Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 36 (1995), No. 2, 211-218.

On a Characterization of Circles and Spheres

H. Groemer *

Department of Mathematics, The University of Arizona
Tucson, AZ 85721, U.S.A., E-mail: groemer@math.arizona.edu

Abstract. It is shown that, among all two-dimensional convex domains, circles can be characterized by the property that for all circumscribed rectangles the midpoints of the sides meet the domain. The proof uses Fourier series, and this method yields not only such characterizations but also corresponding stability results. Moreover, some analogous results for the d-dimensional case ($d \geq 3$) are established.

1. Introduction

Let $\mathbf{E}^d$ denote the d-dimensional euclidean space, and $\|\cdot\|$ its norm. Nonempty compact convex subsets of $\mathbf{E}^d$ will be called *convex bodies* in $\mathbf{E}^d$, and the class of all convex bodies in $\mathbf{E}^d$ will be denoted by $\mathcal{K}^d$. Convex bodies in $\mathbf{E}^2$ will be referred to as *convex domains*. S^{d-1} signifies the $(d-1)$-dimensional unit sphere in $\mathbf{E}^d$ centered at the origin of $\mathbf{E}^d$. Balls (solid spheres) in $\mathbf{E}^d$ are always assumed to be euclidean balls. A *box* in $\mathbf{E}^d$ is defined to be a d-dimensional rectangular parallelotope. $D(K)$ denotes the diameter of K.

Obviously, if K is a ball, then it touches the facets of every circumscribed cube at their respective centers. The present article is a study regarding the converse of this statement and the corresponding stability problem. Expressed more precisely, the question is whether a convex body is a ball if it touches the facets of its circumscribed boxes at their centers, and how much a convex body can deviate from a ball, if the respective distances between the centers of these facets and the points of contact are below a given bound.

The following section concerns the special case $d = 2$. In this situation Fourier series can be used to prove a tight inequality that provides a satisfactory solution of the problems just mentioned. In Section 3 convex bodies of arbitrary dimension will be considered. In this case the results that will be proved are of a less definitive nature.

If $K \in \mathcal{K}^d$ let $h_K(u)$, $H_K(u)$, and $H_K^+(u)$ denote, respectively, the support function, the support plane, and the support half-space of K in the direction u. Thus, $H_K^+(u)$ is a closed half-space with the property that $K \subset H_K^+(u)$ and that its boundary is $H_K(u)$.

* Supported by National Science Foundation Research Grant DMS-9401487

There are various possibilities to measure the distance between two convex bodies. The most important concepts in this respect are the Hausdorff distance δ and the quadratic mean distance $\bar{\delta}$. If $K_1, K_2 \in \mathcal{K}^d$ these are defined by

$$\delta(K_1, K_2) = \max\{|h_{K_1}(u) - h_{K_2}(u)| : u \in S^{d-1}\},$$

and

$$\bar{\delta}(K_1, K_2) = \left(\frac{1}{\sigma_d}\int_{S^{d-1}} (h_{K_1}(u) - h_{K_2}(u))^2 \, d\sigma(u)\right)^{1/2},$$

where σ refers to the spherical Lebesgue measure on S^{d-1} and σ_d denotes the surface area of the d-dimensional unit ball. Besides the trivial inequality

$$\bar{\delta}(K_1, K_2) \leq \delta(K_1, K_2)$$

there are inequalities going in the other direction; see [4], [7], and the survey [3]. (The latter article contains also a general discussion of the concept of stability for geometric inequalities.) For the present purpose only the following inequality concerning domains $K \in \mathcal{K}^2$ will be of importance. If $B(K)$ denotes the Steiner disc of K, i. e., the circular disc whose center is the Steiner point of K and whose diameter equals the mean with of K, then

$$\delta(K, B(K)) \leq 2\pi(400\pi)^{1/5} D(K)^{1/5} \bar{\delta}(K, B(K))^{4/5}. \tag{1}$$

This inequality is the special case $d = 2$ of an inequality proved in [4] (Lemma 1).

2. The Two-Dimensional Case

If $\omega \in [-\pi, \pi]$ let u, or $u(\omega)$, denote the unit vector $(\cos\omega, \sin\omega)$ in $\mathbf{E}^2$. Then, $h_K(u)$, $H_K(u)$ and $H_K^+(u)$ can be perceived as functions of ω and instead of writing $h_K(u(\omega))$, $H_K(u(\omega))$ and $H_K^+(u(\omega))$ it will usually be more convenient to denote these functions by, respectively, $h(\omega)$, $H(\omega)$, and $H^+(\omega)$. The line segment

$$L_K(\omega) = H_K^+(\omega + \tfrac{\pi}{2}) \cap H_K^+(\omega - \tfrac{\pi}{2}) \cap H_K(\omega)$$

will be called the *support segment* of K corresponding to ω. Alternatively, the support segment $L_K(\omega)$ can be defined as one side of a suitable circumscribed rectangle, or as the orthogonal projection of K unto the support line $H_K(\omega)$. A support segment of K will be said to be *regular* if the support set $K \cap H_K(\omega)$ consists of only one point. This point, say $q_K(\omega)$, will then be called the *support point* of K corresponding to $L_K(\omega)$. Furthermore, we let Ω_K denote the set of all angles $\omega \subset [-\pi, \pi]$ for which the corresponding support segment $L_K(\omega)$ is regular. Since for any $c > 0$ the number of line segment in the boundary of K of length greater that c must be finite it follows that $\Omega_K = [-\pi, \pi] \setminus C$, where C is countable. Letting $m_K(\omega)$ denote the midpoint of the support segment $L_K(\omega)$, we use the following two expressions as measures for the deviation of the support point from the midpoint of $L_K(\omega)$:

$$e(K) = \sup\{\|m_K(\omega) - q_K(\omega)\| : \omega \in \Omega_K\}\,,$$

$$\bar{e}(K) = \left(\frac{1}{2\pi}\int_{\Omega_K} \|m_K(\omega) - q_K(\omega)\|^2\right)^{1/2}.$$

(The fact that $\|m_K(\omega) - q_K(\omega)\|^2$ is integrable will be shown in the proof of Theorem 1). In terms of circumscribed rectangles $e(K)$ and $\bar{e}(K)$ are, respectively, the supremum and the quadratic mean value of the deviation of the midpoint from the corresponding support point of the regular sides of circumscribed rectangles. It is obvious that

$$\bar{e}(K) \leq e(K)\,. \tag{2}$$

Now the following theorem, which provides estimates of the deviation of K from a circle in terms of $\bar{e}(K)$ and $e(K)$, will be proved.

Theorem 1. *Let K be a convex domain and $B(K)$ its Steiner disc. Then*

$$\bar{\delta}(K, B(K)) \leq \frac{1}{2}\bar{e}(K) \tag{3}$$

and

$$\delta(K, B(K)) \leq \pi(800\pi)^{1/5} D(K)^{1/5} e(K)^{4/5}\,. \tag{4}$$

Equality holds in (3) *if and only if the support function h of K is of the form*

$$h(\omega) = a_0 + a_1\cos\omega + b_1\sin\omega + a_2\cos 2\omega + b_2\sin 2\omega\,, \tag{5}$$

and in (4) *if and only if K is a circular disc.*

Proof. We start by describing several known facts from the theory of convex bodies. Proofs of these facts can be found in the book [6] of Schneider (Theorem 1.5.1 and Corollary 1.7.3). Let $\tilde{h}$ denote the extended support function of K, i. e., the positively homogeneous function on $\mathbf{E}^2$ whose restriction to the unit circle is $h(u)$. Then $\tilde{h}$ is a convex function on $\mathbf{E}^d$ and h is Lipschitz continuous in the sense that there is a constant λ such that $|\tilde{h}(v) - \tilde{h}(w)| \leq \lambda\|v - w\|$ (for all $v, w \in S^{d-1}$). Since obviously $\|u(\omega_1) - u(\omega_2)\| \leq |\omega_1 - \omega_2|$ it follows that $|\tilde{h}(\omega_1) - \tilde{h}(\omega_2)| \leq \lambda|\omega_1 - \omega_2|$. Hence, $\tilde{h}(\omega)$ is absolutely continuous and this implies that $h'(\omega)$ exists almost everywhere on $[-\pi, \pi]$ and is integrable. For all directions $u(\omega)$ with $\omega \in \Omega_K$, the gradient of $\tilde{h}$ at the point u exists and equals $q_K(\omega)$. If this gradient is expressed in polar coordinates an obvious calculation shows that it equals $h(\omega)u(\omega) + h'(\omega)u'(\omega)$. Hence, for all $\omega \in \Omega_K$ we have

$$q_K(\omega) = h(\omega)u(\omega) + h'(\omega)u'(\omega)\,. \tag{6}$$

Together with the previous remark about the integrability of h' and the obvious fact that $m_K(\omega)$ is continuous this shows in particular that the (evidently bounded) function $\|m_K(\omega) - q_K(\omega)\|^2$ is integrable. Hence $\bar{e}(K)$ is defined for all convex domains.

We now prove inequality (3). From the obvious relation

$$m(\omega) = h(\omega)u(\omega) + \frac{1}{2}\big(h\left(\omega + \tfrac{\pi}{2}\right) - h\left(\omega - \tfrac{\pi}{2}\right)\big)u'(\omega),$$

and (6) it follows that for almost all ω

$$\|m_K(\omega) - q_K(\omega)\| = \left| h'(\omega) - \frac{1}{2}\big(h\left(\omega + \tfrac{\pi}{2}\right) - h\left(\omega - \tfrac{\pi}{2}\right)\big)\right| . \tag{7}$$

Let now

$$h(\omega) \sim \sum_{n=0}^{\infty}(a_n \cos n\omega + b_n \sin n\omega)$$

be the Fourier expansion of h ($b_0 = 0$). Since h is absolutely continuous one can determine the Fourier coefficients of h' using integration by parts. Together with an obvious evaluation of the Fourier coefficients of $\left(\omega + \frac{\pi}{2}\right)$, and $h\left(\omega - \frac{\pi}{2}\right)$, this yields

$$h'(\omega) - \frac{1}{2}\big(h\left(\omega + \tfrac{\pi}{2}\right) - h\left(\omega - \tfrac{\pi}{2}\right)\big) \sim \sum_{n=0}^{\infty}(n - k_n)(b_n \cos n\omega - a_n \sin n\omega) \tag{8}$$

with

$$k_n = \begin{cases} (-1)^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} .$$

From (7), (8), and Parseval's equation it follows that

$$\bar{e}(K)^2 = \frac{1}{2}\sum_{n=2}^{\infty}(n - k_n)^2(a_n^2 + b_n^2). \tag{9}$$

Since it is easily checked that the support function of the Steiner disk $B(K)$ is $a_0 + a_1 \cos\omega + b_1 \sin\omega$ one finds, using again Parseval's equation, that

$$\bar{\delta}(K, B(K))^2 = \frac{1}{2}\sum_{n=2}^{\infty}(a_n^2 + b_n^2). \tag{10}$$

From (9) and (10) it follows that

$$\bar{e}(K)^2 = 4\bar{\delta}(K, B(K))^2 + \frac{1}{2}\sum_{n=2}^{\infty}((n - k_n)^2 - 4)(a_n^2 + b_n^2).$$

Since $n \geq 2$ implies $(n - k_n)^2 - 4 \geq 0$, inequality (3) is an immediate consequence of this relation. Also, since $(n - k_n)^2 - 4 > 0$ if $n > 2$, it follows that in (3) equality holds if only if h is of the form (5). Inequality (4) follows obviously from (1), (2), and (3). If equality holds in (4), it also must hold in (3) and therefore h must be of the form (5). But then (7) and (8) show that $\|m(\omega) - q(\omega)\| = 2|b_2 \cos 2\omega + a_2 \sin 2\omega|$ and if $a_2 \neq 0$ or $b_2 \neq 0$ this is not constant and therefore $\bar{e}(K) < e(K)$. Thus equality holds in (4) only for circular discs, and in this case equality does hold since both sides of (4) vanish. □

Applying Theorem 1 in the case $e(K) = 0$ one obtains immediately the following characterization circles mentioned in the introduction.

Corollary 1. *If for all circumscribed rectangles of a convex domain K the midpoints of the sides of these rectangles meet K, then K is a circular disc.*

It is worth noting that the assumptions of this corollary show that for every pair of parallel support lines there is a line segment in K that is orthogonal to these lines and has an endpoint in each of them. This property is known to characterize convex domains of constant width (see [1], Section 2). Thus, if one only wishes to prove Corollary 1 it suffices to do this for convex domains of constant width. The fact that such domains, if they satisfy the assumptions of the corollary, are circular discs has been proved by Khassa [5].

Concluding this section we note that Theorem 1 may be of some practical value. In view of the fact that it is not possible to determine the deviation of a convex domain from a circle by the usual caliper measurements, i. e. by measurements of its width, this theorem provides a technically simple procedure to determine the deviation of a domain from a circle by measuring the distance between the midpoint and the point of contact of the support segments.

3. Results for Arbitrary Dimension

If $K \in \mathcal{K}^d$ ($d \geq 3$), then, in analogy to the support segment $L_K(\omega)$, one can define the *support strip* of K. This is a subset of $H_K(u)$ bounded by two $(d-2)$-flats that correspond to two parallel support planes of K. More precisely, if u and w are orthogonal unit vectors, we define

$$L_K(u, w) = H_K^+(w) \cap H_K^+(-w) \cap H_K(u)$$

and call $L_K(u, w)$ the support strip of K associated with the directions u, w. (In the case $d = 2$ it was not necessary to specify w since the two directions $\pm w$ are uniquely determined by u.) If it is clear from the context what is meant, we simply write L_K or L instead of $L_K(u, v)$. To the midpoint of a support segment there corresponds in the d-dimensional case the $(d-2)$-flat consisting of all points that are equidistant to the two boundary $(d-2)$-flats of L_K. This set will be denoted by $m(L_K)$. Finally, we let $Q(L_K)$ denote the support set $K \cap L_K$. A support strip L of K will be said to be *regular* if the corresponding support set $Q(L)$ is contained in a $(d-2)$-flat parallel to the boundary flats of L. If L is regular, then each point of $Q(L)$ has the same orthogonal distance to $m(L)$. Let $\Delta(L)$ denote this distance. (This is the analogue to the expression $\|m_K(\omega) - q_K(\omega)\|$ in the preceding section.) Letting $\mathcal{L}_K$ denote the set of all regular support strips of K we define, similarly as in the two-dimensional case, the deviation measure $e(K)$ by

$$e(K) = \sup\{\Delta(L) : L \in \mathcal{L}_K\}\,.$$

Now the following theorem will be proved.

Theorem 2. *If K is a convex body in $\mathbf{E}^d$ $(d \geq 3)$ of diameter $D(K)$ there is a d-dimensional ball B such that*

$$\delta(K, B) \leq \alpha_d D(K)^{1/5} e(K)^{4/5}\,, \tag{11}$$

with $\alpha_d = 3\pi(800\pi)^{1/5}(1+2\sqrt{2})^{d-2}2^{d-3}$. If K is centrally symmetric, then $\alpha_d = 3\pi(800\pi)^{1/5}2^{d-3}$.

Proof. The proof is obtained by applying Theorem 1 to the orthogonal projections of K onto two-dimensional linear subspaces. Let E be such a subspace and let Π_E denote the orthogonal projection operator mapping $\mathbf{E}^d$ onto E. Evidently, $\Pi_E K$ is a convex domain in E, and taking E as the underlying two-dimensional euclidean space one can, as in Section 1, again define the angle ω and the support segment $L_{\Pi_E K}(\omega)$ of $\Pi_E K$. To simplify the notation let us write $L_{\Pi_E K}(\omega) = \bar{L}(\omega) = \bar{L}$. Clearly, if $\bar{L}$ is such a support segment of $\Pi_E K$, then $\Pi_E^{-1}\bar{L}$ is a support strip of K. Moreover, if $\bar{L}$ is regular, then $\Pi_E^{-1} L$ is a regular support strip of K. Hence, the set $\Omega_{\Pi_E K}$ associated with the regular support segments $\bar{L}(\omega)$ of the convex domain $\Pi_E K$ corresponds to the subset $\mathcal{L}_K$ of the regular support strips of K whose boundary flats are orthogonal to E. It also is evident that for any $\omega \in \Omega_{\Pi_E K}$ the distance between $m_{\Pi_E K}(\omega)$ and $q_{\Pi_E K}(\omega)$ is the same as that between $m(\Pi_E^{-1}\bar{L}(\omega))$ and $Q(\Pi_E^{-1}\bar{L}(\omega))$. Hence, for every two-dimensional linear subspace E of $\mathbf{E}^d$ we have

$$e(\Pi_E K) \leq e(K)\,,$$

and it follows from Theorem 1 that

$$\delta(\Pi_E K, B(\Pi_E K)) \leq \pi(800\pi)^{1/5} D(K)^{1/5} e(K)^{4/5}\,. \tag{12}$$

We now need the following result that is proved in [2] (Corollary 3): If $K \in \mathbf{E}^d$ $(d \geq 3)$ and if every orthogonal projection of K onto a $(d-1)$-dimensional subspace of $\mathbf{E}^d$ has Hausdorff distance at most δ_o from a $(d-1)$-dimensional ball, then there is a d-dimensional ball B such that

$$\delta(K, B) \leq (1+2\sqrt{2})\lambda_d \delta_o\,, \tag{13}$$

where $\lambda_3 = 3$ and $\lambda_d = 2$ if $d > 3$. Moreover, if K is centrally symmetric the factor $1+2\sqrt{2}$ can be removed.

If $d = 3$ this result in conjunction with the estimate (12) yields immediately (11). If $d > 3$ one can proceed by induction assuming that (11) holds for dimension $d-1$. Orthogonally projecting K onto linear subspaces of dimension $d-1$ one obtains from the induction assumption that (11) holds for these projections (with d replaced by $d-1$). Since for $d \geq 4$ we have $\alpha_d = 2(1+2\sqrt{2})\alpha_{d-1}$ (or $\alpha_d = 2\alpha_{d-1}$ if K is centrally symmetric) the desired conclusion is now obtained from (13). □

If L is a support strip of K, there obviously is a circumscribed box of K with the property that one of its facets, say F, is contained in L. Furthermore, if L is regular the distance between the center z of F and the support set $K \cap F$ (i. e., the minimum of the distances between z and the points of $K \cap F$) is at least $\Delta(L)$. Hence, we can state the following corollary of Theorem 2.

Corollary 2. *Let K be a convex body in $\mathbf{E}^d$ such that for some $\epsilon \geq 0$ the centers of the facets of the circumscribed boxes of K have distance at most ϵ from the respective support sets of these facets. Then there exist a d-dimensional ball B such that*

$$\delta(K, B) \leq \alpha_d D(K)^{1/5} \epsilon^{4/5}$$

with α_d as in Theorem 2.

In the special case $\epsilon = 0$ this corollary contains the following result which is the generalization of Corollary 1 to higher dimensions.

Corollary 3. *If for all circumscribed boxes of a convex body $K \in \mathcal{K}^d$ the centers of the facets of the boxes meet K, then K is a ball.*

The remark after Corollary 1, regarding convex bodies of constant width, also applies with respect to the present corollary.

As another consequence of Corollary 2 we formulate a corollary regarding projections. If $K \in \mathcal{K}^d$ and $u \in S^{d-1}$ let K_u denote the orthogonal projection of K onto the support plane $H_K(u)$. If K_u is centrally symmetric, and if F is a facet of a circumscribed box of K such that $F \subset H_K(u)$, then F and K_u have obviously the same center. From this remark and Corollary 2, together with the observation that the central symmetry of all K_u implies the central symmetry of K, we obtain immediately the following result.

Corollary 4. *Let K be a convex body in $\mathbf{E}^d$ such that all projections K_u are centrally symmetric. If for some $\epsilon \geq 0$ and every $u \in S^{d-1}$ the center of K_u has distance at most ϵ from the support set $K \cap H_K(u)$, then there exists a d-dimensional ball B such that*

$$\delta(K, B) \leq 3\pi(800\pi)^{1/5} 2^{d-3} D(K)^{1/5} \epsilon^{4/5}\,.$$

We finally note that as a consequence of the results in [2] it is not necessary to consider in Theorem 2 and its corollaries the class of all regular support strips (or, respectively, all boxes or projections) but only suitable subsets of these classes.

References

[1] Chakerian, G. D. and Groemer, H.: *Convex bodies of constant width.* In: Convexity and Its Applications, pp. 49-96, Birkhäuser Verlag, Basel - Boston - Stuttgart, 1983.

[2] Groemer, H.: *Stability theorems for projections of convex sets.* Israel J. Math. **60**, (1987), 177-190.

[3] Groemer, H.: *Stability of Geometric Inequalities.* In: Handbook of Convex Geometry (Section 1.4), North Holland Publ., Amsterdam etc., 1993.

[4] Groemer, H. and Schneider, R.: *Stability estimates for some geometric inequalities.* Bull. Lond. Math. Soc. **23** (1991), 67-74.

[5] Khassa, D. S.: *Relation between maximal chords and symmetry of convex sets.* J. London Math. Soc. **15** (1977), 541-546.

[6] Schneider, R.: *Convex Bodies: The Brunn-Minkowski Theory.* Encyclopedia of mathematics and its applications, Vol. 44. Cambridge University Press, Cambridge - New York - Oakleigh, 1993.

[7] Vitale, R.: L_p *metrics for compact convex sets.* J. Approx. Theory **45** (1985), 280-287.

Received February 1, 1995.