ON CERTAIN DISCONTINUOUS INTEGRALS CONNECTED WITH THE DEVELOPMENT OF THE RADICAL WHICH REPRESENTS THE RECIPROCAL OF THE DISTANCE BETWEEN TWO POINTS

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On certain discontinuous Integrals, connected with the Development of the Radical which represents the Reciprocal of the Distance between two Points. By William Rowan Hamilton, *LL.D., P.R.I.A., Member of several Scientific Societies at Home and Abroad, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland**.

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1. It is well known that the radical

$$(1 - 2xp + x^2)^{-\frac{1}{2}}, \tag{1.}$$

in which x and 1 may represent the radii vectores of two points, while p represents the cosine of the angle between those radii, and the radical represents therefore the reciprocal of the distance of the one point from the other, may be developed in a series of the form

$$\mathrm{P}_0 + \mathrm{P}_1 x + \mathrm{P}_2 x^2 + \ldots + \mathrm{P}_n x^n + \ldots; \tag{2.}$$

the coefficients P_n being functions of p, and possessing many known properties, among which we shall here employ the following only,

$$\mathrm{P}_n = [0]^{-n} \left(\frac{d}{dp}\right)^n \left(\frac{p^2 - 1}{2}\right)^n; \tag{3.}$$

the known notation of factorials being here used, according to which

$$[0]^{-n} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}. \tag{4.}$$

It is proposed to express the sum of the first n terms of the development (2.), which may be thus denoted,

$$\Sigma_{(n)}{}_0^{n-1} \mathrm{P}_n x^n = \mathrm{P}_0 + \mathrm{P}_1 x + \mathrm{P}_2 x^2 + \ldots + \mathrm{P}_{n-1} x^{n-1}. \tag{5.}$$

2. In general, by Taylor's theorem,

$$f(p + q) = \Sigma_{(n)}{}_0^{\infty} [0]^{-n} q^n \left(\frac{d}{dp}\right)^n f(p); \tag{6.}$$

* Communicated by the Author.

hence, by the property (3.), P_n is the coefficient of q^n in the development of

$$\left(\frac{(p+q)^2-1}{2}\right)^n; \tag{7.}$$

it is therefore also the coefficient of q^0 in the development of

$$\left(\frac{p^2-1}{2q}+p+\frac{q}{2}\right)^n \tag{8.}$$

If then we make, for abridgment,

$$\vartheta = p + \frac{p^2}{2}\cos\theta + \sqrt{-1}\left(1-\frac{p^2}{2}\right)\sin\theta, \tag{9.}$$

we shall have the following expression, which perhaps is new, for P_n:

$$\mathrm{P}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vartheta^n \, d\theta; \tag{10.}$$

and hence, immediately, the required sum (5.) may be expressed as follows:

$$\Sigma_{(n)}{}_0^{n-1}\mathrm{P}_n x^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, \frac{1-\vartheta^n x^n}{1-\vartheta x}; \tag{11.}$$

in which it is to be observed that x may be any quantity, real or imaginary.

3. We have therefore, rigorously, for the sum of the n first terms of the series

$$\mathrm{P}_0 + \mathrm{P}_1 + \mathrm{P}_2 + \ldots, \tag{12.}$$

the expression

$$\Sigma_{(n)}{}_0^{n-1}\mathrm{P}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, \frac{1-\vartheta^n}{1-\vartheta}; \tag{13.}$$

of which we propose to consider now the part independent of n, namely,

$$\mathrm{F}(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{1-\vartheta}; \tag{14.}$$

and to examine the form of this function F of p, at least between the limits $p=-1$, $p=1$.

4. A little attention shows that the denominator $1 - \vartheta$ may be decomposed into factors, as follows:

$$1 - \vartheta = -\tfrac{1}{2}(\alpha + e^{\theta\sqrt{-1}})(1 - \beta e^{-\theta\sqrt{-1}}); \tag{15.}$$

in which,

$$\alpha = 2s(1-s), \quad \beta = 2s(1+s), \tag{16.}$$

and

$$p = 1 - 2s^2; \tag{17.}$$

so that s may be supposed not to exceed the limits 0 and 1, since p is supposed not to exceed the limits -1 and 1. Hence

$$\frac{1}{1-\vartheta} = \frac{-2(\alpha + e^{-\theta\sqrt{-1}})(1 - \beta e^{\theta\sqrt{-1}})}{(1 + 2\alpha\cos\theta + \alpha^2)(1 - 2\beta\cos\theta + \beta^2)}; \tag{18.}$$

of which the real part may be put under the form

$$\frac{\lambda}{1 + 2\alpha\cos\theta + \alpha^2} + \frac{\mu}{1 - 2\beta\cos\theta + \beta^2}, \tag{19.}$$

if λ and μ be so chosen as to satisfy the conditions

$$\lambda(1+\beta^2) + \mu(1+\alpha^2) = 2(\beta - \alpha), \tag{20.}$$

$$\lambda\beta - \mu\alpha = 1 - \alpha\beta, \tag{21.}$$

which give

$$\lambda = \frac{1-\alpha^2}{\alpha+\beta}, \quad \mu = \frac{\beta^2 - 1}{\alpha+\beta}. \tag{22.}$$

The imaginary part of the expression (18.) changes sign with θ, and disappears in the integral (14.); that integral therefore reduces itself to the sum of the two following:

$$\mathrm{F}(p) = \frac{1}{4s\pi}\int_0^\pi \frac{(1-\alpha^2)\,d\theta}{1 + 2\alpha\cos\theta + \alpha^2} + \frac{1}{4s\pi}\int_0^\pi \frac{(\beta^2-1)\,d\theta}{1 - 2\beta\cos\theta + \beta^2}; \tag{23.}$$

in which, by (16.), $\alpha + \beta$ has been changed to $4s$. But, in general if $a^2 > b^2$,

$$\int_0^\pi \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}, \tag{24.}$$

the radical being a positive quantity if a be such; therefore in the formula (23.),

$$\int_0^\pi \frac{(1-\alpha^2)\,d\theta}{1 + 2\alpha\cos\theta + \alpha^2} = \pi, \tag{25.}$$

because, by (16.), α cannot exceed the limits 0 and $\frac{1}{2}$, s being supposed not to exceed the limits 0 and 1, so that $1 - \alpha^2$ is positive. On the other hand, β varies from 0 to 4, while s

varies from 0 to 1; and $\beta^2 - 1$ will be positive or negative, according as s is greater or less than the positive root of the equation

$$s^2 + s = \tfrac{1}{2}. \tag{26.}$$

Hence, in (23.), we must make

$$\int_0^\pi \frac{(\beta^2 - 1)\, d\theta}{1 - 2\beta\cos\theta + \beta^2} = \pi, \text{ or } = -\pi, \tag{27.}$$

according as

$$s > \text{ or } < \frac{\sqrt{3} - 1}{2}; \tag{28.}$$

and thus we find, under the same alternative,

$$\mathrm{F}(p) = \frac{1}{4s}(1 \pm 1), \tag{29.}$$

that is,

$$\mathrm{F}(p) = \frac{1}{2s}, \text{ or } = 0. \tag{30.}$$

But, by (17.),

$$s = \sqrt{\frac{1 - p}{2}}; \tag{31.}$$

the function $F(p)$, or the definite integral (14.), receives therefore a sudden change of form when p, in varying from -1 to 1, passes through the critical value

$$p = \sqrt{3} - 1; \tag{32.}$$

in such a manner that we have

$$\mathrm{F}(p) = (2 - 2p)^{-\frac{1}{2}}, \quad \text{if} \quad p < \sqrt{3} - 1; \tag{33.}$$

and, on the other hand,

$$\mathrm{F}(p) = 0 \quad \text{if} \quad p > \sqrt{3} - 1; \tag{34.}$$

For the critical value (32.) itself, we have

$$s = \frac{\sqrt{3} - 1}{2}, \quad \alpha = 2\sqrt{3} - 3, \quad \beta = 1, \tag{35.}$$

and the real part of (18.) becomes

$$\frac{1 - \alpha}{1 + 2\alpha\cos\theta + \alpha^2}; \tag{36.}$$

multiplying therefore by $d\theta$, integrating from $\theta = 0$ to $\theta = \pi$, and dividing by π, we find, by (25.) and (14.), this formula instead of (29.),

$$F(p) = \frac{1}{1 + \alpha} = \frac{1}{4s}, \tag{37.}$$

that is,

$$\mathrm{F}(p) = \tfrac{1}{2}(2 - 2p)^{-\frac{1}{2}}, \quad \text{if} \quad p = \sqrt{3} - 1; \tag{38.}$$

The value of the discontinuous function F is therefore, in this case, equal to the semisum of the two different values which that function receives, immediately before and after the variable p attains its critical value, as usually happens in other similar cases of discontinuity.

5. As verifications of the results (33.), (34.), we may consider the particular values $p = 0$, $p = 1$, which ought to give

$$\mathrm{F}(0) = 2^{-\frac{1}{2}}, \quad \mathrm{F}(1) = 0. \tag{39.}$$

Accordingly, when $p = 0$, the definitions (9.) and (14.) give

$$\vartheta = \sqrt{-1}\sin\theta, \tag{40.}$$

$$\mathrm{F}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{1 - \sqrt{-1}\sin\theta} = \frac{1}{\pi}\int_{0}^{\pi} \frac{d\theta}{1 + \sin\theta^2}; \tag{41.}$$

which easily gives, by (24.),

$$\mathrm{F}(0) = \frac{2}{\pi}\int_{0}^{\pi} \frac{d\theta}{3 - \cos 2\theta} = \frac{1}{\pi}\int_{0}^{2\pi} \frac{d\theta}{3 - \cos\theta} = 2^{-\frac{1}{2}}. \tag{42.}$$

And when $p = 1$, we have

$$1 - \vartheta = -\tfrac{1}{2}(\cos\theta + \sqrt{-1}\sin\theta), \tag{43.}$$

$$\frac{1}{2\pi}\frac{d\theta}{1 - \vartheta} = -\pi^{-1}(\cos\theta - \sqrt{-1}\sin\theta)\, d\theta, \tag{44.}$$

of which the integral, taken from $\theta = -\pi$ to $\theta = \pi$, is $\mathrm{F}(1) = 0$.

6. Let us consider now this other integral,

$$\mathrm{G}(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\vartheta^n\, d\theta}{\vartheta - 1}. \tag{45.}$$

The expression (13.) gives

$$\Sigma_{(n)}{}_0^{n-1}\mathrm{P}_n = \mathrm{F}(p) + \mathrm{G}(p); \tag{46.}$$

therefore, by (34.), we shall have

$$\mathrm{G}(p) = \Sigma_{(n)}{}_0^{n-1}\mathrm{P}_n, \quad \text{if} \quad p > \sqrt{3} - 1. \tag{47.}$$

For instance, let $p = 1$; then multiplying the expression (44.) by

$$-\vartheta^n = -(1 + \tfrac{1}{2}e^{\theta\sqrt{-1}})^n, \tag{48.}$$

the only term which does not vanish when integrated is $\frac{1}{2}n\pi^{-1}\, d\theta$, and this term gives the result

$$\mathrm{G}(1) = n, \tag{49.}$$

which evidently agrees with the formula (47.), because it is well known that

$$\mathrm{P}_n = 1, \quad \text{when} \quad p = 1, \tag{50.}$$

the series (2.) becoming then the development of $(1 - x)^{-1}$.

7. On the other hand, let p be $< \sqrt{3}-1$; then, observing that, by (33.),

$$\mathrm{F}(p) = (2-2p)^{-\frac{1}{2}} = \Sigma_{(n)}{}_0^\infty \mathrm{P}_n, \tag{51.}$$

we find, by the relation (46.) between the functions F and G,

$$\mathrm{G}(p) = -\Sigma_{(n)}{}_n^\infty \mathrm{P}_n = -(\mathrm{P}_n + \mathrm{P}_{n+1} + \mathrm{P}_{n+2} + \ldots). \tag{52.}$$

For instance, let $p = 0$; then, by (40.) and (45.),

$$\mathrm{G}(0) = \frac{-(\sqrt{-1})^n}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta\,(\sin\theta)^n}{1-\sqrt{-1}\sin\theta}; \tag{53.}$$

that is

$$\mathrm{G}(0) = \frac{(-1)^{i+1}}{\pi} \int_0^\pi \int_0^\pi \frac{d\theta\, \sin\theta^{2i}}{1+\sin\theta^2}; \tag{54.}$$

if n be either $= 2i-1$, or $= 2i$. Now, when $p = 0$, P_n is the coefficient of x^n in the development of $(1+x^2)^{-\frac{1}{2}}$; therefore,

$$\mathrm{P}_{2i-1} = 0, \quad \text{when} \quad p = 0, \tag{55.}$$

and, in the notation of factorials,

$$\mathrm{P}_{2i} = [0]^{-i}[-\tfrac{1}{2}]^i = (-1)^i \pi^{-1} \int_0^\pi d\theta\, \sin\theta^{2i}; \tag{56.}$$

so that, by (54.),

$$\mathrm{G}(0) = -(\mathrm{P}_{2i} + \mathrm{P}_{2i+2} + \ldots), \tag{57.}$$

when $p = 0$, and when n is either $2i$ or $2i-1$.

8. For the critical value $p = \sqrt{3}-1$, we have, by (38.),

$$\mathrm{F}(p) = \tfrac{1}{2}\Sigma_{(n)}{}_0^\infty \mathrm{P}_n; \tag{58.}$$

therefore, for the same value of p, by (46.),

$$\begin{aligned} \mathrm{G}(p) &= \tfrac{1}{2}\Sigma_{(n)}{}_0^{n-1} \mathrm{P}_n - \tfrac{1}{2}\Sigma_{(n)}{}_n^\infty \mathrm{P}_n \\ &= \tfrac{1}{2}(\mathrm{P}_0 + \mathrm{P}_1 + \ldots + \mathrm{P}_{n-1} - \mathrm{P}_n - \mathrm{P}_{n+1} - \ldots); \end{aligned} \tag{59.}$$

so that the discontinuous function G, like F, acquires, for the critical value of p, a value which is the semisum of those which it receives immediately before and afterwards.

9. We have seen that the sum of these two discontinuous integrals, F and G, is always equal to the sum of the first n terms of the series (12.), so that

$$\mathrm{F}(p) + \mathrm{G}(p) = \mathrm{P}_0 + \mathrm{P}_1 + \ldots + \mathrm{P}_{n-1}; \tag{60.}$$

and it may not be irrelevant to remark that this sum may be developed under this other form:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \frac{\vartheta^n - 1}{\vartheta - 1} = \Sigma_{(k)}\, {}_1^n [n]^k [0]^{-k} \mathrm{Q}_{k-1}; \tag{61.}$$

in which the factorial expression $[n]^k[0]^{-k}$ denotes the coefficient of x^k in the development of $(1+x)^n$; and

$$\mathrm{Q}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, (\vartheta - 1)^k. \tag{62.}$$

Thus

$$\left.\begin{aligned} &\mathrm{P}_0 = \mathrm{Q}_0; \\ &\mathrm{P}_0 + \mathrm{P}_1 = 2\mathrm{Q}_0 + \mathrm{Q}_1; \\ &\mathrm{P}_0 + \mathrm{P}_1 + \mathrm{P}_2 = 3\mathrm{Q}_0 + 3\mathrm{Q}_1 + \mathrm{Q}_2; \\ &\quad \&\mathrm{c.} \end{aligned}\right\} \tag{63.}$$

and consequently

$$\left.\begin{aligned} &\mathrm{P}_0 = \mathrm{Q}_0; \\ &\mathrm{P}_1 = \mathrm{Q}_0 + \mathrm{Q}_1; \\ &\mathrm{P}_2 = \mathrm{Q}_0 + 2\mathrm{Q}_1 + \mathrm{Q}_2; \\ &\quad \&\mathrm{c.} \end{aligned}\right\} \tag{64.}$$

which last expressions, indeed, follow immediately from the formula (10.).

10. With respect to the calculation of Q_0, Q_1, &c. as functions of p, it may be noted, in conclusion, that, by (15.) and (62.), Q_k is the term independent of θ in the development of

$$2^{-k}(\alpha + e^{\theta\sqrt{-1}})^k (1 - \beta e^{-\theta\sqrt{-1}})^k; \tag{65.}$$

thus

$$\left.\begin{aligned} &\mathrm{Q}_0 = 1, \\ &\mathrm{Q}_1 = 2^{-1}(\alpha - \beta), \\ &\mathrm{Q}_2 = 2^{-2}(\alpha^2 - 4\alpha\beta + \beta^2), \\ &\mathrm{Q}_3 = 2^{-3}(\alpha^3 - 9\alpha^2\beta + 9\alpha\beta^2 - \beta^3), \\ &\quad \&\mathrm{c.} \end{aligned}\right\} \tag{66.}$$

in which the law of formation is evident. It remains to substitute for α, β their values (16.) as functions of s, and then to eliminate s^2 by (17.); and thus we find, for example,

$$\left.\begin{aligned} &\mathrm{Q}_1 = p - 1, \\ &\mathrm{Q}_2 = \tfrac{1}{2}(p-1)(3p-1); \\ &\mathrm{Q}_3 = \tfrac{1}{2}(p-1)^2(5p+1); \\ &\mathrm{Q}_4 = \tfrac{1}{8}(p-1)^2(35p^2 - 10p - 13). \end{aligned}\right\} \tag{67.}$$

This, then, is at least one way, though perhaps not the easiest, of computing the initial values of the successive differences of the function P_n, that is, the quantities

$$\left.\begin{aligned} \mathrm{Q}_0 &= \Delta^0 \mathrm{P}_0 = \mathrm{P}_0, \\ \mathrm{Q}_1 &= \Delta^1 \mathrm{P}_0 = \mathrm{P}_1 - \mathrm{P}_0, \\ \mathrm{Q}_2 &= \Delta^2 \mathrm{P}_0 = \mathrm{P}_2 - 2\mathrm{P}_1 + \mathrm{P}_0, \\ &\&\mathrm{c}. \end{aligned}\right\} \tag{68.}$$

And we see that it is permitted to express generally those differences, as follows:

$$\Delta^k \mathrm{P}_0 = s^k \Sigma_{(i)}{}_0^k (-1)^i ([k]^i [0]^{-i})^2 (1+s)^i (1-s)^{k-i}; \tag{69.}$$

in which

$$s^2 = \tfrac{1}{2}(1-p). \tag{70.}$$

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