Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 36 (1995), No. 2, 185-202.

The Generalized Favard Problem

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Abstract. The present paper deals with a geometrical optimization problem for isoperimetric plane sets contained in a circular sector. For a circle, this problem was solved by J. M. Favard. To obtain structural statements methods of distributional optimal control and constrained optimization are used. With respect to sector angles and perimeter length the optimal value function is discussed and the complete solution is founded.

0. Introduction

In 1929 J. M. Favard has solved the following geometrical extremal problem [6]:

> What is the plane convex set with circumradius R and given area F having maximal perimeter length L?

Favard has shown that the solution must be an inpolyeder of the circumcircle. Moreover, all but one side of the optimal polyeder are equal. This side may be smaller than the other ones. To prove this he has used geometrical methods and investigations of the optimal value function. Favard called this function "une fonction assez compliquée" and meant "... le calcul peut être penible". We follow Favard and denote this function *fonction penetrante.* It plays an important part in our investigations.

Our aim is to discuss some generalizations of this problem. In a later paper, they will enable us to solve a long-standing open problem for plane convex sets which was mentioned in the well-known survey of Bonnesen and Fenchel in 1934 [2].

The above mentioned geometrical problem can be formulated equivalently as an isoperimetric problem for plane convex sets with circumradius R where the area is to minimize. In the present paper we generalize this problem substituting the circumcircle by a sector and solve this problem in three steps. First we use the description of a plane convex and compact set by means of his support function introduced by Minkowski [15] and Blaschke [1]. Thus, we can formulate the geometrical problem as a problem of optimal control. The control appears as a distribution, however. An important tool for solving such problems is Pontryagin's maximum principle. R. Klötzler [10] has posed and proved a generalization of this necessary optimality condition to problems with distributional control.

0138-4821/93 \$ 2.50

In the last years several geometrical problems are successfully discussed by this approach, see for instance [11], [12]. For our problem we get structural statements about the optimal figure. It has to be composed of a partial sector and a polyeder.

Then we formulate a suitable nonlinear optimization problem for admissible polyeders and describe its solution with respect to the given parameters. The corresponding optimal value function depends on the angle of the sector and the perimeter length. This generalized *fonction penetrante* is nonconvex and nonsmooth. Some of its analytical properties will be summarized.

Finally, the use of this function makes it possible to solve the generalized Favard problem. The optimal figure is an almost regular inpolyeder of the sector.

1. Formulation of the problem

To formulate the problem first we give some definitions. Let be $\mathcal{S}_\omega$ the sector with radius R and angle $\omega \in [0, 2\pi]$. By Δ_ω we denote the corresponding isosceles triangle. Without loss of generality we put $R = 1$. We have to distinguish two cases (see fig. 1,2):

1. $\omega \in [0, \pi] : \Delta_\omega \subseteq \mathcal{S}_\omega$ and $\mathcal{S}_\omega$ is a convex set.
2. $\omega \in (\pi, 2\pi] : \Delta_\omega \cup \mathcal{S}_\omega$ is a convex set.

figure 1 figure 2

Definition 1. Let be $\mathcal{B}$ a closed plane set with $\mathcal{S}_\omega \cap \Delta_\omega \subseteq \mathcal{B} \subseteq \mathcal{S}_\omega$ and $\Delta_\omega \cup \mathcal{B}$ convex.Then $\mathcal{B}$ is called an ω-sector-indomain of $\mathcal{S}_\omega$.

For $\omega \in [0, 2\pi)$ the boundary $\partial\mathcal{B}$ consists of two parts. The legs of Δ_ω are the fixed part belonging to the boundary of every ω-sector-indomain of $\mathcal{S}_\omega$. The remaining part differs for different domains $\mathcal{B}$. We denote this *variable boundary curve* by b. For $\omega = 2\pi$ is $b = \partial\mathcal{B}$. A *polyhedral ω-sector-indomain* we denote this polyeder by $\mathcal{P}_\omega$. In this case, b is a polygon and denoted by p_ω. The length of the variable boundary curve b we denote by L. For $\omega \in [0, \pi]$ we have

$$2 \sin \frac{\omega}{2} \leq L \leq \omega.$$

If $\omega \in (\pi, 2\pi]$ we demand

$$2(1-\cos\frac{\omega}{2}) \leq L \leq \omega.$$

Definition 2. A polyhedral ω-sector-indomain $\mathcal{P}_\omega$ is called ω-inpolyeder of $\mathcal{S}_\omega$ if all edges of $\mathcal{P}_\omega$ except the vertex of $\mathcal{S}_\omega$ belong to the sector arc. The corresponding polygon p_ω is called ω-inpolygon of $\mathcal{S}_\omega$.

Definition 3. An ω-inpolygon p_ω of $\mathcal{S}_\omega$ is called almost regular if all but one side of p_ω are equal, this one may be smaller. The polygon p_ω is called regular if all sides are equal. The corresponding ω-inpolyeder $\mathcal{P}_\omega$ is called almost regular and regular, respectively.

Now, we pose the following generalized Favard problem:
What is the minimal possible area of an ω-sector-indomain with given perimeter length? Which set is an optimal one?
This is a parametric optimization problem with the parameters ω and L. In the case $\omega = 2\pi$ we get Favard's problem. For $L \in [4, 2\pi]$ the optimal indomain $\mathcal{B}$ is an inpolyeder of the unit circle.

The existence of the solution follows from the selection theorem of Blaschke [1] for a set of uniformly bounded convex sets because the area depends continuously on the convex sets.

2. Structural analysis

This section is concerned with the formulation of the geometrical optimization problem as a problem of optimal control and the investigation of corresponding neccessary optimality conditions. Our aim is to find structural statements about an optimal ω-sector-indomain.

2.1. The control problem with distributional control

Every nonempty closed convex set $\mathcal{C} \subseteq E^2$ can be descripted by his support function. And conversely, each such function represents a convex set (see [16]). Over the unit ball, this function has the representation

$$h(\varphi) := \operatorname{Max}\left\{e(\varphi)^T c \mid c \in \mathcal{C}\right\}$$

as a function of the polar angle φ. In this, $e(\varphi)$ denotes the unit vector in the direction φ, $\varphi \in [0, 2\pi]$. For further investigations we put

$$h(\varphi) = 0 \quad \text{if} \quad \varphi \notin [0, 2\pi].$$

Then (see [2]), the area of $\mathcal{C}$ is given by

$$F = \frac{1}{2}\int_0^{2\pi}(h^2 - \dot{h}^2)d\varphi,$$

the perimeter by

$$L = \int_0^{2\pi} h d\varphi,$$

and the curvature radius by

$$u = \ddot{h} + h.$$

For ω-sector-indomains analogous formulas hold. We only have to substitute 2π by $\omega \in [0, 2\pi]$.

If the boundary curve b does not contain any straight line then the second derivative of the support function h is summable over $(0, \omega)$. Thus, $\ddot{h}$ belongs to the space $L_1(0, \omega)$ and h to the Sobolev space $W_1^2(0, \omega)$ of real functions over $(0, \omega)$ (for instance, see [3] about Sobolev spaces). We get the following analytical description of our problem:

$$(\mathrm{P}_0^{\omega,\mathrm{L}}): \qquad F(h) = \frac{1}{2}\int_0^{\omega} (h^2 - \dot{h}^2) d\varphi \to \text{Min!}$$

$$\begin{aligned} \ddot{h} + h &= u \quad \text{a.e.} \\ h(\varphi) &\le 1 \quad \text{for all } \varphi \in [0, \omega] \\ u(\varphi) &\ge 0 \quad \text{for all } \varphi \in [0, \omega] \\ \int_0^{\omega} h d\varphi &= L \\ h(0) &= h(\omega) = 1 \\ \dot{h}(0) &= \dot{h}(\omega) = 0. \end{aligned}$$

The support function of a sector-domain with polygonal boundary structure has not a second derivative over $(0, \omega)$. There exists only the first derivative which is essentially bounded. Hence, $\dot{h}$ belongs to the space $L_\infty(0, \omega)$ and h to the Sobolev space $W_\infty^1(0, \omega)$. The second derivative $\ddot{h}$ can be understood in the sense of distributions, only.

Referring to the theory of distributions by L. Schwartz [17] and I. M. Gelfand, G. E. Schilow [9] the distribution is a linear functional over a basic space K. Let be $K = W_{\infty,loc}^1$ the space of real functions $\Psi \in W_\infty^1(R^1)$ having compact support (and therefore $\dot{\Psi} \in L_{\infty,loc}$). We denote $K_+ = \{\Psi \in K \ / \ \Psi \ge 0\}$, $D(K)$ the space of distributions over K and $D_0(K) \subseteq D(K)$ the set of distributions from $D(K)$ with support in $[0, \omega]$.

Then, we explain $\ddot{h}$ by

$$\langle \ddot{h}, \Psi \rangle := -\langle \dot{h}, \dot{\Psi} \rangle_{L_\infty} \ \text{ for all } \Psi \in K$$

and the curvature radius by

$$\langle u, \Psi \rangle := \langle h, \Psi \rangle - \langle \dot{h}, \dot{\Psi} \rangle_{L_\infty} \ \text{ for all } \Psi \in K$$

as distributions over K. In this, we denote by $\langle \cdot, \cdot \rangle$ the linear functionals over K and $\langle \dot{h}, \dot{\Psi} \rangle_{L_\infty} = \int_0^\omega \dot{h}(t)\dot{\Psi}(t)dt$ for $\dot{\Psi} \in L_{\infty,loc}$ remarking that $\dot{h}(t) = 0$ for $t \notin [0, \omega]$. The convexity condition $u \ge 0$ changes into

$$\langle u, \Psi \rangle \ge 0 \ \text{ for all } \Psi \in K_+.$$

To obtain a problem of optimal control in standard form we introduce new variables. We put

$$\begin{aligned} x_1(t) &:= h(t) \\ x_2(t) &:= \dot{h}(t) \\ x_3(t) &:= \int_0^t h(\varphi)d\varphi \end{aligned}$$

and get the following problem with a distributional control u:

$$(\mathrm{P}_1^{\omega,\mathrm{L}}): \qquad I(x,u) = \frac{1}{2}\int_0^{\omega}(x_1(t)^2 - x_2(t)^2)dt \to \mathrm{Min!}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ "\dot{x}_2 &= u - x_1" \\ \dot{x}_3 &= x_1 \\ x_1 &\leq 1 \\ "u &\geq 0" \end{aligned}$$

$$\begin{aligned} x_1(0) &= x_1(\omega) = 1 \\ x_2(0) &= x_2(\omega) = 0 \\ x_3(0) &= 0 \\ x_3(\omega) &= L. \end{aligned}$$

The conditions in quotation marks have to be understood in the sense of distributions, only.

Let be $(\bar{x}^*, \bar{u}^*)$ an optimal process of $(\mathrm{P}_1^{\omega,\mathrm{L}})$ with $L < \omega$ for which the set $\{t \in (0,\omega] \ / \ \bar{x}_1^*(t) = 1\}$ is not an intervall. Since $\bar{x}_1^*$ is a continuous function the set $\bar{N}_1 := \{t / \bar{x}_1^*(t) = 1\}$ is closed. Therefore, it can be represented by

$$\bar{N}_1 = [0,\omega] \setminus \cup_{i\in I}\,(a_i, b_i)$$

where the countable many intervalls (a_i, b_i) do not intersect, and $\bar{x}_1^*(a_i) = \bar{x}_1^*(b_i) = 1$ holds for all $i \in I$. Let be $\bar{N}_1' \subseteq \bar{N}_1$ those subset of $\bar{N}_1$ which is composed by closed intervalls $[b_i, a_{i+1}]$ for $i \in I$ with $b_i \neq a_{i+1}$. $\bar{N}_1'$ may be empty. Suitable change of these intervalls leads to another admissible process with the same value of the objective functional. The corresponding optimal process (x^*, u^*) has the property

$$N_1' = \begin{cases} \emptyset & \text{if } \bar{N}_1' = \emptyset, \\ [\omega', \omega] \text{ for some } \omega' \in (0,\omega) & \text{otherwise.} \end{cases} \tag{I}$$

Geometrically, this corresponds to an ω-sector-indomain containing at most one partial sector $S_{\omega-\omega'}$ of S_ω. Without loss of generality, we only regard such optimal processes.

2.2. Distributional extension of Pontryagin's maximum principle

For the following class of distributional control problems R. Klötzler [10] has proved a generalized maximum principle:

$$(\mathrm{P}): \qquad I(x,u) := \int_0^T [f_0(t,x(t)) + f_1(t)\cdot u(t)]\,dt \to \mathrm{Min!}$$

for $x \in L_2^{loc,n} \subseteq D(K^n)$, $u \in D_0(K^r)$, with respect to state equations

$$\text{"}\dot{x} = A(t,x) + B(t)u\text{"},$$

control restrictions

$$\text{"}Cu \leq b\text{"},$$

state restrictions

$$g_i(t,x(t)) \leq 0 \quad \text{for } i = 1,...,k, \quad \text{a.e. over } [0,T],$$

and linear boundary conditions

$$C_0(x(0)) = 0, \quad C_1(x(T)) = 0.$$

$D(K^n)$ and $D_0(K^r)$ are defined as in section 2.1., and the quotation marks characterize the relations in the sense of distributions. We denote by H the Pontryagin function of (P) and by $C_0'^T$, $C_1'^T$ the transposed Jacobians of C_0, C_1, respectively. The arising quantities have to satisfy the following smoothness assumptions:
1. $f_0 : [0,T] \times E^n \to E^1$, $A : [0,T] \times E^n \to E^n$, $g_i : [0,T] \times E^n \to E^1$ are continuous and, with respect to the second argument continuously differentiable.
2. $B : [0,T] \to E^{n \cdot r}$, $f_1 : [0,T] \to E^r$ are continuously differentiable, rank $B = r$.
3. For every sequence of admissible processes (x^j, u^j) of (P) with $x^j \to x$ in $L_2^n(0,T)$ the sequences $(f_0(\cdot, x^j(\cdot)))$, $(A(\cdot, x^j(\cdot)))$, $(f_{0x}(\cdot, x^j(\cdot)))$ and $(A_x(\cdot, x^j(\cdot)))$ converge weakly in $L_1(0,T)$. Each component x_k^j of x^j for which an i exists with $g_{ix_k} \neq 0$ converges in the sense of W_2^1.

Under the assumptions
(i) to each optimal process (x,u) of (P) there exists a sequence of regular admissible processes (x^j, u^j) with $x^j \in W_{2,loc}^{1,n}$, $u^j \in L^{r,loc}$ and $x^j \to x$ in $L_2^{n,loc}$, $u^j \to u$ in $D(K^r)$,
(ii) the functional I is weakly lower semicontinuous in $L_2^n \times L_2^r$ for all regular admissible processes of (P) with uniformly bounded control, the set of such processes may be nonempty,
Klötzler's distributional extension of Pontryagin's maximum principle holds:

Theorem 1: *If (x,u) is an optimal process of* (P) *then there exists a nonnegative number λ_0, vectors $l^0 \in E^{s_0}$, $l^1 \in E^{s_1}$, a left-continuous vector valued function $y : [0,T] \to E^n$ of bounded variation, and nonnegative regular measures μ_i $(i = 1,...,k)$ on $[0,T]$ which are* *concentrated on* $\{t \in [0,T] \,/\, g_i(t,x(t)) = 0\}$, *such that these so introduced elements do not vanish simultanously and the following necessary optimality conditions are satisfied:*

$$\max_{\substack{v\in L^r_\infty(0,T)\cup D_0(K^r)\\ Cv\le b}} \langle H(\cdot,x(\cdot),v(\cdot),y(\cdot),\lambda_0),\Psi\rangle = \langle H(\cdot,x(\cdot),u(\cdot),y(\cdot),\lambda_0),\Psi\rangle \quad \text{for all } \Psi\in K^1_+ \tag{M}$$

$$\begin{aligned} y(t) = -C_1^{'T}(x(T))l^1 &+ \int_t^T H_\xi(\tau,x(\tau),u(\tau),y(\tau),\lambda_0)d\tau \\ &- \sum_{i=1}^k \int_t^T g_{i\xi}(\tau,x(\tau))\mu_i(d\tau) \end{aligned} \tag{C}$$

$$y(0) = C_0^{'T}(x(0))l^0. \tag{T}$$

Now, we want to apply this theorem to our geometrical problem $(\mathrm{P}_1^{\omega,\mathrm{L}})$.

2.3. Use of Pontryagin's maximum principle

To verify the assumption (i) of theorem 1 for our problem $(\mathrm{P}_1^{\omega,\mathrm{L}})$ we use an approximation property of convex sets. In sense of the Hausdorff metric a convex set can be approximated by strongly convex sets [14]. This property is carried over to the support function and guarantees the existence of regular processes (x^j,u^j) in any vicinity of an optimal process (x,u). Assumption (ii) is fulfilled, also. Because of the simple structure of the problem the required smoothness conditions are true.

Now, we apply the maximum principle to $(\mathrm{P}_1^{\omega,\mathrm{L}})$. The Pontryagin function is given by

$$H(t,\xi,v,\eta,\lambda_0) = -\frac{1}{2}\lambda_0(\xi_1^2-\xi_2^2) + \eta_1\xi_2 + \eta_2(-\xi_1+v) + y_3\xi_1.$$

We study the case $\lambda_0 = 1$. In the case $\lambda_0 = 0$ we get the same structural statements.

Let be (x,u) an optimal process of $(\mathrm{P}_1^{\omega,\mathrm{L}})$ satisfying property (I). Then there exist vectors $l^0,l^1 \in E^3$, a left-continuous vector valued function $y:[0,\omega]\to E^3$ of bounded variation, and a nonnegative regular measure μ on $[0,\omega]$ which is concentrated on the set $\{t\in[0,\omega] \,/\, x_1(t)=1\}$ such that the following conditions hold:
the maximum condition

$$\max_{\substack{v\ge 0\\ v\in L_\infty(0,\omega)\cap D_0(K)}} \langle v\cdot y_2,\Psi\rangle = \langle u\cdot y_2,\Psi\rangle \quad \text{for all } \Psi\in K_+ \tag{M$_1$}$$

the canonical equations

$$\begin{aligned} y_1(t) &= -l_1^1 + \int_t^\omega(-x_1(\tau)-y_2(\tau)+y_3(\tau))d\tau - \int_t^\omega \mu(d\tau) \\ y_2(t) &= -l_2^1 + \int_t^\omega (x_2(\tau)+y_1(\tau))d\tau \\ y_3(t) &= -l_3^1 \end{aligned} \tag{C$_1$}$$

and the transversality conditions

$$\begin{aligned} y_1(0) &= l_1^0 \\ y_2(0) &= l_2^0 \,. \\ y_3(0) &= l_3^0 \end{aligned} \tag{T$_1$}$$

First, we study the maximum condition (M). The maximum exists only if

$$y_2(t) \le 0 \qquad \text{for all } \ t \in [0, \omega] \tag{2.1}$$

It is attained for an u^* which satisfies the complementarity condition

$$\langle u \cdot y_2, \Psi \rangle = 0 \qquad \text{for all } \ \Psi \in K_+. \tag{2.2}$$

The product has sense because y_2 belongs to K. Relation (2.2) means that only zeros of y_2 may be essential points of the distribution u:

$$\operatorname{supp} u \subseteq \{t \in [0, \omega] \ / \ y_2(t) = 0\} \tag{2.3}$$

From the complementarity condition (2.2) and the state equation $\dot{x} = u - x_1$ we get

$$\langle x_1 \cdot y_2, \Psi \rangle + \langle \dot{x}_2 \cdot y_2, \Psi \rangle = 0 \qquad \text{for all } \ \Psi \in K_+.$$

The second expression can be transformed by partial integration

$$\langle (x_2 \cdot y_2), \Psi \rangle = -\langle x_2 \cdot y_2, \dot{\Psi} \rangle.$$

This leads to

$$\langle x_1 \cdot y_2 - x_2 \cdot \dot{y}_2, \Psi \rangle - \langle x_2 \cdot y_2, \dot{\Psi} \rangle = 0 \qquad \text{for all } \ \Psi \in K_+. \tag{2.4}$$

Choosing in this expression $\Psi_i \in K_+$ according

$$\Psi_i(t) = \begin{cases} 1 - i \cdot |t - t_0| & \text{for } t \in [t_0 - \frac{1}{i}, t_0 + \frac{1}{i}] \\ 0 & \text{otherwise} \end{cases}$$

for any $t_0 \in (0, \omega)$ we obtain for $i \to \infty$

$$[x_2(t_0 - 0) - x_2(t_0 + 0)] \, y_2(t_0) = 0. \tag{2.5}$$

Here we have used the continuity of y_2. Likewise it follows from the control restriction $u \ge 0$ and the state equation $\dot{x}_2 = u - x_1$ that

$$x_2(t_0 - 0) \le x_2(t_0 + 0). \tag{2.6}$$

Now, we study the canonical equations (C$_1$). Since μ is a nonnegative measure we have $\mu([t_0, \omega]) \ge \mu((t_0, \omega)) \ge 0$ and consequently

$$y_1(t_0) \le y_1(t_0 + 0) \tag{2.7}$$

for the left-continuous function y_1. The absolutely continuous function y_2 satisfies the canonical differential equation

$$\dot{y}_2 = -x_2 - y_1 \qquad \text{a.e. over } [0,\omega]. \tag{2.8}$$

From (2.6) and (2.7) we get for $t_0 \in [0,\omega]$

$$\dot{y}_2(t_0+0) \leq \dot{y}_2(t_0-0). \tag{2.9}$$

At last, y_3 is constant over $[0,\omega]$.

With respect to the state restriction $x_1(t) \leq 1$ we distinguish *active* ($x(t) = 1$) and *non-active* ($x(t) < 1$) *parts* of the solution. For non-active parts, and even more for any intervall (a,b) with $(a,b) \cap \operatorname{supp} \mu = \emptyset$ the function y_1 satisfies the canonical differential equation

$$\dot{y}_1 = x_1 + y_2 - y_3 \qquad \text{over } (a,b). \tag{2.10}$$

We see that $\dot{y}_1$ is continuous and differentiable a.e. over such an intervall (a,b). Because of (2.8), (2.10) and the state equations y_2 satisfies the distributional differential equation

$$\langle \ddot{y}_2 + y_2 - y_3 + u, \Psi \rangle = 0 \tag{2.11}$$

for $\Psi \in K$ with $\operatorname{supp} \Psi \subseteq (a,b)$. From this discussion, the distinction of the following two cases results.

Case 1: $t_0 \in [0,\omega] \setminus \operatorname{supp} \mu$. (If the parameter L is less than ω then there exists such t_0.)
1. Let be $y_2(t_0) < 0$.
Since y_2 is a continuous function we have $y_2(t) < 0$ over an intervall I_0 containing t_0. From (2.2) follows $u = 0$ over I_0. This intervall corresponds to a vertex, geometrically. Equation (2.11) has the regular solution

$$y_2 = A\cos(t-\alpha) + y_3 \tag{2.12}$$

over I_0.

If this case arises then y_3 must be nonnegative. Otherwise, we would have $y_2(t) < 0$ and $u = 0$ over an intervall of length larger than π. This cannot occur, geometrically.
2. Let be $y_2(t_0) = 0$.
Since y_2 is a continuous function we have the following three cases for zeros of y_2.
2.1. $y_2(t) = 0$ over a closed intervall containing t_0.
From $\ddot{y}_2 = 0$ and (2.11) we get $u = y_3$ over this intervall. Since $u \geq 0$ this corresponds to an arc or a vertex, geometrically.
2.2. t_0 is an isolated zero of y_2.
Therefore $y_2(t) < 0$ in a neighbourhood $U(t_0) \setminus \{t_0\}$ of t_0. Here we know the structure of y_2 which is given piecewise by (2.12). Hence, the derivative $\dot{y}_2$ has a jump in t_0.
From the complementarity condition (2.2) follows $u = 0$ on $U(t_0) \setminus \{t_0\}$. Equation (2.11) leads to $u = d_0 \delta_{t_0}$ over $U(t_0)$ where d_0 is a positive number and δ_{t_0} is the δ-distribution concentrated in t_0. This case corresponds to a straight line limited by two vertices.

2.3. t_0 is an accumulation point of such zeros.

This discussion shows that only arcs, vertices and straight lines occur in any non-active part of the optimal solution. An arc cannot adjoin with a straight line (see case 2.2). Assuming an arc adjoins with a vertex in the inactive part, say on the intervall $(a, c) \subseteq [0, \omega]$ of the solution. Then, u is regular on (a, c) with $u = y_3 > 0$ on (a, b) (with resp. to the arc) and $u = 0$ on (b, c) (with resp. to the vertex). From (2.2) follows $y_2 = 0$ on $[a, b]$ and $y_2 < 0$ on (b, c) with the representation (2.12). We see that the derivative $\dot{y}_2$ is discontinuous in b. Otherwise, from the regularity of u on (a, c) follows the continuity of x_2 and final of $\dot{y}_1$ on (a, c) (see (2.10)). As a consequence of equation (2.8) $\dot{y}_2$ must be continuous on (a, c) which yields a contradiction.

Summarizing the results of case 1 we see that any non-active part of the solution must be polygonal possibly with accumulation points of vertices. From geometrical point of view it is evident that all vertices of the polygonal part must lie on the sector arc. Otherwise we can reduce the area by varying an non-active vertex on an ellipse without changing the perimeter length.

Case 2: $t_0 \in \text{supp}\mu$.
We have supp $\mu \subseteq \{t/x_1(t) = 1\}$.

Lemma. *The support of the measure μ consists at most of isolated points, their accumulation points and an intervall $[\omega', \omega]$ with $x_1(t) = 1$ for $t \in [\omega', \omega]$.*

Proof. The property (I) of our optimal process ensures that at most one closed intervall $[\omega', \omega]$ belongs to the set $\{t/x_1(t) = 1\}$ (see Section 2.1). Since x_1 is continuous this set is closed. Besides of $[\omega', \omega]$ it contains at most isolated points and their accumulation points.

Let us regard the intervall $[\omega', \omega] \subseteq \{t/x_1(t) = 1\}$. Then on (ω', ω) holds $x_2(t) = 0$, $u = x_1 + \dot{x}_2 = 1$. From (2.2.) follows $y_2 = 0$ and $\dot{y}_2 = 0$ on this intervall, consequently. Equation (2.8) yields $y_1 = 0$ on $(\omega', \omega]$. Now, from (C_1) we get

$$\mu([t, \omega]) = -l_{11} + (\omega - t)(y_3 - 1) \geq 0$$

and either $[\omega', \omega] \subseteq$ supp μ or $[\omega', \omega] \cap \text{supp}\mu = \emptyset$ □

This lemma allows us to distinguish the following subcases of case 2.

1. Let be t_0 an isolated point of supp μ.
This corresponds to a vertex on the sector arc since t_0 is an isolated point of $\{t/x_1(t) = 1\}$ and the inactive part of the solution is polygonal.
2. Let be t_0 an accumulation point of isolated elements of supp μ.
This case corresponds to an accumulation point of vertices on the sector arc which we call *polygonal accumulation point.*
3. Let be $t_0 \in [\omega', \omega]$.
In this case the solution coincides with the sector arc.

If the boundary part b of a sector-indomain $\mathcal{B}$ consists of straight lines, vertices and their accumulation points then we say that $\mathcal{B}$ is polyhedral *in the generalized sense.* Our structural investigations led to the following result:

Theorem 2. *The optimal ω-sector-indomain $\mathcal{B}^*$ is a composition of at most two parts $\mathcal{B}' \subseteq \mathcal{S}_{\omega'}$ and $\mathcal{B}'' = \mathcal{S}_{\omega''}$ with $\omega', \omega'' \in [0, \omega]$, $\omega' + \omega'' = \omega$, $L'' = \omega''$, $L' = L - \omega''$ where*

1. *$\mathcal{B}', \mathcal{B}''$ are optimal ω'-, ω''-sector-indomains of $\mathcal{S}_{\omega'}$, $\mathcal{S}_{\omega''}$ with respect to L', L'', respectively.*
2. *$\mathcal{B}'$ is polyhedral in the generalized sense.*
3. *$\mathcal{B}''$ is a partial sector.*

3. Optimal polyhedral structure

In this section we study the structure of polyhedral solutions for all admissible values of the parameters L and ω (see [8] for $\omega = 2\pi$). The corresponding optimal value function is given implicitely, only. We summarize some analytical properties of this *fonction penetrante.*

3.1. Optimal ω-inpolyeder

Let be $\mathcal{S}_\omega$ any sector of radius 1. We pose the following problem:
What is the area-minimal ω-inpolyeder of $\mathcal{S}_\omega$ where the perimeter length is given?

figure 3

We describe an ω-inpolyeder by means of his central angles (see fig. 3) and obtain the following problem formulation:

$(\mathrm{P}_2^{\omega,\mathrm{L}})$:

$$F_\omega(\varphi) := \frac{1}{2} \sum_{i \in I} \sin \varphi_i \to \text{Min!}$$

$$2 \sum_{i \in I} \sin \frac{\varphi_i}{2} = L$$

$$\sum_{i \in I} \varphi_i = \omega$$

$$0 \le \varphi_i \le \omega \qquad \text{for } i \in I$$

where the parameter L is choosen accordingly

$$\omega \ge L \ge \underline{L} := \begin{cases} 2\sin\frac{\omega}{2} & \text{for } \omega \in [0,\pi] \\ \\ 2(1-\cos\frac{\omega}{2}) & \text{for } \omega \in (\pi, 2\pi]. \end{cases} \tag{3.1}$$

Now, we assume that there exists an optimal inpolyeder of S_ω and, equivalently, an optimal number n of central angels corresponding to the given parameters L and ω. The existence of such a finite number n will be discussed below in Section 4.1. For such n we get $I = \{1, ..., n\}$ in $(\mathrm{P}_2^{\omega,\mathrm{L}})$. This problem we denote by $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$. Any optimal solution of $(\mathrm{P}_2^{\omega,\mathrm{L}})$ with n central angles is an optimal solution of $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$.

We start with the evaluation of first order necessary optimality conditions for any solution of $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$ with $\varphi_i \in (0,\omega)$ for all i. The case $\varphi_i = 0$ corresponds to a reduction of n. The other case, say $\varphi_1 = \omega$, can only occur if $\omega \in [0,\pi]$ and $L = \omega$. Then there is only one feasible element in $(\mathrm{P}_2^{\omega,\mathrm{L}})$. For $\omega \in (\pi, 2\pi]$ this case cannot occur since $\varphi_i \in [0,\pi]$ because of (3.1).

With the Lagrange function

$$L(\varphi,\lambda) = \lambda_0 \cdot \frac{1}{2}\sum_{i=1}^{n} \sin\varphi_i + \lambda_1 \left(2\sum_{i=1}^{n} \sin\frac{\varphi_i}{2} - L\right) + \lambda_2 \left(\sum_{i=1}^{n} \varphi_i - \omega\right)$$

of $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$ we evaluate the Fritz-John-conditions for this constrained optimization problem (see [18]). For any optimal solution φ there exist numbers $\lambda_0 \ge 0$ and $\lambda_1, \lambda_2 \in R^1$ not vanishing simultanously such that

$$\frac{1}{2}\lambda_0 \ \cos\varphi_i + \lambda_1 \ \cos\frac{\varphi_i}{2} + \lambda_2 = 0 \qquad \text{for } i = 1, ..., n. \tag{3.2}$$

Case 1: $\lambda_0 = 0$. From (4.2) follows $\lambda_1 \ne 0$ and

$$\cos\frac{\varphi_i}{2} = \frac{\lambda_2}{\lambda_1} \qquad \text{for } i = 1, ..., n,$$

that is

$$\varphi_i = \varphi_1 \qquad \text{for } i = 1, ..., n. \tag{3.3}$$

This case occurs only for $L = 2n\sin\frac{\omega}{2n}$. Then the optimal polyhedral solution is a regular ω-inpolyeder of $\mathcal{S}_\omega$.

Case 2: $\lambda_0 = 1$. The solution of system (3.2) is given by

$$\lambda_1 = \left(\cos\frac{\varphi_2}{2} + \cos\frac{\varphi_1}{2}\right)$$
$$\lambda_2 = \frac{1}{2} + \cos\frac{\varphi_1}{2}\cos\frac{\varphi_2}{2}$$

and

$$\cos\frac{\varphi_1}{2} = \frac{1}{2}\left(\cos\frac{\varphi_1}{2} + \cos\frac{\varphi_2}{2}\right) \pm \frac{1}{2}\left(\cos\frac{\varphi_1}{2} - \cos\frac{\varphi_2}{2}\right)$$

for $i = 1, ..., n$. Hence we have only two different angles

$$\varphi_i = \begin{cases} \varphi_1 & \text{for } i = 1, ..., n_1 \\ \\ \varphi_2 & \text{for } i = n_1 + 1, ..., n \end{cases} \tag{3.4}$$

where $\varphi_1 < \varphi_2$, without loss of generality.

For $\lambda_0 = 1$ we study necessary optimality conditions of second order (see [7]). Any optimal solution φ of structure (3.4) has to satisfy the definiteness condition

$$z^T L_{\varphi\varphi}(\varphi, \lambda) z \geq 0$$

for all z belonging to the tangential space of the feasible domain at φ. Thus we have to varify

$$\left(\cos\frac{\varphi_2}{2} - \cos\frac{\varphi_1}{2}\right)\sin\frac{\varphi_1}{2}\sum_{i=1}^{n_1} z_i^2 + \left(\cos\frac{\varphi_1}{2} - \cos\frac{\varphi_2}{2}\right)\sin\frac{\varphi_2}{2}\sum_{i=n_1+1}^{n} z_i^2 \geq 0 \tag{3.5}$$

for all z with

$$\begin{aligned} \sum_{i=1}^{n_1} z_i &= 0 \\ \sum_{i=n_1+1}^{n} z_i &= 0. \end{aligned} \tag{3.6}$$

If $n_1 = 1$ then (3.6) implies

$$0 \leq \sin\frac{\varphi_2}{2} \cdot \sum_{i=n_1+1}^{n} z_i^2 \tag{3.7}$$

which is equivalent to (3.5) because of $\varphi_1 < \varphi_2$.

If $n_1 \geq 2$ then $z_1 = -z_2 \neq 0$ and $z_i = 0$ for $i = 3, ..., n$ is a special solution of (3.6) with $\sin\frac{\varphi_1}{2}(z_1^2 + z_2^2) > 0$ which contradicts (3.5) since $\cos\frac{\varphi_2}{2} - \cos\frac{\varphi_1}{2} < 0$. Thus, $n_1 \geq 2$ cannot occur. Any solution of problem $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$ must be an almost regular ω-inpolyeder of $\mathcal{S}_\omega$ (see [5]).

The founded structure of optimal solutions of $({}^{\mathrm{n}}\mathrm{P}_2^{\omega,\mathrm{L}})$ involves a unique connection between L and the number n of central angels.

For an optimal n and a solution $\varphi \in R^n$ of $(\mathrm{P}_2^{\omega,\mathrm{L}})$ the isoperimetric condition can be described by

$$L = {}^n l_\omega(\alpha) \tag{3.8}$$

where

$$\alpha := \varphi_2,\ \delta_\omega(\alpha) := \varphi_1 = \frac{1}{2}(\omega - (n-1)\alpha)$$

and

$$^{n}l_\omega(\alpha) := 2(n-1)\sin\frac{\alpha}{2} + 2\sin\frac{\delta_\omega(\alpha)}{2}.$$

Corresponding to a regular ω-inpolyeder P_ω with n central angels we denote

$$\alpha_n^\omega := \frac{\omega}{n}$$
$$l_n^\omega :=^n l_\omega(\alpha_n^\omega) = 2n\sin\frac{\omega}{2n}$$

remarking that $n \geq 1$ for $\omega \in (0,\pi)$ and $n \geq 2$ for $\omega \in [\pi, 2\pi]$. In last case we put

$$\alpha_1^\omega := \pi,\ l_1^\omega := \underline{L}.$$

We ask for a unique pair (n,α) corresponding to the given parameter L via (3.8) (see [4], [8]).

Since $0 < \omega - (n-1)\alpha \leq \alpha$ the function $^{n}l_\omega$ is defined on $[\alpha_n^\omega, \alpha_{n-1}^\omega)$. It is continuous on this intervall with

$$\lim_{\alpha\to\alpha_{n-1}} {}^{n}l_\omega(\alpha) = l_{n-1}^\omega \tag{3.9}$$

and continuously differentiable on $(\alpha_n^\omega, \alpha_{n-1}^\omega)$ with

$$\frac{d}{d\alpha}\,{}^{n}l_\omega(\alpha) = (n-1)\left(\cos\frac{\alpha}{2} - \cos\frac{\delta_\omega(\alpha)}{2}\right) < 0 \tag{3.10}$$

since $\alpha \in [0,\pi]$ and $\delta_\omega(\alpha) < \alpha$.

Now we define the piecewise composed function

$$l_\omega(\alpha) :=^n l_\omega(\alpha) \text{ for } \alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega),\ \ n = 2,3,...$$

over the intervall

$$(0, \mathrm{Min}\{\omega,\pi\}) = \cup_{n=2}^\infty[\alpha_n^\infty, \alpha_{n-1}^\omega).$$

From (3.9) and (3.10) follows that this function is continuous and stictly monotonous decreasing. Thus, the equation (3.8) passes into

$$L = l_\omega(\alpha)$$

and is uniquely solvable with an $\alpha \in [\alpha_n^\omega, \alpha_{n-1}^\omega)$ for a unique number n. With respect to L, this n is determined by

$$L \in (l_{n-1}^\omega, l_n^\omega]. \tag{3.12}$$

We have proved the following theorem.

Theorem 3. *An area-minimal ω-inpolyeder $\mathcal{P}_\omega$ of $\mathcal{S}_\omega$ with given length L of the corresponding inpolygon p_ω is allmost regular. The number n of central angels is uniquely* *determined by L via (3.12). One and only one pair $(\alpha, \delta(\alpha))$ corresponds to L and n and describes uniquely the optimal ω-inpolyeder.*

3.2. Generalized fonction penetrante

We now study the optimal-value function of the parametric optimization problem $(\mathrm{P}_2^{\omega,\mathrm{L}})$ for fixed ω (see [4], [8]). The minimal area of an allmost regular ω-inpolyeder depends on $L \in (\underline{L}, \omega)$ and can be described implicitely by

$$F_\omega(\alpha) := \frac{n-1}{2}\sin\alpha + \frac{1}{2}\sin\delta_\omega(\alpha) \tag{3.13}$$

with n according $L \in (l_{n-1}^\omega, l_n^\omega],\ \delta_\omega(\alpha) = \omega - (n-1)\alpha$ and the unique solution $\alpha = \alpha_\omega(L)$ of the equation

$$L = 2(n-1)\sin\frac{\alpha}{2} + 2\sin\frac{1}{2}(\omega - (n-1)\alpha). \tag{3.14}$$

Definition 4. The function

$$f_\omega(L) := F_\omega(\alpha_\omega(L))$$

is called "ω-fonction penetrante" and for $\omega = 2\pi$ "fonction penetrante".

Some properties of this function will be summarized now: The isoperimetric condition (3.14) can be written as $L = l_\omega(\alpha)$ with $l_\omega(\alpha)$ from (3.11). The continuity of l_ω carries over to the inverse function α_ω and to f_ω, consequently. These functions are piecewise continuously differentiable. By implicite differentiation we get for $\alpha = \alpha_\omega(L)$ and $L \in (l_{n-1}^\omega, l_n^\omega)$

$$f'_\omega(L) = \cos\frac{\alpha}{2} + \cos\frac{\delta_\omega(\alpha)}{2} \tag{3.15}$$

$$f''_\omega(L) = \frac{-\sin\frac{\alpha}{2} + (n-1)\sin\frac{\delta_\omega(\alpha)}{2}}{2l'_\omega(\alpha)} \tag{3.16}$$

$$f'''_\omega(L) = \frac{(n-1)\left((n-1)^2-1\right)\left(1-\cos\frac{\alpha}{2}\cos\frac{\delta_\omega(\alpha)}{2}\right)}{4l'_\omega(\alpha)^3} \tag{3.17}$$

remembering $\alpha \in [0,\pi],\ \delta_\omega(\alpha) \in [0,\alpha]$ and $l'_\omega(\alpha) < 0$ (see 3.10).

If $L > l_2^\omega$ then $f'''_\omega(L) < 0$ and $f'_\omega(L)$ strictly concave on each intervall $(l_{n-1}^\omega, l_n^\omega),\ n \geq 3$. Since

$$f''_\omega(l_{n-1}^\omega + 0) = \frac{-\sin\frac{\alpha_{n-1}^\omega}{2}}{2(n-1)(\cos\frac{\alpha_{n-1}^\omega}{2} - 1)} \geq 0$$

and

$$\lim_{L \to l_n^\omega - 0} f''_\omega(L) = -\infty$$

there exists a unique zero $\lambda_n^\omega \in (l_{n-1}^\omega, l_n^\omega)$ of f_ω'' with $f_\omega''(L) > 0$ for $(l_{n-1}^\omega, \lambda_n^\omega)$ and $f_\omega''(L) < 0$ for $L \in (\lambda_n^\omega, l_n^\omega)$. Therefore $f_\omega(L)$ is convex for $L \in (l_{n-1}^\omega, \lambda_n^\omega)$ and concave for $L \in (\lambda_n^\omega, l_n^\omega)$.

If $\omega < 2\pi$ and $L \in (\underline{L}, l_2^\omega)$ we have for $n = 2$

$$f_\omega'''(L) = 0, \quad f_\omega''(L) = \frac{1}{2}\cot\frac{\omega}{4}, \quad f_\omega'(L) = \frac{L}{2}\cot\frac{\omega}{4}, \tag{3.18}$$

and $f_\omega(L)$ is convex on this intervall.

4. Optimal solution of the generalized Favard problem

Proceeding from Theorem 2 and Theorem 3 we want to solve our geometrical problem. It is possible to show that the variable boundary part b^* of the optimal indomain $\mathcal{B}^*$ contains neither any polygonal accumulation point nor a partial sector arc.

4.1. Exclusion of polygonal accumulation points

From Theorem 2 we have the representation $\mathcal{B}^* = \mathcal{B}' \cup \mathcal{B}''$ where $\mathcal{B}' \subseteq \mathcal{S}_{\omega'}$ is polyhedral in the generalized sense. For the corresponding variable boundary curve holds $b^* = b' \cup b''$.

Theorem 4. *$\mathcal{B}'$ is an almost regular inpolyeder of the sector $\mathcal{S}_{\omega'}$*

Proof. Assuming H is an accumulation point of vertices on the sector arc. Let be $\{E_i\}$ a sequence of neighbouring vertices with $\overline{E_iH} > \overline{E_{i+1}H}$ for all $i \geq 1$ and $\lim_{\tau\to\infty} \overline{E_iH} = 0$, $\mathcal{S}_{\omega_H}$ the partial sector of $\mathcal{S}_\omega'$ between E_1 and H and $\mathcal{S}_{\omega_i}$ the partial sector between E_1 and E_i $(i = 2, 3, \ldots)$.

We regard the ω_H-sector-indomain $\mathcal{B}_H := \mathcal{B}' \cap \mathcal{S}_{\omega_H}$ and the ω_i-sector-indomains $\mathcal{B}_i := \mathcal{B}' \cap \mathcal{S}_{\omega_i}$.

Denote by L_H the length of $b_H := b' \cap \mathcal{S}_{\omega_H}$ and by L_i the length of $b_i := b' \cap \mathcal{S}_{\omega_i}$. It holds

$$\mathcal{B}_1 \subseteq \mathcal{B}_i \subseteq \mathcal{B}_{i+1} \subseteq \mathcal{B}_H \subseteq \mathcal{B}' \quad \text{for all } \; i \geq 2 \tag{4.1}$$

and

$$L_1 \leq L_i \leq L_{i+1} \leq L_H \leq L' \quad \text{for all } \; i \geq 2. \tag{4.2}$$

The optimality of the ω'-sector-indomain $\mathcal{B}'$ for given L' induces the optimality of the partial domains $\mathcal{B}_H$ and $\mathcal{B}_i$ in the sectors $\mathcal{S}_{\omega_H}$ and $\mathcal{S}_{\omega_i}$ for L_H and L_i, respectively. Since b_i is an inpolygon of $\mathcal{S}_{\omega_i}$ we know from Theorem 3 that b_i is almost regular. Taking into account relation (4.1) we get for the length L_i of b_i

$$L_i = (i-2)\cdot b + a \quad \text{for all } \; i \geq 3 \tag{4.3}$$

where $a := \overline{E_1E_2}, \quad b := \overline{E_iE_{i+1}} \geq a \quad$ for $i \geq 2$.

This is a contradiction to (4.2). $\mathcal{B}'$ cannot have any accumulation point of vertices. With theorem 3 the proof is completed. $\square$

4.2. Exclusion of the partial sector

In this section we show that b^* does not contain any sector arc. From section 4.1 we have $\mathcal{B}^* = \mathcal{B}' \cup \mathcal{B}''$ where $\mathcal{B}'$ is an allmost regular inpolygon of $\mathcal{S}_{\omega'}$ and $\mathcal{B}'' = \mathcal{S}_{\omega''}$.

Therefore, the area of $\mathcal{B}^*$ is given by

$$\frac{1}{2}\omega'' + f_{\omega'}(L') \tag{4.4}$$

where $f_{\omega'}$ denotes the *fonction penetrante.*

The parameters ω' and ω'' depends on L'. We have $\omega''(L') = L - L'$ and $\omega'(L') = \omega - L + L'$. Since $\mathcal{B}^*$ is an ω-sector-indomain of minimal area for given L the parameter L' solves the problem

$$\hat{f}(L') := \frac{1}{2}(L - L') + f_{\omega'(L')}(L') \to \mathrm{Min!}_{L' \in [\tilde{L}, L]}$$

where $\tilde{L} \geq 0$ denotes the minimal possible value for L'.

Theorem 5. *The function $\hat{f}$ is monotonously decreasing on $[\tilde{L}, L]$.*

Proof. The function $\hat{f}$ is differentiable a.e. Denote $g(\omega', L) := f_{\omega'}(L')$. Using the representation of the *fonction penetrante* and its derivative with respect to L' (see(3.15)) we get

$$\begin{aligned} \frac{d\hat{f}}{dL'} &= -\frac{1}{2} + \frac{\partial g}{\partial \omega'} \cdot \frac{\partial \omega'}{\partial L'} + \frac{\partial g}{\partial L'} \\ &= -\frac{1}{2} + \left(\cos\frac{\alpha}{2} + \cos\frac{\delta}{2}\right)\left(1 + (n-1)\left(\cos\frac{\alpha}{2} - \cos\frac{\delta}{2} \cdot \frac{\partial \alpha}{\partial \omega'}\right)\right) + \frac{1}{2}\cos\delta \end{aligned} \tag{4.5}$$

with $\alpha = \alpha(L', \omega')$, $\delta = \omega' - (n-1)\alpha(L', \omega')$. The function α is implicitely given by

$$0 = 2(n-1)\sin\frac{\alpha}{2} + 2\sin\frac{\omega' - (n-1)\alpha}{2} - L'$$

see (3.8). Implicite differentiation leads to

$$\frac{\partial \alpha}{\partial \omega'} = -\frac{\cos\frac{\delta}{2}}{(n-1)\left(\cos\frac{\alpha}{2} - \cos\frac{\delta}{2}\right)} \tag{4.6}$$

Putting this in (4.5) we get for $\alpha \in [0, \pi]$, $\delta \in [0, \alpha]$

$$\frac{d\hat{f}}{dL'} = -\frac{1}{2} + \cos\frac{\alpha}{2} + \cos\frac{\delta}{2}\left(1 - \cos\frac{\alpha}{2}\right) - \frac{1}{2} \leq 0. \quad \square \tag{4.7}$$

Consequently, from this theorem follows that $\hat{f}$ attains its minimum at $L' = L$.

We have proved that the optimal solution $\mathcal{B}^*$ of the generalized Favard problem is an almost regular inpolyeder of $\mathcal{S}_\omega$.

Acknowledgement. I would like to thank J. Focke and R. Klötzler for their animation to be engaged in this field and for numerous helpful discussions.

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Received February 10, 1995.