\partial t^2}, n_3> + \|\dot{c}_1\|^2 <\frac{\partial^2 f}{\partial s^2}, n_3> = 0, \tag{2}$$

$$\|c_1\|^2 <\frac{\partial^2 f}{\partial t^2}, n_4> + \|c_1\|^2 <\frac{\partial^2 f}{\partial s^2}, n_4> = 0. \tag{3}$$

We may choose t such that $\|c_1\| = \|\dot{c}_1\|$. Then the last equations become

$$\dot{\beta}(t)(\ddot{\alpha}(t) - \alpha(t)) - \dot{\alpha}(t)(\ddot{\beta}(t) - \beta(t)) = 0, \tag{4}$$

$$\dot{\gamma}(t)(\ddot{\beta}(t) - \beta(t)) - \dot{\beta}(t)(\ddot{\gamma}(t) - \gamma(t)) = 0. \tag{5}$$

By $\|c_1\| = \|\dot{c}_1\|$, one has

$$\dot{\alpha}(t)(\ddot{\alpha}(t) - \alpha(t)) + \dot{\beta}(t)(\ddot{\beta}(t) - \beta(t)) + \dot{\gamma}(t)(\ddot{\gamma}(t) - \gamma(t)) = 0. \tag{6}$$

Consider the system (4)–(6). We have two subsubcases:

a$_{11}$) $\dot{\beta}(t) = 0 \Longrightarrow \beta(t) = 0 \Longrightarrow c_1$ is an orthogonal hyperbola in the plane $x^2 = 0$.

a$_{12}$) If all the components of c_1 are not constant, then by (4)–(6), it follows that

$$\ddot{\alpha}(t) = \alpha(t),$$
$$\ddot{\beta}(t) = \beta(t),$$
$$\ddot{\gamma}(t) = \gamma(t).$$

Then

$$\alpha(t) = \lambda_1 \cosh(t + \mu_1),$$
$$\beta(t) = \lambda_2 \cosh(t + \mu_2),$$
$$\gamma(t) = \lambda_3 \cosh(t + \mu_3).$$

a$_2$) c_1 is spherical, then $\alpha^2 + \beta^2 + \gamma^2 = 1$. Also we may assume c_1 is parametrized by arc length, i.e. $\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 = 1$.

Let $c_2(s) = \rho(s)(\cos s, \sin s)$; then $a(s) = \rho(s)\cos s$, $b(s) = \rho(s)\sin s$. One has

$$\dot{a}(s) = \dot{\rho}(s)\cos s - \rho(s)\sin s, \; \dot{b}(s) = \dot{\rho}(s)\sin s + \rho(s)\cos s,$$

$$\ddot{a}(s) = \ddot{\rho}(s)\cos s - 2\dot{\rho}(s)\sin s - \rho(s)\cos s,$$
$$\ddot{b}(s) = \ddot{\rho}(s)\sin s + 2\dot{\rho}(s)\cos s - \rho(s)\sin s.$$

We have

$$\frac{\partial^2 f}{\partial t^2} = (\ddot{\alpha}(t)a(s), \ddot{\alpha}(t)b(s), \ddot{\beta}(t)a(s), \ddot{\beta}(t)b(s), \ddot{\gamma}(t)a(s), \ddot{\gamma}(t)b(s)),$$
$$\frac{\partial^2 f}{\partial s^2} = (\alpha(t)\ddot{a}(s), \alpha(t)\ddot{b}(s), \beta(t)\ddot{a}(s), \beta(t)\ddot{b}(s), \gamma(t)\ddot{a}(s), \gamma(t)\ddot{b}(s)).$$

Using equations (2) and (3), we get

$$(\rho^2 + \dot{\rho}^2)<\frac{\partial^2 f}{\partial t^2}, n_3> + \rho^2<\frac{\partial^2 f}{\partial s^2}, n_3> = 0, \tag{7}$$
$$(\rho^2 + \dot{\rho}^2)<\frac{\partial^2 f}{\partial t^2}, n_4> + \rho^2<\frac{\partial^2 f}{\partial s^2}, n_4> = 0. \tag{8}$$

The equation (7) becomes

$$(\rho^2 + \dot{\rho}^2)(\ddot{\alpha}(t)\dot{\beta}(t) - \dot{\alpha}(t)\ddot{\beta}(t))(\dot{a}(s)b(s) - a(s)\dot{b}(s))$$
$$+\rho^2(\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))(\dot{a}(s)\ddot{b}(s) - \ddot{a}(s)\dot{b}(s)) = 0,$$

which leads to

$$-(\rho^2 + \dot{\rho}^2)(\ddot{\alpha}(t)\dot{\beta}(t) - \dot{\alpha}(t)\ddot{\beta}(t)) + (\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t))(2\dot{\rho}^2 - \rho\ddot{\rho} + \rho^2) = 0,$$

or equivalently

$$\frac{2\dot{\rho}^2 - \rho\ddot{\rho} + \rho^2}{\rho^2 + \dot{\rho}^2} = \frac{\ddot{\alpha}(t)\dot{\beta}(t) - \dot{\alpha}(t)\ddot{\beta}(t)}{\alpha(t)\dot{\beta}(t) - \dot{\alpha}(t)\beta(t)}. \tag{9}$$

The left hand term is a function of s and right hand term is a function of t, then both should be a constant, say k. Similarly from (8) we find

$$\frac{\ddot{\beta}(t)\dot{\gamma}(t) - \dot{\beta}(t)\ddot{\gamma}(t)}{\beta(t)\dot{\gamma}(t) - \dot{\beta}(t)\gamma(t)} = k. \tag{10}$$

If c_1 has a constant component, which must be 0, then c_1 is a portion of a circle. In this case c_2 is an orthogonal hyperbola (see also [6]).

Otherwise from the equation (9) and (10) we get

$$\frac{\ddot{\alpha}(t) - k\alpha(t)}{\dot{\alpha}(t)} = \frac{\ddot{\beta}(t) - k\beta(t)}{\dot{\beta}(t)} = \frac{\ddot{\gamma}(t) - k\gamma(t)}{\dot{\gamma}(t)} = m(t). \tag{11}$$

Since c_1 is parametrized by arc length, we have $\dot{\alpha}(t)\ddot{\alpha}(t) + \dot{\beta}(t)\ddot{\beta}(t) + \dot{\gamma}(t)\ddot{\gamma}(t) = 0$. Substituting (11) in the last relation, we obtain $m(t) = 0$. Then we get $\ddot{\alpha}(t) = k\alpha(t)$, $\ddot{\beta}(t) = k\beta(t)$, $\ddot{\gamma}(t) = k\gamma(t)$.

c_1 being spherical, we have $k<0$. We put $k=-l^2$, $l>0$. Finally we find

$$\begin{aligned}\alpha(t) &= \varepsilon_1 \cos lt + \eta_1 \sin lt,\\ \beta(t) &= \varepsilon_2 \cos lt + \eta_2 \sin lt,\\ \gamma(t) &= \varepsilon_3 \cos lt + \eta_3 \sin lt.\end{aligned}$$

satisfying $\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2=\eta_1^2+\eta_2^2+\eta_3^2=1$ and $\varepsilon_1\eta_1+\varepsilon_2\eta_2+\varepsilon_3\eta_3=0$.

Also from equation (9) one gets

$$\rho\ddot{\rho}-\dot{\rho}^2=(1+l^2)(\rho^2+\dot{\rho}^2). \tag{12}$$

Putting $w=\frac{\dot{\rho}}{\rho}$, from (12) one obtains $w=\tan[(1+l^2)s+l_1]$, which implies

$$\rho(s)=\frac{l_2}{[\cos(1+l^2)s+l_1]^{\frac{1}{1+l^2}}}\ ,$$

where l_1, l_2 are constant. Thus c_2 is sinusoidal spiral. In particular for $l=1$, c_2 is an orthogonal hyperbola.

Conversely, it is easily seen that in all the above discussed cases, the tensor product immersion $c_1 \otimes c_2$ is minimal.

Summing up, the following theorem is proved.

Theorem 2.1. *The tensor product immersion $c_1 \otimes c_2$ of a Euclidean space curve and a Euclidean plane curve is a minimal surface in $\mathbb{E}^6$ if and only if either*

i) *c_1 is a straight line through 0;*

ii) *c_2 is a straight line through 0;*

iii) *c_1 is a circle centered at 0 and c_2 is an orthogonal hyperbola centered at 0;*

iv) *c_1 is an orthogonal hyperbola centered at 0 and c_2 is a circle centered at 0;*

v) *c_2 is a circle centered at 0 and c_1 is given by*

$$c_1(t)=(\lambda_1\cosh(t+\mu_1),\lambda_2\cosh(t+\mu_2),\lambda_3\cosh(t+\mu_3));$$

vi) *c_1 is given by*

$$c_1(t)=(\varepsilon_1\cos lt+\eta_1\sin lt,\varepsilon_2\cos lt+\eta_2\sin lt,\varepsilon_3\cos lt+\eta_3\sin lt),$$

where $\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2=\eta_1^2+\eta_2^2+\eta_3^2=1$ and $\varepsilon_1\eta_1+\varepsilon_2\eta_2+\varepsilon_3\eta_3=0$, and

$$c_2(s)=\frac{l_2}{[\cos(1+l^2)s+l_1]^{\frac{1}{1+l^2}}}(\cos s,\sin s),$$

with l_1, l_2 =constant.

3. Totally real and slant tensor product surfaces

Let $c_1 : \mathbb{R} \to \mathbb{E}^3$ and $c_2 : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean curves and $f = c_1 \otimes c_2$ their tensor product.

We identify $\mathbb{E}^6$ with $\mathbb{C}^3$ and consider the standard *complex structure* J given by

$$J(y^1, \dots, y^6) = (-y^2, y^1, -y^4, y^3, -y^6, y^5), \; y^1, \dots, y^6 \in \mathbb{R}.$$

Then Imf is a real 2-dimensional submanifold of $\mathbb{C}^3$, which is *totally real*, i.e. the complex structure J of $\mathbb{E}^6$ at each point transforms the tangent space to the surface into the normal space, according to the following result.

Theorem 3.1. *The tensor product immersion $c_1 \otimes c_2$ of a Euclidean space curve and a Euclidean plane curve is totally real in $(\mathbb{C}^3, J)$ if and only if c_1 is spherical or c_2 is a portion of a straight line passing through* 0.

Proof. Imf is a totally real surface if and only if $J(\frac{\partial f}{\partial t})$ is orthogonal to $\frac{\partial f}{\partial s}$ and $J(\frac{\partial f}{\partial s})$ is orthogonal to $\frac{\partial f}{\partial t}$. We have

$$J(\frac{\partial f}{\partial t}) = (-\dot{\alpha}(t)b(s), \dot{\alpha}(t)a(s), -\dot{\beta}(t)b(s), \dot{\beta}(t)a(s), -\dot{\gamma}(t)b(s), \dot{\gamma}(t)a(s)),$$

where $\dot{\alpha}$ means the derivative of α.

By a straightforward calculation, we obtain

$$<J(\frac{\partial f}{\partial t}), \frac{\partial f}{\partial s}> = -<J(\frac{\partial f}{\partial s}), \frac{\partial f}{\partial t}> = 0$$

if and only if

$$\alpha(t)\dot{\alpha}(t) + \beta(t)\dot{\beta}(t) + \gamma(t)\dot{\gamma}(t) = 0$$

or

$$a(s)\dot{b}(s) - b(s)\dot{a}(s) = 0.$$

Integrating these equations, we find that c_1 is spherical or c_2 is a portion of a straight line which contains 0, respectively.

Recall the definition of a slant surface in $(\mathbb{C}^3, J)$ (see [2]). Let M be a surface in $(\mathbb{C}^3, J)$. For a given orthonormal basis $\{e_1, e_2\}$ of T_xM $(x \in M)$, we put

$$\theta(T_xM) = \arccos <Je_1, e_2>,$$

which is independent of the choice of $\{e_1, e_2\}$. M is said to be *slant* if $\theta(T_xM)$ is constant along M. Totally real and complex surfaces are *improper slant* surfaces, with slant angles $\theta = \frac{\pi}{2}$ and $\theta = 0$, respectively.

Let $c_1 : \mathbb{R} \to \mathbb{E}^3$, $c_2 : \mathbb{R} \to \mathbb{E}^2$ be two Euclidean curves. From Theorem 3.1, we know that if c_2 is a portion of a straight line containing 0, $c_1 \otimes c_2$ is an improper slant surface. Otherwise, we consider polar coordinates on c_2. Then

$$c_2(s) = \rho_2(s)(\cos s, \sin s).$$

A straightforward computation leads to

$$<Je_1, e_2> = \frac{[a(s)\dot{b}(s) - b(s)\dot{a}(s)][\alpha(t)\dot{\alpha}(t) + \beta(t)\dot{\beta}(t) + \gamma(t)\dot{\gamma}(t)]}{\sqrt{\|\dot{c}_1\|^2\|c_1\|^2\|\dot{c}_2\|^2\|c_2\|^2 - <c_1, \dot{c}_1>^2<c_2, \dot{c}_2>^2}}.$$

Let $A(t) = \alpha^2(t) + \beta^2(t) + \gamma^2(t)$, $B = \frac{\dot{\alpha}}{\alpha}$ and $R = \frac{\dot{\rho}}{\rho}$. Then

$$\cos\theta = \frac{1}{\sqrt{\rho^2(\rho^2 + \dot{\rho}^2)A^2 - \rho^2\dot{\rho}^2(\frac{\dot{A}}{2})^2}}\rho^2\frac{\dot{A}}{2}.$$

Therefore Imf is a slant surface if and only if

$$B = \text{constant and } R = \text{constant},$$

or equivalently

$$\begin{aligned} A(t) &= k_1 e^{l_1 t}, \\ \rho(s) &= k_2 e^{l_2 s}. \end{aligned}$$

We proved the following

Theorem 3.2. *The tensor product immersion $c_1 \otimes c_2$ of a Euclidean space curve and a Euclidean plane curve is a proper slant surface if and only if c_2 is a logarithmic spiral curve or a circle and c_1 satisfies $\alpha^2(t) + \beta^2(t) + \gamma^2(t) = k_1 e^{l_1 t}$, for all $t \in \mathbb{R}$.*

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