ON QUATERNIONS, OR ON A NEW SYSTEM OF IMAGINARIES IN ALGEBRA

By

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NOTE ON THE TEXT

The paper *On Quaternions; or on a new System of Imaginaries in Algebra*, by Sir William Rowan Hamilton, appeared in 18 instalments in volumes xxv–xxxvi of *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* (3rd Series), for the years 1844–1850. Each instalment (including the last) ended with the words '*To be continued*'.

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articles 68–70	May 1849	vol. xxxiv (1849), pp. 340–343,
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articles 88–90	April 1850	vol. xxxvi (1850), pp. 305–306.

(Articles 51–55 appeared in the supplementary number of the *Philosophical Magazine* which appeared at the end of 1847.)

Various errata noted by Hamilton have been corrected. In addition, the following corrections have been made:—

in the footnote to article 19, the title of the paper by Cayley in the *Philosophical Magazine* in February 1845 was wrongly given as "On Certain Results respecting Quaternions", but has been corrected to "On Certain Results related to Quaternions" (see the *Philosophical Magazine* (3rd series) vol. XXVI (1845), pp. 141–145);

a sentence has been added to the footnote to article 70, giving the date of the meeting of the Royal Irish Academy at which the relevant communication in fact took place (see the *Proceedings of the Royal Irish Academy*, vol. IV (1850), pp. 324–325).

In this edition, 'small capitals' (A, B, C, etc.) have been used throughout to denote points of space. (This is the practice in most of the original text in the *Philosophical Magazine*, but some of the early articles of this paper used normal-size roman capitals to denote points of space.)

The spelling 'co-ordinates' has been used throughout. (The hyphen was present in most instances of this word in the original text, but was absent in articles 12, 15 and 16.)

The paper *On Quaternions; or on a new System of Imaginaries in Algebra*, is included in *The Mathematical Papers of Sir William Rowan Hamilton*, vol. iii (Algebra), edited for the Royal Irish Academy by H. Halberstam and R. E. Ingram (Cambridge University Press, Cambridge, 1967).

David R. Wilkins
Dublin, March 2000

On Quaternions; or on a new System of Imaginaries in Algebra. By* Sir WILLIAM ROWAN HAMILTON, *LL.D., P.R.I.A., F.R.A.S., Hon. M. R. Soc. Ed. and Dub., Hon. or Corr. M. of the Royal or Imperial Academies of St. Petersburgh, Berlin, Turin, and Paris, Member of the American Academy of Arts and Sciences, and of other Scientific Societies at Home and Abroad, Andrews' Prof. of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

[*The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 1844–1850.]

1. Let an expression of the form

$$\mathrm{Q} = w + ix + jy + kz$$

be called a *quaternion*, when w, x, y, z, which we shall call the four *constituents* of the quaternion Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call *imaginary units*, and shall suppose to be unconnected by any linear relation with each other; in such a manner that if there be another expression of the same form,

$$\mathrm{Q}' = w' + ix' + jy' + kz',$$

the supposition of an equality between these two quaternions,

$$\mathrm{Q} = \mathrm{Q}',$$

shall be understood to involve four separate equations between their respective constituents, namely, the four following,

$$w = w', \quad x = x', \quad y = y', \quad z = z'.$$

It will then be natural to define that the *addition* or *subtraction* of quaternions is effected by the formula

$$\mathrm{Q} \pm \mathrm{Q}' = w \pm w' + i(x \pm x') + j(y \pm y') + k(z \pm z');$$

or, in words, by the rule, that *the sums or differences of the constituents of any two quaternions, are the constituents of the sum or difference of those two quaternions themselves.* It

* A communication, substantially the same with that here published, was made by the present writer to the Royal Irish Academy, at the first meeting of that body after the last summer recess, in November 1843.

will also be natural to define that the product QQ$'$, of the multiplication of Q as a multiplier into Q$'$ as a multiplicand, is capable of being thus expressed:

$$\begin{aligned}\mathrm{QQ}' = \quad & ww' + iwx' + jwy' + kwz' \\ & + ixw' + i^2xx' + ijxy' + ikxz' \\ & + jyw' + jiyx' + j^2yy' + jkyz' \\ & + kzw' + kizx' + kjzy' + k^2zz';\end{aligned}$$

but before we can reduce this product to an expression of the quaternion form, such as

$$\mathrm{QQ}' = \mathrm{Q}'' = w'' + ix'' + jy'' + kz'',$$

it is necessary to fix on quaternion-expressions (or on real values) for the nine squares or products,

$$i^2, \quad ij, \quad ik, \quad ji, \quad j^2, \quad jk, \quad ki, \quad kj, \quad k^2.$$

2. Considerations, which it might occupy too much space to give an account of on the present occasion, have led the writer to adopt the following system of values or expressions for these nine squares or products:

$$i^2 = j^2 = k^2 = -1; \tag{A.}$$

$$ij = k, \quad jk = i, \quad ki = j; \tag{B.}$$

$$ji = -k, \quad kj = -i, \quad ik = -j; \tag{C.}$$

though it must, at first sight, seem strange and almost unallowable, to define that the product of two imaginary factors in one order differs (in sign) from the product of the same factors in the opposite order ($ji = -ij$). It will, however, it is hoped, be allowed, that in entering on the discussion of a new system of imaginaries, it may be found necessary or convenient to surrender *some* of the expectations suggested by the previous study of products of real quantities, or even of expressions of the form $x + iy$, in which $i^2 = -1$. And whether the choice of the system of definitional equations, (A.), (B.), (C.), has been a judicious, or at least a happy one, will probably be judged by the event, that is, by trying whether those equations conduct to results of sufficient consistency and elegance.

3. With the assumed relations (A.), (B.), (C.), we have the four following expressions for the four constituents of the product of two quaternions, as functions of the constituents of the multiplier and multiplicand:

$$\left.\begin{aligned} w'' &= ww' - xx' - yy' - zz', \\ x'' &= wx' + xw' + yz' - zy', \\ y'' &= wy' + yw' + zx' - xz', \\ z'' &= wz' + zw' + xy' - yx'. \end{aligned}\right\} \tag{D.}$$

These equations give

$$w''^2 + x''^2 + y''^2 + z''^2 = (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2);$$

and therefore

$$\mu'' = \mu\mu', \tag{E.}$$

if we introduce a system of expressions for the constituents, of the forms

$$\left.\begin{aligned} w &= \mu\cos\theta, \\ x &= \mu\sin\theta\cos\phi, \\ y &= \mu\sin\theta\sin\phi\cos\psi, \\ z &= \mu\sin\theta\sin\phi\sin\psi, \end{aligned}\right\} \tag{F.}$$

and suppose each μ to be positive. Calling, therefore, μ the *modulus* of the quaternion Q, we have this theorem: that *the modulus of the product* Q$''$ *of any two quaternions* Q *and* Q$'$, *is equal to the product of their moduli.*

4. The equations (D.) give also

$$\begin{aligned} w'w'' + x'x'' + y'y'' + z'z'' &= w(w'^2 + x'^2 + y'^2 + z'^2), \\ ww'' + xx'' + yy'' + zz'' &= w'(w^2 + x^2 + y^2 + z^2); \end{aligned}$$

combining, therefore, these results with the first of those equations (D.), and with the trigonometrical expressions (F.), and the relation (E.) between the moduli, we obtain the three following relations between the angular co-ordinates $\theta\ \phi\ \psi$, $\theta'\ \phi'\ \psi'$, $\theta''\ \phi''\ \psi''$ of the two factors and the product:

$$\left.\begin{aligned} \cos\theta'' &= \cos\theta\cos\theta' - \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\psi - \psi')), \\ \cos\theta &= \cos\theta'\cos\theta'' + \sin\theta'\sin\theta''(\cos\phi'\cos\phi'' + \sin\phi'\sin\phi''\cos(\psi' - \psi'')), \\ \cos\theta' &= \cos\theta''\cos\theta + \sin\theta''\sin\theta(\cos\phi''\cos\phi + \sin\phi''\sin\phi\cos(\psi'' - \psi)). \end{aligned}\right\} \tag{G.}$$

These equations (G.) admit of a simple geometrical construction. Let $x\,y\,z$ be considered as the three rectangular co-ordinates of a point in space, of which the radius vector is $= \mu\sin\theta$, the longitude $= \psi$, and the co-latitude $= \phi$; and let these three latter quantities be called also the *radius vector*, the *longitude* and the *co-latitude of the quaternion* Q; while θ shall be called the *amplitude* of that quaternion. Let R be the point where the radius vector, prolonged if necessary, intersects the spheric surface described about the origin of co-ordinates with a radius = unity, so that ϕ is the co-latitude and ψ is the longitude of R; and let this point R be called the *representative point* of the quaternion Q. Let R$'$ and R$''$ be, in like manner, the representative points of Q$'$ and Q$''$; then the equations (G.) express that *in the spherical triangle* R R$'$ R$''$, *formed by the representative points of the two factors and the product* (in any multiplication of two quaternions), *the angles are respectively equal to the amplitudes of* *those two factors, and the supplement of the amplitude of the product* (to two right angles); in such a manner that we have the three equations:

$$\mathrm{R} = \theta, \quad \mathrm{R}' = \theta', \quad \mathrm{R}'' = \pi - \theta''. \tag{H.}$$

5. The system of the four very simple and easily remembered equations (E.) and (H.), may be considered as equivalent to the system of the four more complex equations (D.), and as containing within themselves a sufficient expression of the rules of multiplication of quaternions; with this exception, that they leave undetermined the *hemisphere* to which the point R'' belongs, or the *side* of the arc $\mathrm{R}\,\mathrm{R}'$ on which that *product-point* R'' falls, after the *factor-points* R and R', and the amplitudes θ and θ' have been assigned. In fact, the equations (E.) and (H.) have been obtained, not immediately from the equations (D.), but from certain combinations of the last-mentioned equations, which combinations would have been unchanged, if the signs of the three functions,

$$yz' - zy', \quad zx' - xz', \quad xy' - yx',$$

had all been changed together. This latter change would correspond to an alteration in the assumed conditions (B.) and (C.), such as would have consisted in assuming $ij = -k$, $ji = +k$, &c., that is, in taking the cyclical order $k\ j\ i$ (instead of $i\ j\ k$), as that in which the product of any two imaginary units (considered as multiplier and multiplicand) is equal to the imaginary unit following, taken positively. With this remark, it is not difficult to perceive that *the product-point* R'' *is always to be taken to the right, or always to the left of the multiplicand-point* R', *with respect to the multiplier-point* R, according as the semiaxis of $+z$ is to the right or left of the semiaxis of $+y$, with respect to the semiaxis of $+x$; or, in other words, according as the positive direction of rotation in longitude is to the right or to the left. This *rule of rotation*, combined with the *law of the moduli* and with the *theorem of the spherical triangle*, completes the transformed system of conditions, connecting the product of any two quaternions with the factors, and with their order.

6. It follows immediately from the principles already explained, that if $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''$ be any spherical triangle, and if $\alpha\,\beta\,\gamma$ be the rectangular co-ordinates of R, $\alpha'\,\beta'\,\gamma'$ of R', and $\alpha''\,\beta''\,\gamma''$ of R'', the centre O of the sphere being origin and the radius unity, and the positive semiaxis of z being so chosen as to lie to the right or left of the positive semiaxis of y, with respect to the positive semiaxis of x, according as the radius $\mathrm{O}\,\mathrm{R}''$ lies to the right or left of $\mathrm{O}\,\mathrm{R}'$ with respect to $\mathrm{O}\,\mathrm{R}$, then the following imaginary or symbolic *formula of multiplication* of quaternions will hold good:

$$\begin{aligned}\{\cos\mathrm{R} + (i\alpha + j\beta + k\gamma)\sin\mathrm{R}\}\{\cos\mathrm{R}' + (i\alpha' + j\beta' + k\gamma')\sin\mathrm{R}'\}& \\ = -\cos\mathrm{R}'' + (i\alpha'' + j\beta'' + k\gamma'')\sin\mathrm{R}'';&\end{aligned} \tag{I.}$$

the squares and products of the three imaginary units, i, j, k, being determined by the nine equations of definition, assigned in a former article, namely,

$$i^2 = j^2 = k^2 = -1; \tag{A.}$$

$$ij = k, \quad jk = i, \quad ki = j; \tag{B.}$$

$$ji = -k, \quad kj = -i, \quad ik = -j. \tag{C.}$$

Developing and decomposing the imaginary formula (I.) by these conditions, it resolves itself into the four following real equations of spherical trigonometry:

$$\left.\begin{aligned} -\cos \mathrm{R}'' &= \cos \mathrm{R} \cos \mathrm{R}' - (\alpha\alpha' + \beta\beta' + \gamma\gamma') \sin \mathrm{R} \sin \mathrm{R}', \\ \alpha'' \sin \mathrm{R}'' &= \alpha \sin \mathrm{R} \cos \mathrm{R}' + \alpha' \sin \mathrm{R}' \cos \mathrm{R} + (\beta\gamma' - \gamma\beta') \sin \mathrm{R} \sin \mathrm{R}', \\ \beta'' \sin \mathrm{R}'' &= \beta \sin \mathrm{R} \cos \mathrm{R}' + \beta' \sin \mathrm{R}' \cos \mathrm{R} + (\gamma\alpha' - \alpha\gamma') \sin \mathrm{R} \sin \mathrm{R}', \\ \gamma'' \sin \mathrm{R}'' &= \gamma \sin \mathrm{R} \cos \mathrm{R}' + \gamma' \sin \mathrm{R}' \cos \mathrm{R} + (\alpha\beta' - \beta\alpha') \sin \mathrm{R} \sin \mathrm{R}'; \end{aligned}\right\} \tag{K.}$$

of which indeed the first answers to the well-known relation (already employed in this paper), connecting a side with the three angles of a spherical triangle. The three other equations (K.) correspond to and contain a theorem (perhaps new), which may be enunciated thus: that if three forces be applied at the centre of the sphere, one equal to $\sin \mathrm{R} \cos \mathrm{R}'$ and directed to the point R, another equal to $\sin \mathrm{R}' \cos \mathrm{R}$ and directed to R', and the third equal to $\sin \mathrm{R} \sin \mathrm{R}' \sin \mathrm{R}\,\mathrm{R}'$ and directed to that pole of the arc $\mathrm{R}\,\mathrm{R}'$ which lies towards the same side of this arc as the point R'', the resultant of these three forces will be directed to R'', and will be equal to $\sin \mathrm{R}''$. It is not difficult to prove this theorem otherwise, but it may be regarded as interesting to see that the four real equations (K.) are all included so simply in the single imaginary formula (I.), and can so easily be deduced from that formula by the rules of the *multiplication of quaternions*; in the same manner as the fundamental theorems of plane trigonometry, for the cosine and sine of the sum of any two arcs, are included in the well-known formula for the multiplication of *couples*, that is, expressions of the form $x + iy$, or more particularly $\cos\theta + i \sin\theta$, in which $i^2 = -1$. A new sort of algorithm, or *calculus for spherical trigonometry*, would seem to be thus given or indicated.

And if we suppose the spherical triangle $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''$ to become indefinitely small, by each of its corners tending to coincide with the point of which the co-ordinates are 1, 0, 0, then each co-ordinate α will tend to become $= 1$, while each β and γ will ultimately vanish, and the sum of the three angles will approach indefinitely to the value π; the formula (I.) will therefore have for its limit the following,

$$(\cos \mathrm{R} + i \sin \mathrm{R})(\cos \mathrm{R}' + i \sin \mathrm{R}') = \cos(\mathrm{R} + \mathrm{R}') + i \sin(\mathrm{R} + \mathrm{R}'),$$

which has so many important applications in the usual theory of imaginaries.

7. In that theory there are only two different square roots of negative unity, and they differ only in their signs. In the theory of quaternions, in order that the square of $w + ix + jy + kz$ should be equal to -1, it is necessary and sufficient that we should have

$$w = 0, \quad x^2 + y^2 + z^2 = 1;$$

for, in general, by the expressions (D.) of this paper for the constituents of a product, or by the definitions (A.), (B.), (C.), we have

$$(w + ix + jy + kz)^2 = w^2 - x^2 - y^2 - z^2 + 2w(ix + jy + kz).$$

There are, therefore, in this theory, *infinitely many different square roots of negative one*, which have all one common modulus $= 1$, and one common amplitude $= \frac{\pi}{2}$, being all of the form

$$\sqrt{-1} = i \cos\phi + j \sin\phi \cos\psi + k \sin\phi \sin\psi, \tag{L.}$$

but which admit of all varieties of *directional co-ordinates*, that is to say, co-latitude and longitude, since ϕ and ψ are arbitrary; and we may call them all *imaginary units*, as well as the three original imaginaries, i, j, k, from which they are derived. To distinguish one such root or unit from another, we may denote the second member of the formula (L.) by $i_{\phi,\psi}$, or more concisely by i_{R}, if R denote (as before) that particular point of the spheric surface (described about the origin as centre with a radius equal to unity) which has its co-latitude $= \phi$, and its longitude $= \psi$. We shall then have

$$i_{\mathrm{R}} = i\alpha + j\beta + k\gamma, \quad i_{\mathrm{R}}^2 = -1, \tag{L$'$.}$$

in which

$$\alpha = \cos\phi, \quad \beta = \sin\phi\cos\psi, \quad \gamma = \sin\phi\sin\psi,$$

α, β, γ being still the rectangular co-ordinates of R, referred to the centre as their origin. The formula (I.) will thus become, for any spherical triangle,

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}') = -\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}''. \tag{I$'$.}$$

8. To *separate the real and imaginary parts* of this last formula, it is only necessary to effect a similar separation for the product of the two imaginary units which enter into the first member. By changing the angles R and R$'$ to right angles, without changing the points R and R$'$ upon the sphere, the imaginary units i_{R} and $i_{\mathrm{R}'}$ undergo no change, but the angle R$''$ becomes equal to the arc R R$'$, and the point R$''$ comes to coincide with the *positive pole* of that arc, that is, with the pole to which the least rotation from R$'$ round R is positive. Denoting then this pole by P$''$, we have the equation

$$i_{\mathrm{R}} i_{\mathrm{R}'} = -\cos \mathrm{R}\,\mathrm{R}' + i_{\mathrm{P}''} \sin \mathrm{R}\,\mathrm{R}', \tag{M.}$$

which is included in the formula (I$'$.), and reciprocally serves to transform it; for it shows that while the comparison of the real parts reproduces the known equation

$$\cos \mathrm{R} \cos \mathrm{R}' - \sin \mathrm{R} \sin \mathrm{R}' \cos \mathrm{R}\,\mathrm{R}' = -\cos \mathrm{R}'', \tag{K$'$.}$$

the comparison of the imaginary parts conducts to the following symbolic expression for the theorem of the 6th article:

$$i_{\mathrm{R}} \sin \mathrm{R} \cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}' \cos \mathrm{R} + i_{\mathrm{P}''} \sin \mathrm{R} \sin \mathrm{R}' \sin \mathrm{R}\,\mathrm{R}' = i_{\mathrm{R}''} \sin \mathrm{R}''. \tag{K$''$.}$$

As a verification we may remark, that when the triangle (and with it the arc R R$'$) tends to vanish, the two last equations tend to concur in giving the property of the plane triangle,

$$\mathrm{R} + \mathrm{R}' + \mathrm{R}'' = \pi.$$

9. The expression (M.) for the *product of any two imaginary units*, which admits of many applications, may be immediately deduced from the fundamental definitions (A.), (B.), (C.), respecting the squares and products of the three original imaginary units, i, j, k, by putting it under the form

$$\begin{aligned} &(i\alpha + j\beta + k\gamma)(i\alpha' + j\beta' + k\gamma') \\ &= -(\alpha\alpha' + \beta\beta' + \gamma\gamma') + i(\beta\gamma' - \gamma\beta') + j(\gamma\alpha' - \alpha\gamma') + k(\alpha\beta' - \beta\alpha'); \end{aligned} \qquad \text{(M}'\text{.)}$$

and it is evident, either from this last form or from considerations of rotation such as those already explained, that if the *order* of any two pure imaginary factors be changed, the real part of the product remains unaltered, but the imaginary part changes sign, in such a manner that the equation (M.) may be combined with this other analogous equation,

$$i_{\mathrm{R}'} i_{\mathrm{R}} = -\cos \mathrm{R\,R}' - i_{\mathrm{P}''} \sin \mathrm{R\,R}'. \qquad \text{(N.)}$$

In fact, we may consider $-i_{\mathrm{P}''}$ as $= i_{\mathrm{P}''_{\prime}}$, if $\mathrm{P}''_{\prime}$ be the point diametrically opposite to P'', and consequently the positive pole of the reversed arc $\mathrm{R}'\mathrm{R}$ (in the sense already determined), though it is the *negative pole* of the arc $\mathrm{R\,R}'$ taken in its former direction.

And since in general the product of any two quaternions, which differ only in the signs of their imaginary parts, is real and equal to the square of the modulus, or in symbols,

$$\mu(\cos\theta + i_{\mathrm{R}} \sin\theta) \times \mu(\cos\theta - i_{\mathrm{R}} \sin\theta) = \mu^2, \qquad \text{(O.)}$$

we see that the product of the two different products, (M.) and (N.), obtained by multiplying any two imaginary units together in different orders, is real and equal to unity, in such a manner that we may write

$$i_{\mathrm{R}} i_{\mathrm{R}'} \,.\, i_{\mathrm{R}'} i_{\mathrm{R}} = 1; \qquad \text{(P.)}$$

and the two quaternions, represented by the two products $i_{\mathrm{R}} i_{\mathrm{R}'}$ and $i_{\mathrm{R}'} i_{\mathrm{R}}$, may be said to be *reciprocals* of each other. For example, it follows immediately from the fundamental definitions (A.), (B.), (C.), that

$$ij \,.\, ji = k \times -k = -k^2 = 1;$$

the products ij and ji are therefore reciprocals, in the sense just now explained. By supposing the two imaginary factors, i_{R} and $i_{\mathrm{R}'}$, to be mutually *rectangular*, that is, the arc $\mathrm{R\,R}' =$ a quadrant, the two products (M.) and (N.) become $\pm i_{\mathrm{P}''}$; and thus, or by a process more direct, we might show that if two imaginary units be mutually *opposite* (one being the negative of the other), they are also mutually *reciprocal.*

10. The equation (P.), which we shall find to be of use in the *division of quaternions*, may be proved in a more purely algebraical way, or at least in one more abstracted from considerations of directions in space, as follows. It will be found that, in virtue of the definitions (A.), (B.), (C.), every equation of the form

$$\iota \,.\, \kappa\lambda = \iota\kappa \,.\, \lambda$$

is true, if the three factors ι, κ, λ, whether equal or unequal among themselves, be equal, each, to one or other of the three imaginary units i, j, k; thus, for example,

$$i \,.\, jk = (\, i \,.\, i = -1 = k \,.\, k =)\, ij \,.\, k,$$

$$j \,.\, ji = (\, j \,.\, -k = -i = -1 \,.\, i =)\, jj \,.\, i.$$

Hence, whatever three quaternions may be denoted by Q, Q′, Q″, we have the equation

$$\mathrm{Q} \,.\, \mathrm{Q}'\mathrm{Q}'' = \mathrm{QQ}' \,.\, \mathrm{Q}''; \tag{Q.}$$

and in like manner, for any four quaternions,

$$\mathrm{Q} \,.\, \mathrm{Q}'\mathrm{Q}''\mathrm{Q}''' = \mathrm{QQ}' \,.\, \mathrm{Q}''\mathrm{Q}''' = \mathrm{QQ}'\mathrm{Q}'' \,.\, \mathrm{Q}''', \tag{Q′.}$$

and so on for any number of factors; the notation $\mathrm{QQ}'\mathrm{Q}''$ being employed, in the formula (Q′.), to denote that one determined quaternion which, in virtue of the theorem (Q.), is obtained, whether we first multiply Q″ as a multiplicand by Q′ as a multiplier, and then multiply the product $\mathrm{Q}'\mathrm{Q}''$ as a multiplicand by Q as a multiplier; or multiply first Q′ by Q, and then Q″ by QQ′. With the help of this principle we may easily prove the equation (P.), by observing that

$$i_{\mathrm{R}} i_{\mathrm{R}'} \,.\, i_{\mathrm{R}'} i_{\mathrm{R}} = i_{\mathrm{R}} \,.\, i_{\mathrm{R}'}^2 \,.\, i_{\mathrm{R}} = -i_{\mathrm{R}}^2 = +1.$$

11. The theorem expressed by the formulæ (Q.), (Q′), &c., is of great importance in the *calculus of quaternions*, as tending (so far as it goes) to assimilate this system of calculations to that employed in ordinary algebra. In ordinary multiplication we may distribute any factor into any number of parts, real or imaginary, and collect the partial products; and the same process is allowed in operating on quaternions: *quaternion-multiplication* possesses therefore the *distributive* character of multiplication commonly so called, or in symbols,

$$\mathrm{Q}(\mathrm{Q}' + \mathrm{Q}'') = \mathrm{QQ}' + \mathrm{QQ}'', \quad (\mathrm{Q} + \mathrm{Q}')\mathrm{Q}'' = \mathrm{QQ}'' + \mathrm{Q}'\mathrm{Q}'', \quad \text{\&c.}$$

But in ordinary algebra we have also

$$\mathrm{QQ}' = \mathrm{Q}'\mathrm{Q};$$

which equality of products of factors, taken in opposite orders, does *not* in general hold good for quaternions ($ji = -ij$); the *commutative* character of ordinary multiplication is therefore in general *lost* in passing to the new operation, and $\mathrm{QQ}' - \mathrm{Q}'\mathrm{Q}$, instead of being a symbol of zero, comes to represent a pure imaginary, but not (in general) an evanescent quantity. On the other hand, for quaternions as for ordinary factors, we *may* in general *associate the factors among themselves, by groups, in any manner which does not alter their order*; for example,

$$\mathrm{Q} \,.\, \mathrm{Q}'\mathrm{Q}'' \,.\, \mathrm{Q}'''\mathrm{Q}^{\mathrm{IV}} = \mathrm{QQ}' \,.\, \mathrm{Q}''\mathrm{Q}'''\mathrm{Q}^{\mathrm{IV}};$$

this, therefore, which may be called the *associative* character of the operation, is (like the distributive character) *common* to the multiplication of quaternions, and to that of ordinary quantities, real or imaginary.

12. A quaternion, Q, divided by its modulus, μ, may in general (by what has been shown) be put under the form,

$$\mu^{-1}\mathrm{Q} = \cos\theta + i_{\mathrm{R}} \sin\theta;$$

in which θ is a real quantity, namely the amplitude of the quaternion; and i_{R} is an imaginary unit, or square root of a negative one, namely that particular root, or unit, which is distinguished from all others by its two directional co-ordinates, and is constructed by a straight line drawn from the origin of co-ordinates to the representative point R; this point R being on the spheric surface which is described about the origin as centre, with a radius equal to unity. Comparing this expression for $\mu^{-1}\mathrm{Q}$ with the formula (M.) for the product of any two imaginary units, we see that if with the point R as a positive pole, we describe on the same spheric surface an arc $\mathrm{P}'\mathrm{P}''$ of a great circle, and take this arc $= \pi - \theta =$ the supplement of the amplitude of Q; and then consider the points P' and P'' as the representative points of two new imaginary units $i_{\mathrm{P}'}$ and $i_{\mathrm{P}''}$, we shall have the following *general transformation for any given quaternion*,

$$\mathrm{Q} = \mu i_{\mathrm{P}'} i_{\mathrm{P}''};$$

the arc $\mathrm{P}'\mathrm{P}''$ being given in length and in direction, except that it may turn round in its own plane (or on the great circle to which it belongs), and may be increased or diminished by any whole number of circumferences, without altering the value of Q.

13. Consider now the *product of several successive quaternion factors* $\mathrm{Q}_1, \mathrm{Q}_2, \ldots$ under the condition that their amplitudes $\theta_1, \theta_2, \ldots$ shall be respectively equal to the angles of the spherical polygon which is formed by their representative points $\mathrm{R}_1, \mathrm{R}_2, \ldots$ taken in their order. To fix more precisely what is to be understood in speaking here of these angles, suppose that R_m is the representative point of the mth quaternion factor, or the mth corner of the polygon, the next preceding corner being R_{m-1}, and the next following being R_{m+1}; and let the angle, or (more fully) the *internal angle*, of the polygon, at the point R_m, be denoted by the same symbol R_m, and be defined to be the least angle of rotation through which the arc $\mathrm{R}_m\mathrm{R}_{m+1}$ must revolve in the positive direction round the point R_m, in order to come into the direction of the arc $\mathrm{R}_m\mathrm{R}_{m-1}$. Then, the rotation $2\pi - \mathrm{R}_m$ would bring $\mathrm{R}_m\mathrm{R}_{m-1}$ to coincide in direction with $\mathrm{R}_m\mathrm{R}_{m+1}$; and therefore the rotation $\pi - \mathrm{R}_m$, performed in the same sense or in the opposite, according as it is positive or negative, would bring the *prolongation* of the preceding arc $\mathrm{R}_{m-1}\,\mathrm{R}_m$ to coincide in direction with the following arc $\mathrm{R}_m\mathrm{R}_{m+1}$; on which account we shall call this angle $\pi - \mathrm{R}_m$, taken with its proper sign, the *external angle* of the polygon at the point R_m. The same rotation $\pi - \mathrm{R}_m$ would bring the positive pole, which we shall call P_{m+1}, of the preceding side $\mathrm{R}_{m-1}\,\mathrm{R}_m$ of the polygon, to coincide with the positive pole P_{m+2} of the following side $\mathrm{R}_m\mathrm{R}_{m+1}$ thereof, by turning round the corner R_m as a pole, in an arc of a great circle, and in a positive or negative direction of rotation according as the external angle $\pi - \mathrm{R}_m$ of the polygon is itself positive or negative; consequently, by the last article, we shall have the formula

$$\mu_m^{-1}\mathrm{Q}_m = \cos\mathrm{R}_m + i_{\mathrm{R}_m} \sin\mathrm{R}_m = i_{\mathrm{P}_{m+1}} i_{\mathrm{P}_{m+2}}.$$

Multiplying together in their order the n formulæ of this sort for the n corners of the polygon, and attending to the *associative* character of quaternion multiplication, which gives, as an extension of the formula (P.), the following,

$$i_{\mathrm{P}_1} i_{\mathrm{P}_2} \,.\, i_{\mathrm{P}_2} i_{\mathrm{P}_3} \,.\,.\,.\,.\, i_{\mathrm{P}_n} i_{\mathrm{P}_1} = (-1)^n, \tag{P$'$.}$$

we see that under the supposed conditions as to the amplitudes we have this expression for the product of the n quaternion factors,

$$\mathrm{Q}_1 \mathrm{Q}_2 \mathrm{Q}_3 \ \ldots\ \mathrm{Q}_n = (-1)^n \mu_1 \mu_2 \mu_3 \ \ldots\ \mu_n;$$

from which it follows, that for *any spherical polygon* $\mathrm{R}_1 \mathrm{R}_2 \ldots \mathrm{R}_n$, (even with salient and re-entrant angles), this general equation holds good:

$$(\cos \mathrm{R}_1 + i_{\mathrm{R}_1} \sin \mathrm{R}_1)(\cos \mathrm{R}_2 + i_{\mathrm{R}_2} \sin \mathrm{R}_2) \ \ldots\ (\cos \mathrm{R}_n + i_{\mathrm{R}_n} \sin \mathrm{R}_n) = (-1)^n. \tag{R.}$$

14. For the case of a spherical triangle $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''$, this relation becomes

$$(\cos \mathrm{R} + i_{\mathrm{R}} \sin \mathrm{R})(\cos \mathrm{R}' + i_{\mathrm{R}'} \sin \mathrm{R}')(\cos \mathrm{R}'' + i_{\mathrm{R}''} \sin \mathrm{R}'') = -1; \tag{I$''$.}$$

and reproduces the formula (I$'$.), when we multiply each member, as multiplier, into $\cos \mathrm{R}'' - i_{\mathrm{R}''} \sin \mathrm{R}''$ as multiplicand. The restriction, mentioned in a former article, on the direction of the positive semiaxis of one co-ordinate after those of the two other co-ordinates had been chosen, was designed merely to enable us to consider the three angles of the triangle as being each positive and less than two right angles, according to the usage commonly adopted by writers on spherical trigonometry. It would not have been difficult to deduce reciprocally the theorem (R.) for any spherical polygon, from the less general relation (I$'$.) or (I$''$.) for the case of a spherical triangle, by assuming any point P upon the spherical surface as the common vertex of n triangles which have the sides of the polygon for their n bases, and by employing the associative character of multiplication, together with the principle that codirectional quaternions, when their moduli are supposed each equal to unity, are multiplied by adding their amplitudes. This last principle gives also, as a verification of the formula (R.), for the case of an infinitely small, or in other words, a *plane polygon*, the known equations,

$$\cos \Sigma \mathrm{R} = (-1)^n, \quad \sin \Sigma \mathrm{R} = 0.$$

15. The associative character of multiplication, or the formula (Q.), shows that if we assume any three quaternions Q, Q$'$, Q$''$, and derive two others $\mathrm{Q}_{\prime}$, $\mathrm{Q}_{\prime\prime}$ from them, by the equations

$$\mathrm{Q}\mathrm{Q}' = \mathrm{Q}_{\prime}, \quad \mathrm{Q}'\mathrm{Q}'' = \mathrm{Q}_{\prime\prime},$$

we shall have also the equations

$$\mathrm{Q}_{\prime}\mathrm{Q}'' = \mathrm{Q}\mathrm{Q}_{\prime\prime} = \mathrm{Q}''',$$

Q$'''$ being a third derived quaternion, namely the ternary product $\mathrm{Q}\mathrm{Q}'\mathrm{Q}''$. Let $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''$ $\mathrm{R}_{\prime}\,\mathrm{R}_{\prime\prime}\,\mathrm{R}'''$ be the six representative points of these six quaternions, on the same spheric surface as before; then, by the general construction of a product assigned in a former article*, we shall have the following expressions for the six amplitudes of the same six quaternions:

$$\begin{aligned}
\theta &= \mathrm{R}'\mathrm{R}\mathrm{R}_{\prime} = \mathrm{R}_{\prime\prime}\,\mathrm{R}\,\mathrm{R}'''; &\quad \theta_{\prime} &= \mathrm{R}''\mathrm{R}_{\prime}\mathrm{R}''' = \pi - \mathrm{R}\,\mathrm{R}_{\prime}\mathrm{R}';\\
\theta' &= \mathrm{R}_{\prime}\mathrm{R}'\mathrm{R} = \mathrm{R}''\mathrm{R}'\mathrm{R}_{\prime\prime}; &\quad \theta_{\prime\prime} &= \mathrm{R}'''\mathrm{R}_{\prime\prime}\,\mathrm{R} = \pi - \mathrm{R}'\mathrm{R}_{\prime\prime}\,\mathrm{R}'';\\
\theta'' &= \mathrm{R}_{\prime\prime}\,\mathrm{R}''\mathrm{R}' = \mathrm{R}'''\mathrm{R}''\mathrm{R}_{\prime}; &\quad \theta''' &= \pi - \mathrm{R}_{\prime}\mathrm{R}'''\mathrm{R}'' = \pi - \mathrm{R}\,\mathrm{R}'''\mathrm{R}_{\prime\prime};
\end{aligned}$$

$\mathrm{R}'\,\mathrm{R}\,\mathrm{R}_{\prime}$ being the spherical angle at R, measured from $\mathrm{R}\,\mathrm{R}'$ to $\mathrm{R}\,\mathrm{R}_{\prime}$, and similarly in other cases. But these equations between the spherical angles of the figure are precisely those which are requisite in order that the two points $\mathrm{R}_{\prime}$ and $\mathrm{R}_{\prime\prime}$ should be the *two foci of a spherical conic inscribed in the spherical quadrilateral* $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''\,\mathrm{R}'''$, or touched by the four great circles of which the arcs $\mathrm{R}\,\mathrm{R}'$, $\mathrm{R}'\,\mathrm{R}''$, $\mathrm{R}''\mathrm{R}'''$, $\mathrm{R}'''\mathrm{R}$ are parts; this geometrical relation between the six representative points $\mathrm{R}\,\mathrm{R}'\,\mathrm{R}''\,\mathrm{R}_{\prime}\,\mathrm{R}_{\prime\prime}\,\mathrm{R}'''$ of the six quaternions Q, Q′, Q″, QQ′, Q′Q″, QQ′Q″, which may conveniently be thus denoted,

$$\mathrm{R}_{\prime}\mathrm{R}_{\prime\prime}(\,.\,.\,)\mathrm{R}\mathrm{R}'\mathrm{R}''\mathrm{R}''', \tag{Q″.}$$

is therefore a consequence, and may be considered as an interpretation, of the very simple algebraical theorem for three quaternion factors,

$$\mathrm{QQ}'\,.\,\mathrm{Q}'' = \mathrm{Q}\,.\,\mathrm{Q}'\mathrm{Q}''. \tag{Q.}$$

It follows at the same time, from the theory of spherical conics, that the two straight lines, or *radii vectores*, which are drawn from the origin of co-ordinates to the points $\mathrm{R}_{\prime}$, $\mathrm{R}_{\prime\prime}$, and *which construct the imaginary parts of the two binary quaternion products* QQ′, Q′Q″, *are the two focal lines of a cone of the second degree, inscribed in the pyramid which has for its four edges the four radii which construct the imaginary parts of the three quaternion factors* Q, Q′, Q″, *and of their continued (or ternary) product* QQ′Q″.

16. We had also, by the same associative character of multiplication, analogous formulæ for any four independent factors,

$$\mathrm{Q}\,.\,\mathrm{Q}'\mathrm{Q}''\mathrm{Q}''' = \mathrm{QQ}'\,.\,\mathrm{Q}''\mathrm{Q}''' = \&\text{c.}; \tag{Q′.}$$

if then we denote this continued product by Q^{IV}, and make

$$\mathrm{QQ}' = \mathrm{Q}_{\prime}, \quad \mathrm{Q}'\mathrm{Q}'' = \mathrm{Q}_{\prime}', \quad \mathrm{Q}''\mathrm{Q}''' = \mathrm{Q}_{\prime}'',$$

$$\mathrm{QQ}'\mathrm{Q}'' = \mathrm{Q}_{\prime}''', \quad \mathrm{Q}'\mathrm{Q}''\mathrm{Q}''' = \mathrm{Q}_{\prime}^{\mathrm{IV}},$$

and observe that whenever E and F are foci of a spherical conic inscribed in a spherical quadrilateral ABCD, so that, in the notation recently proposed,

$$\mathrm{EF}(\,.\,.\,)\mathrm{ABCD},$$

* In the Number of this Magazine for July 1844, S. 3. vol. XXV.

then also we may write

$$\text{FE}(\,.\,.\,)\text{ABCD}, \quad \text{and} \quad \text{EF}(\,.\,.\,)\text{BCDA},$$

we shall find, without difficulty, by the help of the formula (Q″.), the five following geometrical relations, in which each R is the representative point of the corresponding quaternion Q:

$$\left.\begin{array}{l} \text{R}_{\prime}\text{R}'_{\prime}(\,.\,.\,)\text{RR}'\text{R}''\text{R}'''_{\prime}; \\ \text{R}'_{\prime}\text{R}''_{\prime}(\,.\,.\,)\text{R}'\text{R}''\text{R}'''\text{R}^{\text{IV}}_{\prime}; \\ \text{R}''_{\prime}\text{R}'''_{\prime}(\,.\,.\,)\text{R}''\text{R}'''\text{R}^{\text{IV}}\text{R}_{\prime}; \\ \text{R}'''_{\prime}\text{R}^{\text{IV}}_{\prime}(\,.\,.\,)\text{R}'''\text{R}^{\text{IV}}\text{RR}'_{\prime}; \\ \text{R}^{\text{IV}}_{\prime}\text{R}_{\prime}(\,.\,.\,)\text{R}^{\text{IV}}\text{RR}'\text{R}''_{\prime}. \end{array}\right\} \qquad (\text{Q}'''.)$$

These five formulæ establish a remarkable *connexion between one spherical pentagon and another* (when constructed according to the foregoing rules), through the medium of *five spherical conics*; of which five curves each touches two sides of one pentagon and has its foci at two corners of the other. If we suppose for simplicity that each of the ten moduli is $= 1$, the dependence of six quaternions by multiplication on four (as their three binary, two ternary, and one quaternary product, all taken without altering the order of succession of the factors) will give eighteen distinct equations between the ten amplitudes and the twenty polar co-ordinates of the ten quaternions here considered; it is therefore in general permitted to assume at pleasure twelve of these co-ordinates, or to choose six of the ten points upon the sphere. Not only, therefore, may we in general take *one of the two pentagons arbitrarily*, but also, at the same time, may assume *one corner of the other pentagon* (subject of course to exceptional cases); and, after a suitable choice of the ten amplitudes, the five relations (Q‴.), between the two pentagons and the five conics, will still hold good.

17. A very particular (or rather limiting) yet not inelegant case of this theorem is furnished by the consideration of the plane and regular pentagon of elementary geometry, as compared with that other and interior pentagon which is determined by the intersections of its five diagonals. Denoting by $\text{R}_{\prime}$ that corner of the interior pentagon which is nearest to the side $\text{R}\,\text{R}'$ of the exterior one; by $\text{R}'_{\prime}$ that corner which is nearest to $\text{R}'\text{R}''$, and so on to $\text{R}^{\text{IV}}_{\prime}$; the relations (Q‴) are satisfied, the symbol (. .) now denoting that the two points written before it are foci of an ordinary (or plane) ellipse, inscribed in the plane quadrilateral whose corners are the four points written after it. We may add, that (in this particular case) two points of contact for each of the five quadrilaterals are corners of the interior pentagon; and that the axis major of each of the five inscribed ellipses is equal to a side of the exterior figure.

18. The separation of the real and imaginary parts of a quaternion is an operation of such frequent occurrence, and may be regarded as being so fundamental in this theory, that it is convenient to introduce symbols which shall denote concisely the two separate results of this operation. The algebraically *real* part may receive, according to the question in which it occurs, all values contained on the one *scale* of progression of number from negative to positive infinity; we shall call it therefore the *scalar part*, or simply the *scalar* of the quaternion, and shall form its symbol by prefixing, to the symbol of the quaternion, the characteristic Scal., or

simply S., where no confusion seems likely to arise from using this last abbreviation. On the other hand, the algebraically *imaginary* part, being geometrically constructed by a straight line, or radius vector, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the *vector part*, or simply the *vector* of the quaternion; and may be denoted by prefixing the characteristic Vect., or V. We may therefore say that *a quaternion is in general the sum of its own scalar and vector parts*, and may write

$$\mathrm{Q} = \mathrm{Scal.\,Q} + \mathrm{Vect.\,Q} = \mathrm{S\,.\,Q} + \mathrm{V\,.\,Q},$$

or simply

$$\mathrm{Q} = \mathrm{SQ} + \mathrm{VQ}.$$

By detaching the characteristics of operation from the signs of the operands, we may establish, for this notation, the general formulæ:

$$1 = \mathrm{S} + \mathrm{V}; \quad 1 - \mathrm{S} = \mathrm{V}; \quad 1 - \mathrm{V} = \mathrm{S};$$

$$\mathrm{S\,.\,S} = \mathrm{S}; \quad \mathrm{S\,.\,V} = 0; \quad \mathrm{V\,.\,S} = 0; \quad \mathrm{V\,.\,V} = \mathrm{V};$$

and may write

$$(\mathrm{S} + \mathrm{V})^n = 1,$$

if n be any positive whole number. The scalar or vector of a sum or difference of quaternions is the sum or difference of the scalars or vectors of those quaternions, which we may express by writing the formulæ:

$$\mathrm{S}\Sigma = \Sigma\mathrm{S}; \quad \mathrm{S}\Delta = \Delta\mathrm{S}; \quad \mathrm{V}\Sigma = \Sigma\mathrm{V}; \quad \mathrm{V}\Delta = \Delta\mathrm{V}.$$

19. Another general decomposition of a quaternion, into factors instead of summands, may be obtained in the following way:—Since the square of a scalar is always positive, while the square of a vector is always negative, the algebraical excess of the former over the latter square is always a positive number; if then we make

$$(\mathrm{TQ})^2 = (\mathrm{SQ})^2 - (\mathrm{VQ})^2,$$

and if we suppose TQ to be always a real and positive or absolute number, which we may call the *tensor* of the quaternion Q, we shall not thereby diminish the generality of that quaternion. The *tensor* is what was called in former articles the *modulus*;* but there seem to be some conveniences in not obliging ourselves to retain here a term which has been

* The writer believes that what originally led him to use the terms "modulus" and "amplitude," was a recollection of M. Cauchy's nomenclature respecting the usual imaginaries of algebra. It was the use made by his friend John T. Graves, Esq., of the word "constituents," in connexion with the ordinary imaginary expressions of the form $x + \sqrt{-1}\,y$, which led Sir William Hamilton to employ the same term in connexion with his own imaginaries. And if he had not come to prefer to the word "modulus," in this theory, the name "tensor," which suggested the characteristic T, he would have borrowed the symbol M, with the same signifi-

used in several other senses by writers on other subjects; and the word tensor has (it is conceived) some reasons in its favour, which will afterwards more fully appear. Meantime we may observe, as some justification of the use of this word, or at least as some assistance to the memory, that it enables us to say that *the tensor of a pure imaginary*, or vector, is the number expressing the *length* or linear *extension of the straight line* by which that algebraical imaginary is geometrically constructed. If such an imaginary be divided by its own tensor, the quotient is an imaginary or vector *unit*, which marks the *direction* of the constructing line, or the region of space towards which that line is *turned*; hence, and for other reasons, we propose to call this quotient the *versor* of the pure imaginary: and generally to say that *a quaternion is the product of its own tensor and versor factors*, or to write

$$\mathrm{Q} = \mathrm{TQ} \,.\, \mathrm{UQ},$$

using U for the characteristic of versor, as T for that of tensor. This is the other general decomposition of a quaternion, referred to at the beginning of the present article; and in the same notation we have

$$\mathrm{T} \,.\, \mathrm{TQ} = \mathrm{TQ}; \quad \mathrm{T} \,.\, \mathrm{UQ} = 1; \quad \mathrm{U} \,.\, \mathrm{TQ} = 1; \quad \mathrm{U} \,.\, \mathrm{UQ} = \mathrm{UQ};$$

so that the tensor of a versor, or the versor of a tensor, is unity, as it was seen that the scalar of a vector, or the vector of a scalar, is zero.

The tensor of a positive scalar is equal to that scalar itself, but the tensor of a negative scalar is equal to the positive opposite thereof. The versor of a positive or negative scalar is equal to positive or negative unity; and in general, by what has been shown in the 12th article, the versor of a quaternion is the product of two imaginary units. The tensor and versor of a vector have been considered in the present article. A tensor cannot become equal to a versor, except by each becoming equal to positive unity; as a scalar and a vector cannot be equal to each other, unless each reduces itself to zero.

20. If we call two quaternions *conjugate* when they have the same scalar part, but have opposite vector parts, then because, by the last article,

$$(\mathrm{TQ})^2 = (\mathrm{SQ} + \mathrm{VQ})(\mathrm{SQ} - \mathrm{VQ}),$$

we may say that the *product of two conjugate quaternions*, $\mathrm{SQ} + \mathrm{VQ}$ and $\mathrm{SQ} - \mathrm{VQ}$, is equal to the *square of their common tensor*, TQ; from which it follows that *conjugate versors are*

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cation, from the valuable paper by Mr. Cayley, "On Certain Results relating to Quaternions," which appeared in the Number of this Magazine for February 1845. It will be proposed by the present writer, in a future article, to call the *logarithmic modulus* the "mensor" of a quaternion, and to denote it by the foregoing characteristic M; so as to have

$$\mathrm{MQ} = \log.\,\mathrm{TQ}, \quad \mathrm{M} \,.\, \mathrm{QQ}' = \mathrm{MQ} + \mathrm{MQ}'.$$

the reciprocals of each other, one quaternion being called the *reciprocal* of another when their product is positive unity. If Q and Q′ be any two quaternions, the two products of their vectors, taken in opposite orders, namely $\mathrm{VQ}\,.\,\mathrm{VQ}'$ and $\mathrm{VQ}'\,.\,\mathrm{VQ}$, are conjugate quaternions, by the definition given above, and by the principles of the 9th article; and the conjugate of the sum of any number of quaternions is equal to the sum of their conjugates; therefore the products

$$(\mathrm{SQ}+\mathrm{VQ})(\mathrm{SQ}'+\mathrm{VQ}') \quad \text{and} \quad (\mathrm{SQ}'-\mathrm{VQ}')(\mathrm{SQ}-\mathrm{VQ})$$

are conjugate; therefore $\mathrm{T}\,.\,\mathrm{QQ}'$, which is the tensor of the first, is equal to the square root of their product, that is, of

$$(\mathrm{SQ}+\mathrm{VQ})(\mathrm{TQ}')^2(\mathrm{SQ}-\mathrm{VQ}), \quad \text{or of} \quad (\mathrm{TQ})^2(\mathrm{TQ}')^2;$$

we have therefore the formula

$$\mathrm{T}\,.\,\mathrm{QQ}' = \mathrm{TQ}\,.\,\mathrm{TQ}',$$

which gives also

$$\mathrm{U}\,.\,\mathrm{QQ}' = \mathrm{UQ}\,.\,\mathrm{UQ}';$$

that is to say, the *tensor of the product* of any two quaternions is equal to the *product of the tensors*, and in like manner the *versor of the product* is equal to the *product of the versors.* Both these results may easily be extended to any number of factors, and by using Π as the characteristic of a product, we may write, generally,

$$\mathrm{T}\Pi\mathrm{Q} = \Pi\mathrm{TQ}; \quad \mathrm{U}\Pi\mathrm{Q} = \Pi\mathrm{UQ}.$$

It was indeed shown, so early as in the 3rd article, that the modulus of a product is equal to the product of the moduli; but the process by which an equivalent result has been here deduced does not essentially depend upon that earlier demonstration: it has also the advantage of showing that *the continued product of any number of quaternion factors is conjugate to the continued product of the respective conjugates of those factors, taken in the opposite order*; so that we may write

$$(\mathrm{S}-\mathrm{V})\,.\,\mathrm{QQ}'\mathrm{Q}''\,\ldots = \ldots\,(\mathrm{SQ}''-\mathrm{VQ}'')(\mathrm{SQ}'-\mathrm{VQ}')(\mathrm{SQ}-\mathrm{VQ}),$$

a formula which, when combined with this other,

$$(\mathrm{S}+\mathrm{V})\,.\,\mathrm{QQ}'\mathrm{Q}''\,\ldots = (\mathrm{SQ}+\mathrm{VQ})(\mathrm{SQ}'+\mathrm{VQ}')(\mathrm{SQ}''+\mathrm{VQ}'')\,\ldots,$$

enables us easily to develope $\mathrm{S}\Pi\mathrm{Q}$ and $\mathrm{V}\Pi\mathrm{Q}$, that is, the scalar and vector of any product of quaternions, in terms of the scalars and vectors of the several factors of that product. For example, if we agree to use, in these calculations, the small Greek letters α, β, &c., with or without accents, as symbols of vectors (with the exception of π, and with a few other exceptions, which shall be either expressly mentioned as they occur, or clearly indicated by the context), we may form the following table:—

$$\begin{array}{rr}
2\mathrm{S}\,.\,\alpha = \alpha - \alpha = 0; & 2\mathrm{V}\,.\,\alpha = \alpha + \alpha = 2\alpha; \\
2\mathrm{S}\,.\,\alpha\alpha' = \alpha\alpha' + \alpha'\alpha; & 2\mathrm{V}\,.\,\alpha\alpha' = \alpha\alpha' - \alpha'\alpha; \\
2\mathrm{S}\,.\,\alpha\alpha'\alpha'' = \alpha\alpha'\alpha'' - \alpha''\alpha'\alpha; & 2\mathrm{V}\,.\,\alpha\alpha'\alpha'' = \alpha\alpha'\alpha'' + \alpha''\alpha'\alpha; \\
\&\text{c}. & \&\text{c}.
\end{array}$$

of which the law is evident.

21. The fundamental rules of multiplication in this calculus give, in the recent notation, for the scalar and vector parts of the product of any two vectors, the expressions,

$$\mathrm{S}\,.\,\alpha\alpha' = -(xx' + yy' + zz');$$
$$\mathrm{V}\,.\,\alpha\alpha' = i(yz' - zy') + j(zx' - xz') + k(xy' - yz');$$

if we make

$$\alpha = ix + jy + kz, \quad \alpha' = ix' + jy' + kz',$$

x, y, z and x', y', z' being real and rectangular co-ordinates, while i, j, k are the original imaginary units of this theory. The *geometrical meanings* of the symbols $\mathrm{S}\,.\,\alpha\alpha'$, $\mathrm{V}\,.\,\alpha\alpha'$, are therefore fully known. The former of these two symbols will be found to have an intimate connexion with the theory of *reciprocal polars*; as may be expected, if it be observed that the equation

$$\mathrm{S}\,.\,\alpha\alpha' = -a^2$$

expresses that *with reference to the sphere of which the equation is*

$$\alpha^2 = -a^2,$$

that is, with reference to the sphere of which the centre is at the origin of vectors, and of which the radius has its length denoted by a, *the vector α' terminates in the polar plane of the point which is the termination of the vector α.* The latter of the same two symbols, namely $\mathrm{V}\,.\,\alpha\alpha'$, denotes, or may be constructed by a straight line, which is in direction perpendicular to both the lines denoted by α and α', being also such that the rotation round it from α to α' is positive; and bearing, in length, to the unit of length, the same ratio which the area of the *parallelogram under the two factor lines* bears to the unit of area. The *volume of the parallelepipedon* under any three coinitial lines, or the *sextuple volume of the tetrahedron* of which those lines are conterminous edges, may easily be shown, on the same principles, to be equal to the *scalar of the product of the three vectors* corresponding; this scalar $\mathrm{S}\,.\,\alpha\alpha'\alpha''$, which is equal to $\mathrm{S}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\alpha'')$, being positive or negative according as α'' makes an obtuse or an acute angle with $\mathrm{V}\,.\,\alpha\alpha'$, that is, according as the rotation round α'' from α' towards α is positive or negative. To express that two proposed lines α, α' are rectangular, we may write the following *equation of perpendicularity,*

$$\mathrm{S}\,.\,\alpha\alpha' = 0; \quad \text{or} \quad \alpha\alpha' + \alpha'\alpha = 0.$$

To express that two lines are similar or opposite in direction, we may write the following *equation of coaxality*, or of parallelism,

$$\mathrm{V}\,.\,\alpha\alpha' = 0; \quad \text{or} \quad \alpha\alpha' - \alpha'\alpha = 0.$$

And to express that three lines are in or parallel to one common plane, we may write the *equation of coplanarity,*

$$\mathrm{S}\,.\,\alpha\alpha'\alpha'' = 0; \quad \text{or} \quad \alpha\alpha'\alpha'' - \alpha''\alpha'\alpha = 0;$$

either because the volume of the parallelepipedon under the three lines then vanishes, or because one of the three vectors is then perpendicular to the vector part of the product of the other two.

22. The *geometrical considerations* of the foregoing article may often suggest *algebraical transformations* of functions of the new imaginaries which enter into the present theory. Thus, if we meet the function

$$\alpha S \,.\, \alpha'\alpha'' - \alpha' S \,.\, \alpha''\alpha, \tag{1.}$$

we may see, in the first place, that in the recent notation this function is algebraically a pure imaginary, or *vector form*, which may be constructed geometrically in this theory by a straight line having length and direction in space; because the three symbols α, α', α'' are supposed to be themselves such vector forms, or to admit of being constructed by three such lines; while $\mathrm{S} \,.\, \alpha'\alpha''$ and $\mathrm{S} \,.\, \alpha''\alpha$ are, in the same notation, two *scalar forms*, and denote some two real numbers, positive negative, or zero. We may therefore equate the proposed function (1.) to a new small Greek letter, accented or unaccented, for example to α''', writing

$$\alpha''' = \alpha \, S \,.\, \alpha'\alpha'' - \alpha' \, S \,.\, \alpha''\alpha. \tag{2.}$$

Multiplying this equation by α'', and taking the scalar parts of the two members of the product, that is, operating on it by the characteristic $\mathrm{S} \,.\, \alpha''$; and observing that, by the properties of scalars,

$$\begin{aligned} \mathrm{S} \,.\, \alpha''\alpha \, \mathrm{S} \,.\, \alpha'\alpha'' &= \mathrm{S} \,.\, \alpha''\alpha \,.\, \mathrm{S} \,.\, \alpha'\alpha'' \\ &= \mathrm{S} \,.\, \alpha''\alpha' \,.\, \mathrm{S} \,.\, \alpha''\alpha = \mathrm{S} \,.\, \alpha''\alpha' \, \mathrm{S} \,.\, \alpha''\alpha, \end{aligned}$$

in which the notation $\mathrm{S} \,.\, \alpha''\alpha \, \mathrm{S} \,.\, \alpha'\alpha''$ is an abridgment for $\mathrm{S}(\alpha''\alpha \, \mathrm{S} \,.\, \alpha'\alpha'')$, and the notation $\mathrm{S} \,.\, \alpha''\alpha \,.\, \mathrm{S} \,.\, \alpha'\alpha''$ is abridged from $(\mathrm{S} \,.\, \alpha''\alpha) \,.\, (\mathrm{S} \,.\, \alpha'\alpha'')$, while $\mathrm{S} \,.\, \alpha'\alpha''$ is a symbol equivalent to $\mathrm{S}(\alpha'\alpha'')$, and also, by article 20, to $\mathrm{S}(\alpha''\alpha')$, or to $\mathrm{S} \,.\, \alpha''\alpha'$, although $\alpha'\alpha''$ and $\alpha''\alpha'$ are not themselves equivalent symbols; we are conducted to the equation

$$\mathrm{S} \,.\, \alpha''\alpha''' = 0, \tag{3.}$$

which shows, by comparison with the general *equation of perpendicularity* assigned in the last article, that *the new vector α''' is perpendicular to the given vector α''*, or that these two vector forms represent two rectangular straight lines in space. Again, because the squares of vectors are scalars (being real, though negative numbers), we have

$$\begin{aligned} \alpha(\alpha \, \mathrm{S} \,.\, \alpha'\alpha'') \,.\, \alpha' &= \alpha^2\alpha' \, \mathrm{S} \,.\, \alpha'\alpha'' = \alpha'(\alpha \, \mathrm{S} \,.\, \alpha'\alpha'') \,.\, \alpha, \\ \alpha'(\alpha' \, \mathrm{S} \,.\, \alpha''\alpha) \,.\, \alpha &= \alpha'^2\alpha \, \mathrm{S} \,.\, \alpha''\alpha = \alpha(\alpha' \, \mathrm{S} \,.\, \alpha''\alpha) \,.\, \alpha'; \end{aligned}$$

therefore the equation (2.) gives also

$$\alpha\alpha'''\alpha' = \alpha'\alpha'''\alpha; \tag{4.}$$

a result which, when compared with the general *equation of coplanarity* assigned in the same preceding article, shows that *the new vector α''' is coplanar with the two other given vectors*, α and α'; it is therefore perpendicular to the vector of their product, $\mathrm{V} \,.\, \alpha\alpha'$, which is perpendicular to both those given vectors. We have therefore two known vectors, namely $\mathrm{V} \,.\, \alpha\alpha'$ and α'', to both of which the sought vector α''' is perpendicular; it is therefore parallel to, or coaxal with, the vector of the product of the known vectors last mentioned, or is equal to this vector of their product, multiplied by some scalar coefficient x; so that we may write the transformed expression,

$$\alpha''' = x\,\mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\alpha''). \tag{5.}$$

And because the function α''' is, by the equation (2.), homogeneous of the dimension unity with respect to each separately of the three vectors α, α', α'', while the function $\mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\alpha'')$ is likewise homogeneous of the same dimension with respect to each of those three vectors, we see that the scalar coefficient x must be either an entirely constant number, or else a homogeneous function of the dimension zero, with respect to each of the same three vectors; we may therefore assign to these vectors any arbitrary lengths which may most facilitate the determination of this scalar coefficient x. Again, the two expressions (2.) and (5.) both vanish if α'' be perpendicular to the plane of α and α'; in order therefore to determine x, we are permitted to suppose that α, α', α'' are three coplanar vectors: and, by what was just now remarked, we may suppose their lengths to be each equal to the assumed unit of length. In this manner we are led to seek the value of x in the equation

$$x\,\mathrm{V}\,.\,\alpha\alpha'\,.\,\alpha'' = \alpha S\,.\,\alpha'\alpha'' - \alpha' S\,.\,\alpha''\alpha, \tag{6.}$$

under the conditions

$$\mathrm{S}\,.\,\alpha\alpha'\alpha'' = 0, \tag{7.}$$

and

$$\alpha^2 = \alpha'^2 = \alpha''^2 = -1; \tag{8.}$$

so that α, α', α'' are here *three coplanar and imaginary units.* Multiplying each member of the equation (6.), as a multiplier, *into* $-\alpha''$ as a multiplicand, and taking the vector parts of the two products; observing also that

$$\mathrm{V}\,.\,\alpha'\alpha'' = -\mathrm{V}\,.\,\alpha''\alpha', \quad \text{and} \quad -\mathrm{V}\,.\,\alpha\alpha'' = \mathrm{V}\,.\,\alpha''\alpha;$$

we obtain this other equation,

$$x\,\mathrm{V}\,.\,\alpha\alpha' = \mathrm{V}\,.\,\alpha''\alpha\,.\,\mathrm{S}\,.\,\alpha'\alpha'' - \mathrm{V}\,.\,\alpha''\alpha'\,.\,\mathrm{S}\,.\,\alpha''\alpha; \tag{9.}$$

in which the three vectors $\mathrm{V}\,.\,\alpha''\alpha$, $\mathrm{V}\,.\,\alpha''\alpha'$, $\mathrm{V}\,.\,\alpha\alpha'$ are coaxal, being each perpendicular to the common plane of the three vectors α, α', α''; they bear therefore scalar ratios to each other, and are proportional (by the last article) to the areas of the parallelograms under the three pairs of unit-vectors, α'' and α, α'' and α', and α and α', respectively; that is, to the sines of the angles a, a', and $a' - a$, if a be the rotation from α'' to α, and a' the rotation from α'' to α', in the common plane of these three vectors. At the same time we have (by the principles of the same article) the expressions:

$$-\mathrm{S}\,.\,\alpha''\alpha = \cos a; \quad \mathrm{S}\,.\,\alpha''\alpha' = -\cos a';$$

so that the equation (9.) reduces itself to the following very simple form,

$$x\sin(a' - a) = \sin a' \cos a - \sin a \cos a', \tag{10.}$$

and gives immediately

$$x = 1. \tag{11.}$$

Such then is the value of the coefficient x in the transformed expression (5.); and by comparing this expression with the proposed form (1.), we find that we may write, for *any three vectors*, α, α', α'', not necessarily subject to any conditions such as those of being equal in length and coplanar in direction (since those conditions were not used in *discovering the form* (5.), but only in *determining the value* (11.),) the following *general transformation*:

$$\alpha\,\mathrm{S}\,.\,\alpha'\alpha'' - \alpha'\,\mathrm{S}\,.\,\alpha''\alpha = \mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\alpha''); \tag{12.}$$

which will be found to have extensive applications.

23. But although it is possible thus to employ geometrical considerations, in order to *suggest* and even to *demonstrate* the validity of many general transformations, yet it is always desirable to know how to obtain the same *symbolic results*, from the *laws of combination of the symbols*: nor ought the *calculus of quaternions* to be regarded as complete, till all such *equivalences of form* can be deduced from such symbolic laws, by the fewest and simplest principles. In the example of the foregoing article, the symbolic transformation may be effected in the following way.

When a scalar form is multiplied by a vector form, or a vector by a scalar, the product is a vector form; and the sum or difference of two such vector forms is itself a vector form; therefore the expression (1.) of the last article is a vector form, and may be equated as such to a small Greek letter; or in other words, the equation (2.) is allowed. But every vector form is equal to its own vector part, or undergoes no change of signification when it is operated on by the characteristic V; we have therefore this other expression, after interchanging, as is allowed, the places of the two vector factors α' α'' of a binary product under the characteristic S,

$$\alpha''' = \mathrm{V}(\alpha\,\mathrm{S}\,.\,\alpha''\alpha' - \alpha'\,\mathrm{S}\,.\,\alpha''\alpha). \tag*{(1.)$'$}$$

Substituting here for the characteristic S, that which is, by article 18, symbolically equivalent thereto, namely the characteristic $1 - \mathrm{V}$, and observing that

$$0 = \mathrm{V}(\alpha\alpha''\alpha' - \alpha'\alpha''\alpha), \tag*{(2.)$'$}$$

because, by article 20, $\alpha\alpha''\alpha' - \alpha'\alpha''\alpha$ is a scalar form, we obtain this other expression,

$$\alpha''' = \mathrm{V}(\alpha'\mathrm{V}\,.\,\alpha''\alpha - \alpha\,\mathrm{V}\,.\,\alpha''\alpha'). \tag*{(3.)$'$}$$

The expression (1.)$'$ may be written under the form

$$\alpha''' = \mathrm{V}(\alpha\,\mathrm{S}\,.\,\alpha'\alpha'' - \alpha'\,\mathrm{S}\,.\,\alpha\alpha''); \tag*{(4.)$'$}$$

and (3.)$'$ under the form

$$\alpha''' = \mathrm{V}(\alpha\,\mathrm{V}\,.\,\alpha'\alpha'' - \alpha'\,\mathrm{V}\,.\,\alpha\alpha''), \tag*{(5.)$'$}$$

obtained by interchanging the places of two vector-factors in each of two binary products under the sign V, and by then changing the signs of those two products; taking then the semisum of these two forms (4.)′, (5.)′, and using the symbolic relation of article 18, S+V = 1, we find

$$\begin{aligned} \alpha''' &= \tfrac{1}{2}\mathrm{V}(\alpha\alpha'\alpha'' - \alpha'\alpha\alpha'') \\ &= \mathrm{V}\left(\tfrac{1}{2}(\alpha\alpha' - \alpha'\alpha) \,.\, \alpha''\right); \end{aligned} \tag{6.$'$}$$

in which, by article 20, $\frac{1}{2}(\alpha\alpha' - \alpha'\alpha) = \mathrm{V} \,.\, \alpha\alpha'$; we have therefore finally

$$\alpha''' = \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \alpha'') : \tag{7.$'$}$$

that is, we are conducted by this purely symbolical process, from laws of combination previously established, to the transformed expression (12.) of the last article.

24. A relation of the form (4.), art. 22, that is an *equation between the two ternary products of three vectors taken in two different and opposite orders*, or an evanescence of the scalar part of such a ternary product, may (and in fact does) present itself in several researches; and although we know, by art. 21, the *geometrical interpretation* of such a symbolic relation between three vector forms, namely that it is the condition of their representing *three coplanar lines*, which interpretation may suggest a transformation of one of them, as a *linear function with scalar coefficients*, of the two other vectors, because any one straight line in any given plane may be treated as the diagonal of a parallelogram of which two adjacent sides have any two given directions in the same given plane; yet it is desirable, for the reason mentioned at the beginning of the last article, to know how to obtain the same general transformation of the same symbolic relation, without having recourse to geometrical considerations.

Suppose then that any research has conducted to the relation,

$$\alpha\alpha'\alpha'' - \alpha''\alpha'\alpha = 0, \tag{1.}$$

which is not in this theory an identity, and which it is required to transform. [We propose for convenience to commence from time to time a new numbering of the *formulæ*, but shall take care to avoid all danger of confusion of reference, by naming, where it may be necessary, the *article* to which a formula belongs; and when no such reference to an article is made, the formula is to be understood to belong to the *current series* of formulæ, connected with the existing investigation.] By article 20, we may write the recent relation (1.) under the form,

$$\mathrm{S} \,.\, \alpha\alpha'\alpha'' = 0; \tag{2.}$$

and because generally, for any three vectors, we have the formula (12.) of art. 22, if we make, in that formula, $\alpha'' = \mathrm{V} \,.\, \beta\beta'$, and observe that $\mathrm{S}(\mathrm{V} \,.\, \beta\beta' \,.\, \alpha) = \mathrm{S}(\alpha \,\mathrm{V} \,.\, \beta\beta') = \mathrm{S} \,.\, \alpha\beta\beta'$, we find, for *any four vectors* $\alpha\ \alpha'\ \beta\ \beta'$, the equation:

$$\mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \mathrm{V} \,.\, \beta\beta') = \alpha \,\mathrm{S} \,.\, \alpha'\beta\beta' - \alpha' \,\mathrm{S} \,.\, \alpha\beta\beta'; \tag{3.}$$

making then, in this last equation, $\beta = \alpha'$, $\beta' = \alpha''$, we find, for *any three vectors*, $\alpha\ \alpha'\ \alpha''$, the formula:

$$\mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\alpha'\alpha'') = -\alpha'\,\mathrm{S}\,.\,\alpha\alpha'\alpha''. \tag{4.}$$

If then the scalar of the product $\alpha\alpha'\alpha''$ be equal to zero, that is, if the condition (2.) or (1.) of the present article be satisfied, the product of the two vectors $\mathrm{V}\,.\,\alpha\alpha'$ and $\mathrm{V}\,.\,\alpha'\alpha''$ is a scalar, and therefore the latter of these two vectors, or the opposite vector $\mathrm{V}\,.\,\alpha''\alpha'$, is in general equal to the former vector $\mathrm{V}\,.\,\alpha\alpha'$, multiplied by some scalar coefficient b; we may therefore write, under this condition (1.), the equation

$$\mathrm{V}\,.\,\alpha''\alpha' = b\,\mathrm{V}\,.\,\alpha\alpha', \tag{5.}$$

that is,

$$\mathrm{V}\,.\,(\alpha'' - b\alpha)\alpha' = 0, \tag{6.}$$

so that the one vector factor $\alpha'' - b\alpha$ of this last product must be equal to the other vector factor α' multiplied by some new scalar b'; and we may write the formula,

$$\alpha'' = b\alpha + b'\alpha', \tag{7.}$$

as a transformation of (1.) or of (2.). We may also write, more symmetrically, the equation

$$a\alpha + a'\alpha' + a''\alpha'' = 0, \tag{8.}$$

introducing *three* scalar coefficients a, a', a'', which have however only *two arbitrary ratios*, as a symbolic transformation of the proposed equation $\alpha\alpha'\alpha'' - \alpha''\alpha'\alpha = 0$. And it is remarkable that while we have thus *lowered by two units the dimension of that proposed equation*, considered as involving three variable vectors, α, α', α'', we have at the same time *introduced* (what may be regarded as) *two arbitrary constants*, namely the two ratios of a, a', a''. A converse process would have served to *eliminate two arbitrary constants*, such as these two ratios, or the two scalar coefficients b and b', from a linear equation of the form (8.) or (7.), between three variable vectors, at the same time *elevating the dimension of the equation by two units*, in the passage to the form (2.) or (1.). And the analogy of these two converse transformations to *integrations and differentiations of equations* will appear still more complete, if we attend to the intermediate stage (5.) of either transformation, which is of an *intermediate degree*, or dimension, and involves *one arbitrary constant* b; that is to say, *one more* than the equation of the highest dimension (1.), and *one fewer* than the equation of lowest dimension (7.).

25. As the equation $\mathrm{S}\,.\,\alpha\alpha'\alpha'' = 0$ has been seen to express that the three vectors $\alpha\ \alpha'\ \alpha''$ represent coplanar lines, or that any one of these three lines, for example the line represented by the vector α, is in the plane determined by the other two, when they diverge from a common origin; so, if we make for abridgment

$$\left.\begin{aligned} \beta &= \mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\alpha'''\alpha^{\mathrm{IV}}), \\ \beta' &= \mathrm{V}(\mathrm{V}\,.\,\alpha'\alpha''\,.\,\mathrm{V}\,.\,\alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}), \\ \beta'' &= \mathrm{V}(\mathrm{V}\,.\,\alpha''\alpha'''\,.\,\mathrm{V}\,.\,\alpha^{\mathrm{V}}\alpha), \end{aligned}\right\} \tag{1.}$$

the equation

$$\mathrm{S} \,.\, \beta\beta'\beta'' = 0 \tag{2.}$$

may easily be shown to express that *the six vectors* $\alpha\ \alpha'\ \alpha''\ \alpha'''\ \alpha^{\mathrm{IV}}\ \alpha^{\mathrm{V}}$ *are homoconic*, or represent *six edges of one cone of the second degree*, if they be supposed to be all drawn from one common origin of vectors. For if we regard the five vectors $\alpha'\ \alpha''\ \alpha'''\ \alpha^{\mathrm{IV}}\ \alpha^{\mathrm{V}}$ as given, and the remaining vector α as variable, then first the equation (2.) will give for the locus of this variable vector α, some cone of the second degree; because, by the definitions (1.) of β, β', β'', if we change α to $a\alpha$, a being any scalar, each of the two vectors β and β'' will also be multiplied by a, while β' will not be altered: and therefore the function $\mathrm{S} \,.\, \beta\beta'\beta''$ will be multiplied by a^2, that is by the square of the scalar a, by which the vector α is multiplied. In the next place, this conical locus of α will contain the given vector α'; because if we suppose $\alpha = \alpha'$, we have $\beta = 0$, and the equation (2.) is satisfied: and in like manner the locus of α contains the vector α^{V}, because the supposition $\alpha = \alpha^{\mathrm{V}}$ gives $\beta'' = 0$. In the third place, the cone contains α'' and α^{IV}; for if we suppose $\alpha = \alpha''$, then, by the principle contained in the formula (4.) of the last article, we have

$$\beta'' = -\mathrm{V}(\mathrm{V} \,.\, \alpha^{\mathrm{V}}\alpha'' \,.\, \mathrm{V} \,.\, \alpha''\alpha''') = \alpha''\,\mathrm{S} \,.\, \alpha^{\mathrm{V}}\alpha''\alpha''';$$

and by the same principle, under the same condition,

$$\begin{aligned}\mathrm{V} \,.\, \beta\beta' &= \mathrm{V}(\mathrm{V}(\mathrm{V} \,.\, \alpha'''\alpha^{\mathrm{IV}} \,.\, \mathrm{V} \,.\, \alpha'\alpha'') \,.\, \mathrm{V}(\mathrm{V} \,.\, \alpha'\alpha'' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}))\\ &= -\mathrm{V} \,.\, \alpha'\alpha'' \,.\, \mathrm{S}(\mathrm{V} \,.\, \alpha'''\alpha^{\mathrm{IV}} \,.\, \mathrm{V} \,.\, \alpha'\alpha'' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}});\end{aligned}$$

but $\mathrm{S}(\mathrm{V} \,.\, \alpha'\alpha'' \,.\, \alpha'') = \mathrm{S} \,.\, \alpha'\alpha''\alpha'' = 0$; therefore $\mathrm{S} \,.\, \beta\beta'\beta'' = \mathrm{S}(\mathrm{V} \,.\, \beta\beta' \,.\, \beta'') = 0$; and in like manner this last condition is satisfied, if $\alpha = \alpha^{\mathrm{IV}}$, because β and $\mathrm{V} \,.\, \beta'\beta''$ then differ only by scalar coefficients from α^{IV} and $\mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}$, respectively, so that the scalar of their product is zero. Finally, the conical locus of α contains also the remaining vector α''', because if we suppose $\alpha = \alpha'''$, we have

$$\beta = \alpha'''\,\mathrm{S} \,.\, \alpha'\alpha'''\alpha^{\mathrm{IV}}, \quad \beta'' = \alpha'''\,\mathrm{S} \,.\, \alpha''\alpha'''\alpha^{\mathrm{V}},$$

and therefore in this case $\mathrm{S} \,.\, \beta\beta'\beta'' = 0$ because the scalar of the product of α''' and $\beta'\alpha'''$ is zero. The locus of α is therefore a cone of the second degree, containing the five vectors α', α'', α''', α^{IV}, α^{V}; and in exactly the same manner it may be shown without difficulty that *whichever of the six vectors* $\alpha \ldots \alpha^{\mathrm{V}}$ *may be regarded as the variable vector, its locus assigned by the equation* (2.), of the present article, *is a cone of the second degree, containing the five other vectors.* We may therefore say that this equation,

$$\mathrm{S} \,.\, \beta\beta'\beta'' = 0,$$

when the symbols β, β', β'' have the meanings assigned by the definitions (1.), or (substituting for those symbols their values) we may say that the following equation

$$\mathrm{S}\{\mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \mathrm{V} \,.\, \alpha'''\alpha^{\mathrm{IV}}) \,.\, \mathrm{V}(\mathrm{V} \,.\, \alpha'\alpha'' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}) \,.\, \mathrm{V}(\mathrm{V} \,.\, \alpha''\alpha''' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{V}}\alpha)\} = 0, \tag{3.}$$

is the *equation of homoconicism*, or of *uniconality*, expressing that, when it is satisfied, one common cone of the second degree passes through all the six vectors $\alpha\ \alpha'\ \alpha''\ \alpha'''\ \alpha^{\mathrm{IV}}\ \alpha^{\mathrm{V}}$, and enabling us to deduce from it all the properties of this common cone.

26. The considerations employed in the foregoing article might leave a doubt whether *no other* cone of the same degree could pass through the same six vectors; to remove which doubt, by a method consistent with the spirit of the present theory, we may introduce the following considerations respecting conical surfaces in general.

Whatever four vectors may be denoted by α, α', β, β', we have

$$\mathrm{V}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\beta\beta') + \mathrm{V}(\mathrm{V}\,.\,\beta\beta'\,.\,\mathrm{V}\,.\,\alpha\alpha') = 0; \tag{1.}$$

substituting then for the first of these two opposite vector functions the expression (3.) of art. 24, and for the second the expression formed from this by interchanging each α with the corresponding β, we find, for any four vectors,

$$\alpha\,\mathrm{S}\,.\,\alpha'\beta\beta' - \alpha'\,\mathrm{S}\,.\,\alpha\beta\beta' + \beta\,\mathrm{S}\,.\,\beta'\alpha\alpha' - \beta'\,\mathrm{S}\,.\,\beta\alpha\alpha' = 0. \tag{2.}$$

Again, it follows easily from principles and results already stated, that the scalar of the product of three vectors changes sign when any two of those three factors change places among themselves, so that

$$\mathrm{S}\,.\,\alpha\beta\gamma = -\mathrm{S}\alpha\gamma\beta = \mathrm{S}\gamma\alpha\beta = -\mathrm{S}\gamma\beta\alpha = \mathrm{S}\beta\gamma\alpha = -\mathrm{S}\beta\alpha\gamma. \tag{3.}$$

Assuming therefore any three vectors, ι, κ, λ, of which the scalar of the product does not vanish, we may express any fourth vector α in terms of these three vectors, and of the scalars of the three products $\alpha\kappa\lambda$, $\iota\alpha\lambda$, $\iota\kappa\alpha$, by the formula:

$$\alpha\,\mathrm{S}\,.\,\iota\kappa\lambda = \iota\,\mathrm{S}\,.\,\alpha\kappa\lambda + \kappa\,\mathrm{S}\,.\,\iota\alpha\lambda + \lambda\,\mathrm{S}\,.\,\iota\kappa\alpha. \tag{4.}$$

Let α be supposed to be a vector function of one scalar variable t, which supposition may be expressed by writing the equation

$$\alpha = \phi(t); \tag{5.}$$

and make for abridgment

$$\frac{\mathrm{S}\,.\,\alpha\kappa\lambda}{\mathrm{S}\,.\,\iota\kappa\lambda} = f_1(t); \quad \frac{\mathrm{S}\,.\,\iota\alpha\lambda}{\mathrm{S}\,.\,\iota\kappa\lambda} = f_2(t); \quad \frac{\mathrm{S}\,.\,\iota\kappa\alpha}{\mathrm{S}\,.\,\iota\kappa\lambda} = f_3(t); \tag{6.}$$

the forms of these three scalar functions f_1 f_2 f_3 depending on the form of the vector function ϕ, and on the three assumed vectors ι κ λ, and being connected with these and with each other by the relation

$$\phi(t) = \iota f_1(t) + \kappa f_2(t) + \lambda f_3(t). \tag{7.}$$

Conceive t to be eliminated between the expressions for the ratios of the three scalar functions f_1 f_2 f_3, and an equation of the form

$$F(f_1(t), f_2(t), f_3(t)) = 0 \tag{8.}$$

to be thus obtained, in which the function F is scalar (or real), and homogeneous; it will then be evident that while the equation (5.) may be regarded as the *equation of a curve in space* (equivalent to a system of three real equations between the three co-ordinates of a curve of double curvature and an auxiliary variable t, which latter variable may represent the *time*, in a motion along this curve), the *equation of the cone* which passes through this arbitary curve, and has its vertex at the origin of vectors, is

$$F(\mathrm{S}\,.\,\alpha\kappa\lambda,\ \mathrm{S}\,.\,\iota\alpha\lambda,\ \mathrm{S}\,.\,\iota\kappa\alpha) = 0. \tag{9.}$$

Such being a form in this theory for the equation of an *arbitrary conical surface*, we may write, in particular, as a *definition of the cone of the* nth *degree*, the equation:

$$\Sigma(A_{p,q,r}(\mathrm{S}\,.\,\alpha\kappa\lambda)^p\,.\,(\mathrm{S}\,.\,\iota\alpha\lambda)^q\,.\,(\mathrm{S}\,.\,\iota\kappa\alpha)^r) = 0; \tag{10.}$$

p, q, r being any three whole numbers, positive or null, of which the sum is n; $A_{p,q,r}$ being a scalar function of these three numbers; and the summation indicated by Σ extending to all their systems of values consistent with the last-mentioned conditions, which may be written thus:

$$\left.\begin{array}{r} \sin p\pi = \sin q\pi = \sin r\pi = 0; \\ p \geq 0, \quad q \geq 0, \quad r \geq 0; \\ p + q + r = n. \end{array}\right\} \tag{11.}$$

When $n = 2$, these conditions can be satisfied only by *six* systems of values of p, q, r; therefore, in this case, there enter only six coefficients A into the equation (10.); consequently *five* scalar ratios of these six coefficients are sufficient to particularize a cone of the second degree; and these can in general be found, by ordinary elimination between five equations of the first degree, when five particular vectors are given, such as α', α'', α''', α^{IV}, α^{V}, through which the cone is to pass, or which its surface must contain upon it. Hence, as indeed is known from other considerations, it is in general a determined problem to find the particular cone of the second degree which contains on its surface five given straight lines: and the general solution of this problem is contained in the equation of homoconicism, assigned in the preceding article. The proof there given that the six vectors $\alpha \ldots \alpha^{\mathrm{V}}$ are homoconic, when they satisfy that equation, does not involve any property of conic sections, nor even any property of the circle: on the contrary, that equation having once been established, by the proof just now referred to, might be used as the basis of a complete theory of conic sections, and of cones of the second degree.

27. To justify this assertion, without at present attempting to effect the actual development of such a theory, it may be sufficient to deduce from the equation of homoconicism assigned in article 25, that great and fertile property of the circle, or of the cone with circular base, which was discovered by the genius of Pascal. And this deduction is easy; for the three auxiliary vectors β, β', β'', introduced in the equations (1.) of the 25th article, are evidently, by the principles stated in other recent articles of this paper, the respective lines of intersection of three pairs of planes, as follows:—The planes of $\alpha\alpha'$ and $\alpha'''\alpha^{\mathrm{IV}}$ intersect in β, those of $\alpha'\alpha''$ and $\alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}$ in β'; and those of $\alpha''\alpha'''$ and $\alpha^{\mathrm{V}}\alpha$ in β''; and in the form (2.), article 25, of the equation of homoconicism, expresses that these three lines, $\beta\ \beta'\ \beta''$, are coplanar. *If then a hexahedral angle be inscribed in a cone of the second degree, and if each of the six plane faces be prolonged* (if necessary) *so as to meet its opposite in a straight line, the three lines of meeting of opposite faces*, thus obtained, *will be situated in one common plane*: which is a form of the theorem of Pascal.

28. The known and purely *graphic* property of the cone of the second degree which constitutes the theorem of Pascal, and which expresses the coplanarity of the three lines of meeting of opposite plane faces of an inscribed hexahedral angle, may be transformed into another known but purely *metric* property of the same cone of the second degree, which is a form of the theorem of M. Chasles, respecting the constancy of an anharmonic ratio. This transformation may be effected without difficulty, on the plan of the present paper; for if we multiply into $\mathrm{V}\,.\,\gamma\gamma'$ both members of the equation (3.) of the 24th article, and then operate by the characteristic S., attending to the general properties of scalars of products, we find, for *any six vectors* $\alpha\ \alpha'\ \beta\ \beta'\ \gamma\ \gamma'$, the formula

$$\mathrm{S}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\beta\beta'\,.\,\mathrm{V}\,.\,\gamma\gamma') = \mathrm{S}\,.\,\alpha\gamma\gamma'\,.\,\mathrm{S}\,.\,\alpha'\beta\beta' - \mathrm{S}\,.\,\alpha'\gamma\gamma'\,.\,\mathrm{S}\,.\,\alpha\beta\beta'; \tag{1.}$$

which gives, for any *five* vectors $\alpha\ \alpha'\ \alpha''\ \gamma\ \gamma'$, this other:

$$\mathrm{S}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\alpha'\alpha''\,.\,\mathrm{V}\,.\,\gamma\gamma') = \mathrm{S}\,.\,\alpha\alpha'\alpha''\,.\,\mathrm{S}\,.\,\gamma\alpha'\gamma'. \tag{2.}$$

If, then, we take six arbitrary vectors $\alpha\ \alpha'\ \alpha''\ \alpha'''\ \alpha^{\mathrm{IV}}\ \alpha^{\mathrm{V}}$, and deduce nine other vectors from them by the expressions

$$\left.\begin{array}{lll} \alpha_0 = \mathrm{V}\,.\,\alpha\alpha', & \alpha_1 = \mathrm{V}\,.\,\alpha'\alpha'', & \alpha_2 = \mathrm{V}\,.\,\alpha''\alpha''', \\ \alpha_3 = \mathrm{V}\,.\,\alpha'''\alpha^{\mathrm{IV}}, & \alpha_4 = \mathrm{V}\,.\,\alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}, & \alpha_5 = \mathrm{V}\,.\,\alpha^{\mathrm{V}}\alpha, \\ \beta = \mathrm{V}\,.\,\alpha_0\alpha_3, & \beta' = \mathrm{V}\,.\,\alpha_1\alpha_4, & \beta'' = \mathrm{V}\,.\,\alpha_2\alpha_5; \end{array}\right\} \tag{3.}$$

we shall have, *generally*,

$$\left.\begin{aligned} \mathrm{S}\,.\,\beta\beta'\beta'' &= \mathrm{S}\,.\,\alpha_0\alpha_2\alpha_5\,.\,\mathrm{S}\,.\,\alpha_3\alpha_1\alpha_4 - \mathrm{S}\,.\,\alpha_3\alpha_2\alpha_5\,.\,\mathrm{S}\,.\,\alpha_0\alpha_1\alpha_4 \\ &= \mathrm{S}\,.\,\alpha_0\alpha_1\alpha_4\,.\,\mathrm{S}\,.\,\alpha_2\alpha_3\alpha_5 - \mathrm{S}\,.\,\alpha_3\alpha_4\alpha_1\,.\,\mathrm{S}\,.\,\alpha_5\alpha_0\alpha_2 \\ &= \mathrm{S}\,.\,\alpha\alpha'\alpha''\,.\,\mathrm{S}\,.\,\alpha^{\mathrm{IV}}\alpha'\alpha^{\mathrm{V}}\,.\,\mathrm{S}\,.\,\alpha''\alpha'''\alpha^{\mathrm{IV}}\,.\,\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'''\alpha \\ &\quad - \mathrm{S}\,.\,\alpha'''\alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}\,.\,\mathrm{S}\,.\,\alpha'\alpha^{\mathrm{IV}}\alpha''\,.\,\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha\alpha'\,.\,\mathrm{S}\,.\,\alpha''\alpha\alpha''' \\ &= \mathrm{S}\,.\,\alpha\alpha'\alpha''\,.\,\mathrm{S}\,.\,\alpha''\alpha'''\alpha^{\mathrm{IV}}\,.\,\mathrm{S}\,.\,\alpha\alpha'''\alpha^{\mathrm{V}}\,.\,\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'\alpha^{\mathrm{IV}} \\ &\quad - \mathrm{S}\,.\,\alpha\alpha'''\alpha''\,.\,\mathrm{S}\,.\,\alpha''\alpha'\alpha^{\mathrm{IV}}\,.\,\mathrm{S}\,.\,\alpha\alpha'\alpha^{\mathrm{V}}\,.\,\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'''\alpha^{\mathrm{IV}}. \end{aligned}\right\} \tag{4.}$$

Thus if, in particular, the six vectors $\alpha \ldots \alpha^{\mathrm{V}}$ are such as to satisfy the condition

$$\mathrm{S}\,.\,\beta\beta'\beta'' = 0, \tag{5.}$$

they will satisfy also this other condition, or this other form of the same condition:

$$\frac{\mathrm{S}\,.\,\alpha\alpha'\alpha''}{\mathrm{S}\,.\,\alpha\alpha'''\alpha''}\cdot\frac{\mathrm{S}\,.\,\alpha''\alpha'''\alpha^{\mathrm{IV}}}{\mathrm{S}\,.\,\alpha''\alpha'\alpha^{\mathrm{IV}}} = \frac{\mathrm{S}\,.\,\alpha\alpha'\alpha^{\mathrm{V}}}{\mathrm{S}\,.\,\alpha\alpha'''\alpha^{\mathrm{V}}}\cdot\frac{\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'''\alpha^{\mathrm{IV}}}{\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'\alpha^{\mathrm{IV}}}; \tag{6.}$$

and reciprocally the former of these two conditions will be satisfied if the latter be so.

These two equations (5.) and (6.), express, therefore, each in its own way, the existence of one and the same geometrical relation between the six vectors $\alpha\ \alpha'\ \alpha''\ \alpha'''\ \alpha^{\mathrm{IV}}\ \alpha^{\mathrm{V}}$: and a slight study of the *forms* of these equations suffices to render evident that they both agree in expressing that these six vectors are *homoconic*, in the sense of the 25th article; or in other words, that the six vectors are sides (or edges) of one common cone of the second degree. Indeed the equation (5.) of the present article, in virtue of the definitions (3.), coincides with the equation (2.) of the article just cited, the symbols β, β', β'' retaining in the one the significations which they had received in the other. The recent transformations show, therefore, that the *equation of homoconicism*, assigned in article 25, may be put under the form (6.) of the present article, which is different, and in *some* respects simpler. The former expresses a *graphic* property, or relation between *directions*, namely that the three lines β, β', β'', which are the respective intersections of the three pairs of planes $(\alpha\alpha', \alpha'''\alpha^{\mathrm{IV}})$, $(\alpha'\alpha'', \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}})$, $(\alpha''\alpha''', \alpha^{\mathrm{V}}\alpha)$, are all situated in one common plane, if the six homoconic vectors be supposed to diverge from one common origin; the latter expresses the *metric* property, or relation between *magnitudes*, that the ratio compounded of the two ratios of the two pyramids $(\alpha\alpha'\alpha'')$ $(\alpha''\alpha'''\alpha^{\mathrm{IV}})$ to the two other pyramids $(\alpha\alpha'''\alpha'')$ $(\alpha''\alpha'\alpha^{\mathrm{IV}})$, or that the product of the volumes of the first pair of pyramids divided by the product of the volumes of the second pair, does not vary, when the vector α'', which is the common edge of these four pyramids, is changed to the new but homoconic vector α^{V}, as their new common edge, the four remaining homoconic and coinitial edges α α' α''' α^{IV} of the pyramids being supposed to undergo no alteration. The one is the expression of the property of the *mystic hexagram* of Pascal; the other is an expression of the constancy of the *anharmonic ratio* of Chasles.* The calculus of Quaternions (or the method of scalars and vectors) enables us, as we have seen, to pass, by a very short and simple symbolical transition, from either to the other of these two great and known properties of the cone of the second degree.

29. If we denote by α and β two constant vectors, and by ρ a variable vector, all drawn from one common origin; if also we denote by u and v two variable scalars, depending on the foregoing vectors α, β, ρ by the relations

$$\left.\begin{aligned} u &= 2\mathrm{S} \, . \, \alpha\rho = \alpha\rho + \rho\alpha; \\ v^2 &= -4(\mathrm{V} \, . \, \beta\rho)^2 = -(\beta\rho - \rho\beta)^2; \end{aligned}\right\} \qquad (1.)$$

* Although the foregoing process of calculation, and generally the method of treating geometrical problems by quaternions, which has been extended by the writer to questions of dynamics and thermology, appears to him to be new, yet it is impossible for him, in mentioning here the name of Chasles, to abstain from acknowledging the deep intellectual obligations under which he feels himself to be, for the information, and still more for the impulse given to his mind by the perusal of that very interesting and excellent History of Geometrical Science, which is so widely known by its own modest title of *Aperçu Historique* (Brussels, 1837). He has also endeavoured to profit by a study of the Memoirs by M. Chasles, on Spherical Conics and Cones of the Second Degree, which have been translated, with Notes and an Appendix, by the Rev. Charles Graves (Dublin 1841); and desires to take this opportunity of adding, that he conceives himself to have derived assistance, as well as encouragement, in his geometrical researches generally, from the frequent and familiar intercourse which he has enjoyed with the last-mentioned gentleman.

we may then represent the central surfaces of the second degree by equations of great simplicity, as follows:—

An ellipsoid, with three unequal axes, may be represented by the equation

$$u^2 + v^2 = 1. \tag{2.}$$

One of its circumscribing cylinders of revolution has for equation

$$v^2 = 1; \tag{3.}$$

the plane of the ellipse of contact is represented by

$$u = 0; \tag{4.}$$

and the system of the two tangent planes of the ellipsoid, parallel to the plane of this ellipse, by

$$u^2 = 1. \tag{5.}$$

A hyperboloid of one sheet, touching the same cylinder in the same ellipse, is denoted by the equation

$$u^2 - v^2 = -1; \tag{6.}$$

its asymptotic cone by

$$u^2 - v^2 = 0; \tag{7.}$$

and a hyperboloid of two sheets, with the same asymptotic cone (7.), and with the two tangent planes (5.), is represented by this other equation,

$$u^2 - v^2 = 1. \tag{8.}$$

By changing ρ to $\rho - \gamma$, where γ is a third arbitrary but constant vector, we introduce an arbitrary origin of vectors, or an arbitrary position of the centre of the surface, as referred to such an origin. And the general problem of determining that individual surface of the second degree (supposed to have a centre, until the calculation shall show in any particular question that it has none), which shall pass through *nine given points*, may thus be regarded as equivalent to the problem of finding *three constant vectors*, α, β γ, which shall, for nine given values of the variable vector ρ, satisfy one equation of the form

$$\{\alpha(\rho - \gamma) + (\rho - \gamma)\alpha\}^2 \pm \{\beta(\rho - \gamma) - (\rho - \gamma)\beta\}^2 = \pm 1; \tag{9.}$$

with suitable selections of the two ambiguous signs, depending on, and in their turn determining, the particular species of the surface.

30. The equation of the ellipsoid with three unequal axes, referred to its centre as the origin of vectors, may thus be presented under the following form (which was exhibited to the Royal Irish Academy in December 1845):

$$(\alpha\rho + \rho\alpha)^2 - (\beta\rho - \rho\beta)^2 = 1; \tag{1.}$$

and which decomposes itself into two factors, as follows:

$$(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta)(\alpha\rho + \rho\alpha - \beta\rho + \rho\beta) = 1. \tag{2.}$$

These two factors are not only separately linear with respect to the variable vector ρ, but are also (by art. 20, Phil. Mag. for July 1846) *conjugate quaternions*; they have therefore a common *tensor*, which must be equal to unity, so that we may write the equation of the ellipsoid under this other form,

$$\mathrm{T}(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta) = 1; \tag{3.}$$

if we use, as in the 19th article, Phil. Mag., July 1846, the characteristic T to denote the operation of taking the tensor of a quaternion. Let σ be an auxiliary vector, connected with the vector ρ of the ellipsoid by the equation

$$\sigma = \rho(\alpha - \beta)\rho^{-1}; \tag{4.}$$

we shall then have, by (3.), and by the general law for the tensor of a product,

$$\mathrm{T}(\alpha + \beta + \sigma) \,.\, \mathrm{T}\rho = 1; \tag{5.}$$

but also

$$(\alpha - \beta + \sigma)\rho = (\alpha - \beta)\rho + \rho(\alpha - \beta), \tag{6.}$$

where the second member is scalar; therefore, using the characteristic U to denote the operation of taking the *versor* of a quaternion, as in the same art. 19, we have the equation

$$\mathrm{U}(\alpha - \beta + \sigma) \,.\, \mathrm{U}\rho = \mp 1; \tag{7.}$$

and the dependence of the variable vector ρ of the ellipsoid on the auxiliary vector σ is expressed by the formula

$$\rho = \pm\frac{\mathrm{U}(\alpha - \beta + \sigma)}{\mathrm{T}(\alpha + \beta + \sigma)}. \tag{8.}$$

Besides, the length of this auxiliary vector σ is constant, and equal to that of $\alpha - \beta$, because the equation (4.) gives

$$\mathrm{T}\sigma = \mathrm{T}(\alpha - \beta); \tag{9.}$$

we may therefore regard $\alpha - \beta$ as the vector of the centre C of a certain auxiliary sphere, of which the surface passes through the centre A of the ellipsoid; and may regard the vector $\alpha - \beta + \sigma$ as a variable and auxiliary *guide-chord* AD of the same *guide-sphere*, which chord determines the (exactly similar or exactly opposite) direction of the variable radius vector

AE (or ρ) of the ellipsoid. At the same time, the constant vector -2β, drawn from the same constant origin as before, namely the centre A of the ellipsoid, will determine the position of a certain fixed point B, having this remarkable property, that its *distance* from the extremity D of the variable guide-chord from A, will represent the *reciprocal of the length of the radius vector* ρ, or *the proximity* $(\text{AE})^{-1}$ of the point E on the surface of the ellipsoid to the centre (the use of this word "proximity" being borrowed from Sir John Herschel). Supposing then, for simplicity, that the fixed point B is external to the fixed sphere, which does not essentially diminish the generality of the question; and taking, for the unit of length, the length of a tangent to that sphere from that point; we may regard AE and BD′ as two equally long lines, or may write the equation

$$\overline{\text{AE}} = \overline{\text{BD}'}, \tag{10.}$$

if D′ be the other point of intersection of the straight line BD with the sphere.

31. Hence follows this very simple *construction** *for an ellipsoid* (with three unequal axes), by means of a sphere and an external point, to which the author was led by the foregoing process, but which may also be deduced from principles more generally known. From a fixed point A on the surface of a sphere, draw a variable chord AD; let D′ be the second point of intersection of the spheric surface with the secant BD, drawn to the variable extremity D of this chord AD from a fixed external point B; take the radius vector AE equal in length to BD′, and in direction either coincident with, or opposite to, the chord AD; *the locus of the point* E*, thus constructed, will be an ellipsoid*, which will pass through the point B.

32. We may also say that *if of a quadrilateral* (ABED′), *of which one side* (AB) *is given in length and in position, the two diagonals* (AE, BD′) *be equal to each other in length, and intersect* (in D) *on the surface of a given sphere* (with centre C), *of which a chord* (AD′) *is a side of the quadrilateral adjacent to the given side* (AB), *then the other side* (BE), *adjacent to the same given side, is a chord of a given ellipsoid.* The form, position, and magnitude of an *ellipsoid* (with three unequal axes), may thus be made to depend on the form, position, and magnitude of a *generating triangle* ABC. Two sides of this triangle, namely BC and CA, are perpendicular to the *two planes of circular section*; and the third side AB is perpendicular to *one of the two planes of circular projection* of the ellipsoid, because it is the axis of revolution of one of the two circumscribed circular cylinders. This *triple reference to circles* is perhaps the cause of the extreme facility with which it will be found that many fundamental properties of the ellipsoid may be deduced from this mode of generation. As an example of such deduction, it may be mentioned that the known proportionality of the difference of the

* This construction has already been printed in the Proceedings of the Royal Irish Academy for 1846; but it is conceived that its being reprinted here may be acceptable to some of the readers of the London, Edinburgh and Dublin Philosophical Magazine; in which periodical (namely in the Number for July 1844) the first *printed* publication of the fundamental equations of the theory of quaternions ($i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$) took place, although those equations had been communicated to the Royal Irish Academy in November 1843, and had been exhibited at a meeting of the Council during the preceding month.

squares of the reciprocals of the semiaxes of a diametral section to the product of the sines of the inclinations of its plane to the two planes of circular section, presents itself under the form of a proportionality of the same difference of squares to the rectangle under the projections of the two sides BC and CA of the generating triangle on the plane of the elliptic section.

33. For the sake of those mathematical readers who are familiar with the method of co-ordinates, and not with the method of quaternions, the writer will here offer an investigation, by the former method, of that general property of the ellipsoid to which he was conducted by the latter method, and of which an account was given in a recent Number of this Magazine (for June 1847).

Let $x\ y\ z$ denote, as usual, the three rectangular co-ordinates of a point, and let us introduce two real functions of these three co-ordinates, and of six arbitrary but real constants, $l\ m\ n\ l'\ m'\ n'$, which functions shall be denoted by u and v, and shall be determined by the two following relations:

$$u(ll' + mm' + nn') = l'x + m'y + n'z;$$

$$v^2(ll' + mm' + nn')^2 = (ly - mx)^2 + (mz - ny)^2 + (nx - lz)^2;$$

then the equation

$$u^2 + v^2 = 1 \tag{1.}$$

will denote (as received principles suffice to show) that the curved surface which is the locus of the point $x\ y\ z$ is an ellipsoid, having its centre at the origin of co-ordinates; and conversely this equation $u^2 + v^2 = 1$ may represent any such ellipsoid, by a suitable choice of the six real constants $l\ m\ n\ l'\ m'\ n'$. At the same time the equation

$$u^2 = 1$$

will represent a system of two parallel planes, which touch the ellipsoid at the extremities of the diameter denoted by the equation

$$v = 0;$$

and this diameter will be the axis of revolution of a certain circumscribed cylinder, namely of the cylinder denoted by the equation

$$v^2 = 1;$$

the equation of the plane of the ellipse of contact, along which this circular cylinder envelopes the ellipsoid, being, in the same notation,

$$u = 0;$$

all which may be inferred from ordinary principles, and agrees with what was remarked in the 29th article of this paper.

34. This being premised, let us next introduce three new constants, p, q, r, depending on the six former constants by the three relations

$$2p = l + l', \quad 2q = m + m', \quad 2r = n + n'.$$

We shall then have

$$l'x + m'y + n'z = 2(px + qy + rz) - (lx + my + nz);$$

and the equation (1.) of the ellipsoid will become

$$\begin{aligned}(ll' + mm' + nn')^2 &= (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - 4(lx + my + nz)(px + qy + rz) \\ &\quad + 4(px + qy + rz)^2 \\ &= (x^2 + y^2 + z^2)\{(l - x')^2 + (m - y')^2 + (n - z')^2\},\end{aligned}$$

if we introduce three new variables, x', y', z', depending on the three old variables x, y, z, or rather on their ratios, and on the three new constants p, q, r, by the conditions,

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{2(px + qy + rz)}{x^2 + y^2 + z^2}.$$

These three last equations give, by elimination of the two ratios of x, y, z, the relation

$$x'^2 + y'^2 + z'^2 = 2(px' + qy' + rz');$$

the new variables x', y', z', are therefore co-ordinates of a new point, which has for its locus a certain spheric surface, passing through the centre of the ellipsoid; and the same new point is evidently contained on the radius vector drawn from that centre of the ellipsoid to the point $x\,y\,z$, or on that radius vector prolonged. We see, also, that the length of this radius vector of the ellipsoid, or the distance of the point $x\,y\,z$ from the origin of the co-ordinates, is inversely proportional to the distance of the new point $x'\,y'\,z'$ of the spheric surface from the point $l\,m\,n$, which latter is a certain fixed point upon the surface of the ellipsoid. This result gives already an easy and elementary mode of generating the latter surface, which may however be reduced to a still greater degree of simplicity by continuing the analysis as follows.

35. Let the straight line which connects the two points $x'\,y'\,z'$ and $l\,m\,n$ be prolonged, if necessary, so as to cut the same spheric surface again in another point $x''\,y''\,z''$; we shall then have the equation

$$x''^2 + y''^2 + z''^2 = 2(px'' + qy'' + rz''),$$

from which the new co-ordinates x'', y'', z'' may be eliminated by substituting the expressions

$$x'' = l + t(x' - l), \quad y'' = m + t(y' - m), \quad z'' = n + t(z' - n);$$

and the root that is equal to unity is then to be rejected, in the resulting quadratic for t. Taking therefore for t the product of the roots of that quadratic, we find

$$t = \frac{l^2 + m^2 + n^2 - 2(lp + mq + nr)}{(x' - l)^2 + (y' - m)^2 + (z' - n)^2};$$

therefore also, by the last article,

$$t = \frac{x^2 + y^2 + z^2}{l^2 + m^2 + n^2 - 2(lp + mq + nr)};$$

consequently

$$t^2 = \frac{x^2 + y^2 + z^2}{(x' - l)^2 + (y' - m)^2 + (z' - n)^2};$$

and finally,

$$(x'' - l)^2 + (y'' - m)^2 + (z'' - n)^2 = x^2 + y^2 + z^2. \tag{2.}$$

Denoting by A, B, C, the three fixed points of which the co-ordinates are respectively $(0, 0, 0)$, (l, m, n), (p, q, r); and by D, D$'$, E, the three variable points of which the co-ordinates are (x', y', z'), (x'', y'', z''), (x, y, z); A B E D$'$ may be regarded as a plane quadrilateral, of which the diagonals AE and BD$'$ intersect each other in a point D on a fixed spheric surface, which has its centre at C, and passes through A and D$'$; so that one side D$'$A of the quadrilateral, adjacent to the fixed side AB, is a chord of this fixed sphere. And the equation (2.) expresses that the *other side* BE *of the same plane quadrilateral, adjacent to the same fixed side* AB, *is a chord of a fixed ellipsoid, if the two diagonals* AE, BD$'$ *of the quadrilateral be equally long*; so that a general and characteristic property of the ellipsoid, sufficient for the construction of that surface, and for the investigation of all its properties, is included in the remarkably simple and eminently geometrical formula

$$\overline{\text{AE}} = \overline{\text{BD}'}; \tag{3.}$$

the locus of the point E being an ellipsoid, which passes through B, and has its centre at A, when this condition is satisfied.

This formula (3.), which has already been printed in this Magazine as the equation (10.) of article 30 of this paper, may therefore be deduced, as above, from generally admitted principles, by the Cartesian method of co-ordinates; although it had not been known to geometers, so far as the present writer has hitherto been able to ascertain, until he was led to it, in the summer of 1846*, by an entirely different method; namely by applying his calculus of quaternions to the discussion of one of those new forms for the equations of central surfaces of the second order, which he had communicated to the Royal Irish Academy in December 1845.

* See the Proceedings of the Royal Irish Academy.

36. As an example (already alluded to in the 32nd article of this paper) of the *geometrical* employment of the formula (3.), or of the equality which it expresses as existing between the lengths of the two diagonals of a certain plane quadrilateral connected with that new construction of the ellipsoid to which the writer was thus led by quaternions, let us now propose to investigate geometrically, by the help of that equality of diagonals, the difference of the squares of the reciprocals of the greatest and least semi-diameters of any plane and diametral section of an ellipsoid (with three unequal axes). Conceive then that the ellipsoid, and the auxiliary sphere employed in the above-mentioned construction, are both cut by a plane $\mathrm{AB'C'}$, on which $\mathrm{B'}$ and $\mathrm{C'}$ are the orthogonal projections of the fixed points B and C; the auxiliary point D may thus be conceived to move on the circumference of a circle, which passes through A, and has its centre at $\mathrm{C'}$; and since AE, being equal in length to $\mathrm{BD'}$ (because these are the two equal diagonals of the quadrilateral in the construction), must vary inversely as BD (by an elementary property of the sphere), we are to seek the difference of the squares of the extreme values of BD, or of $\mathrm{B'D}$, because the square of the perpendicular $\mathrm{BB'}$ is constant for the section. But the longest and shortest straight lines, $\mathrm{B'D_1}$, $\mathrm{B'D_2}$, which can thus be drawn to the auxiliary circle round $\mathrm{C'}$, from the fixed point $\mathrm{B'}$ in its plane, are those drawn to the extremities of that diameter $\mathrm{D_1C'D_2}$ of this circle which passes through or tends towards $\mathrm{B'}$; so that the four points $\mathrm{D_1\,C'\,D_2\,B'}$ are on one straight line, and the difference of the squares of $\mathrm{B'D_1}$, $\mathrm{B'D_2}$ is equal to four times the rectangle under $\mathrm{B'C'}$ and $\mathrm{C'D_1}$, or under $\mathrm{B'C'}$ and $\mathrm{C'A}$. We see therefore that the shortest and longest semi-diameters $\mathrm{AE_1}$, $\mathrm{AE_2}$ of the diametral section of the ellipsoid, are perpendicular to each other, because (by the construction above-mentioned) they coincide in their directions respectively with the two supplementary chords $\mathrm{AD_1}$, $\mathrm{AD_2}$ of the section of the auxiliary sphere, and an angle in a semicircle is a right angle; and at the same time we see also that the difference of the squares of the reciprocals of these two rectangular semiaxes of a diametral section of the ellipsoid varies, in passing from one such section to another, proportionally to the rectangle under the projections, $\mathrm{B'C'}$ and $\mathrm{C'A}$, of the two fixed lines BC and CA, on the plane of the variable section. The difference of the squares of these reciprocals of the semi-axes of a section therefore varies (as indeed it is well-known to do) proportionally to the product of the sines of the inclinations of the plane of the section to two fixed diametral planes, which cut the ellipsoid in circles; and we see that the normals to these two latter or cyclic planes have precisely the directions of the sides BC, CA of the *generating triangle* ABC, which has for its corners the three fixed points employed in the foregoing construction: so that the auxiliary and *diacentric sphere*, employed in the same construction, touches one of those two cyclic planes at the centre A of the ellipsoid. If we take, as we are allowed to do, the point B external to this sphere, then the distance BC of this external point B from the centre C of the sphere is (by the construction) the semisum of the greatest and least semiaxes of the ellipsoid, while the radius CA of the sphere is the semidifference of the same two semiaxes: and (by the same construction) these greatest and least semiaxes of the ellipsoid, or their prolongations, intersect the surface of the same diacentric sphere in points which are respectively situated on the finite straight line BC itself, and on the prolongation of that line. The remaining side AB of the same fixed or generating triangle ABC is a semidiameter of the ellipsoid, drawn in the direction of the axis of one of the two circumscribed cylinders of revolution; a property which was mentioned in the 32nd article, and which may be seen to hold good, not only from the recent analysis conducted by the Cartesian method, but also and more simply from the geometrical consideration that

the constant rectangle under the two straight lines BD and AE, in the construction, exceeds the double area of the triangle ABE, and therefore exceeds the rectangle under the fixed line AB and the perpendicular let fall thereon from the variable point E of the ellipsoid, except at the limit where the angle ADB is right; which last condition determines a circular locus for D, and an elliptic locus for E, namely that ellipse of contact along which a cylinder of revolution round AB envelops the ellipsoid, and which here presents itself as a section of the cylinder by a plane. The radius of this cylinder is equal to the line BG, if G be the point of intersection, distinct from A, of the side AB of the generating triangle with the surface of the diacentric sphere; which line BG is also easily shown, on similar geometrical principles, as a consequence of the same construction, to be equal to the common radius of the two circular sections, or to the mean semiaxis of the ellipsoid, which is perpendicular to the greatest and the least. Hence also the side AB of the generating triangle is, in length, a fourth proportional to the three semiaxes, that is to the mean, the least, and the greatest, or to the mean, the greatest, and the least, of the three principal and rectangular semidiameters of the ellipsoid.

37. Resuming now the quaternion form of the equation of the ellipsoid,

$$(\alpha\rho + \rho\alpha)^2 - (\beta\rho - \rho\beta)^2 = 1, \tag{1.}$$

and making

$$\alpha + \beta = \frac{\iota}{\iota^2 - \kappa^2}, \quad \alpha - \beta = \frac{\kappa}{\iota^2 - \kappa^2}, \tag{2.}$$

and

$$\frac{\iota\rho + \rho\kappa}{\iota^2 - \kappa^2} = \mathrm{Q}, \quad \frac{\rho\iota + \kappa\rho}{\iota^2 - \kappa^2} = \mathrm{Q}', \tag{3.}$$

the two linear factors of the first member of the equation (1.) become the two conjugate quaternions Q and Q′, so that the equation itself becomes

$$\mathrm{QQ}' = 1. \tag{4.}$$

But by articles 19 and 20 (Phil. Mag. for July 1846), the product of any two conjugate quaternions is equal to the square of their common tensor; this common tensor of the two quaternions Q and Q′ is therefore equal to unity. Using, therefore, as in those articles, the letter T as the characteristic of the operation of *taking the tensor* of a quaternion, the equation of the ellipsoid reduces itself to the form

$$\mathrm{TQ} = 1; \tag{5.}$$

or, substituting for Q its expression (3.),

$$\mathrm{T}\left(\frac{\iota\rho + \rho\kappa}{\iota^2 - \kappa^2}\right) = 1; \tag{6.}$$

which latter form might also have been obtained, by the substitutions (2.), from the equation (3.) of the 30th article (Phil. Mag., June 1847), namely from the following:*

$$\mathrm{T}(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta) = 1. \tag{7.}$$

* See equation (35.) of the Abstract in the Proceedings of the Royal Irish Academy for July 1846. The equation of the ellipsoid marked (1.) in article 37 of the present paper, was communicated to the Academy in December 1845, and is numbered (21.) in the Proceedings of that date.

38. In the geometrical construction or generation of the ellipsoid, which was assigned in the preceding articles of this paper (see the Numbers of the Philosophical Magazine for June and September 1847), the significations of some of the recent symbols are the following. The two constant vectors ι and κ may be regarded as denoting, respectively, (in lengths and in directions,) the two sides of the generating triangle ABC, which are drawn from the centre C of the auxiliary and diacentric sphere, to the fixed superficial point B of the ellipsoid, and to the centre A of the same ellipsoid; the third side of the triangle, or the vector from A to B, being therefore denoted (in length and in direction) by $\iota - \kappa$: while ρ is the radius vector of the ellipsoid, drawn from the centre A to a variable point E of the surface; so that the constant vector $\iota - \kappa$ is, by the construction, a particular value of this variable vector ρ. The vector from A to C, being the opposite of that from C to A, is denoted by $-\kappa$; and if D be still the same auxiliary point on the surface of the auxiliary sphere, which was denoted by the same letter in the account already printed of the construction, then the vector from C to D, which may be regarded as being (in a sense to be hereafter more fully considered) the *reflexion* of $-\kappa$ with respect to ρ, is $= -\rho\kappa\rho^{-1}$; and consequently the vector from D to B is $= \iota + \rho\kappa\rho^{-1}$. The lengths of the two straight lines BD, and AE, are therefore respectively denoted by the two tensors $\mathrm{T}(\iota + \rho\kappa\rho^{-1})$ and $T\rho$; and the rectangle under those two lines is represented by the product of these two tensors, that is by the tensor of the product, or by $\mathrm{T}(\iota\rho + \rho\kappa)$. But by the fundamental equality of the lengths of the diagonals, AE, BD$'$, of the plane quadrilateral ABED$'$ in the construction, this rectangle under BD and AE is equal to the constant rectangle under BD and BD$'$, that is under the whole secant and its external part, or to the square on the tangent from B, if the point B be supposed external to the auxiliary sphere, which has its centre at C, and passes through D, D$'$ and A. Thus $\mathrm{T}(\iota\rho+\rho\kappa)$ is equal to $(\mathrm{T}\iota)^2 - (\mathrm{T}\kappa)^2$, or to $\kappa^2 - \iota^2$, which difference is here a positive scalar, because it is supposed that CB is longer than CA, or that

$$\mathrm{T}\iota > \mathrm{T}\kappa; \tag{8.}$$

and the quaternion equation (6.) of the ellipsoid reproduces itself, as a result of the geometrical construction, under the slightly simplified form*

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2. \tag{9.}$$

And to verify that this equation relative to ρ is satisfied (as we have seen that it ought to be) by the particular value

$$\rho = \iota - \kappa, \tag{10.}$$

which corresponds to the particular position B of the variable point E on the surface of the ellipsoid, we have only to observe that, identically,

$$\iota(\iota - \kappa) + (\iota - \kappa)\kappa = \iota^2 - \iota\kappa + \iota\kappa - \kappa^2 = \iota^2 - \kappa^2 = -(\kappa^2 - \iota^2);$$

and that (by article 19) the tensor of a negative scalar is equal to the positive opposite thereof.

* See the Proceedings of the Royal Irish Academy for July 1846, equation (44.).

39. The foregoing article contains a sufficiently simple process for the *retranslation* of the geometrical construction* of the ellipsoid described in article 31, into the language of the calculus of quaternions, from the construction itself had been originally derived, in the manner stated in the 30th article of this paper. Yet it may not seem obvious to readers unfamiliar with this calculus, why the expression $-\rho\kappa\rho^{-1}$ was taken, in that foregoing article 38, as one denoting, in length and in direction, that radius of the auxiliary sphere which was drawn from C to D; not in what sense, and for what reason, this expression $-\rho\kappa\rho^{-1}$ has been said to represent the reflexion of the vector $-\kappa$ with respect to ρ. As a perfectly clear answer to each of these questions, or a distinct justification of each of the assumptions or assertions thus referred to, may not only be useful in connexion with the present mode of considering the ellipsoid, but also may throw light on other applications of quaternions to the treatment of geometrical and physical problems, we shall not think it an irrelevant digression to enter here into some details respecting this expression $-\rho\kappa\rho^{-1}$, and respecting the ways in which it might present itself in calculations such as the foregoing. Let us therefore now denote by σ the vector, whatever it may be, from C to D in the construction (C being still the centre of the sphere); and let us propose to find an expression for this sought vector σ, as a function of ρ and of κ, by the principles of the calculus of quaternions.

40. For this purpose we have first the equation between tensors,

$$\mathrm{T}\sigma = \mathrm{T}\kappa; \tag{11.}$$

which expresses that the two vectors σ and κ are equally long, as being both radii of one common auxiliary sphere, namely those drawn from the centre C to the points D and A. And secondly, we have the equation

$$\mathrm{V}\,.\,(\sigma - \kappa)\rho = 0, \tag{12.}$$

where V is the characteristic of the operation of *taking the vector* of a quaternion; which equation expresses immediately that the product of the two vectors $\sigma - \kappa$ and ρ is scalar, and therefore that these two vector-factors are either exactly similar or exactly opposite in direction; since otherwise their product would be a quaternion, having always a vector part, although the scalar part of this quaternion-product $(\sigma - \kappa)\rho$ might vanish, namely by the factors becoming perpendicular to each other. Such being the immediate and general signification of equation (12.), the justification of our establishing it in the present question is derived from the consideration that the radius vector ρ, drawn from the centre A to the surface E of the ellipsoid, has, by the construction, a direction either exactly similar or exactly opposite to the direction of that *guide-chord* of the auxiliary sphere which is drawn from A

* The brevity and novelty of this rule for constructing that important surface may perhaps justify the reprinting it here. It was as follows: From a fixed point A on the surface of a sphere, draw a variable chord AD; let D′ be the second point of intersection of the spheric surface with the secant BD, drawn to the variable extremity D of the chord AD from a fixed external point B; take the radius vector AE equal in length to BD′, and in direction either coincident with, or opposite to, the chord AD; the locus of the point E, thus constructed, will be an ellipsoid, which will pass through the point B (and will have its centre at A). See Proceedings of the Royal Irish Academy for July 1846.

to D, that is, from the end of the radius denoted by κ to the end of the radius denoted by σ. For, that the chord so drawn is properly denoted, in length and in direction, by the symbol $\sigma - \kappa$, follows from principles respecting *addition and subtraction of directed lines*, which are indeed *essential*, but are *not peculiar*, to the geometrical applications of quaternions; had occurred, in various ways, to several independent inquirers, before quaternions (as *products of quotients of directed lines in space*) were thought of; and are now extensively received.

41. The two equations (11.) and (12.) are evidently both satisfied when we suppose $\sigma = \kappa$; but because the point D is in general different from A, we must endeavour to find another value of the vector σ, distinct from κ, which shall satisfy the same two equations. Such a value, or expression, for this sought vector σ may be found at once, so far as the equation (12.) is concerned, by observing that, in virtue of this latter equation, $\sigma - \kappa$ must bear some scalar ratio to ρ, or must be equal to this vector ρ multiplied by some scalar coefficient x, so that we may write

$$\sigma = \kappa + x\rho; \tag{13.}$$

and then, on substituting this expression for σ in the former equation (11.), we find that x must satisfy the condition

$$\mathrm{T}(\kappa + x\rho) = \mathrm{T}\kappa, \tag{14.}$$

in which this sought coefficient x is supposed to be some scalar different from zero, that is, in other words, some positive or negative number. Squaring both members of this last condition, and observing that by article 19 the square of the tensor of a vector is equal to the negative of the square of that vector, we find the new equation

$$-(\kappa + x\rho)^2 = -\kappa^2. \tag{15.}$$

But also, generally, if κ and ρ be vectors and x a scalar,

$$(\kappa + x\rho)^2 = \kappa^2 + x(\kappa\rho + \rho\kappa) + x^2\rho^2;$$

adding therefore κ^2 to both members of (15.), dividing by $-x$, and then eliminating x by (13.), which is done by merely changing $\kappa\rho + x\rho^2$ to $\sigma\rho$, we find the equation

$$\sigma\rho + \rho\kappa = 0; \tag{16.}$$

and finally

$$\sigma = -\rho\kappa\rho^{-1}: \tag{17.}$$

so that the expression already assigned for the vector from C to D, presents itself as the result of this analysis. And in fact the tensor of this expression (17.) is equal to $\mathrm{T}\kappa$, by the general rule for the tensor of a product, or because $(-\rho\kappa\rho^{-1})^2 = \rho\kappa\rho^{-1}\rho\kappa\rho^{-1} = \rho\kappa^2\rho^{-1} = \kappa^2$, since κ^2 is a (negative) scalar; while the product $(\sigma - \kappa)\rho$, being $= -(\kappa\rho + \rho\kappa)$, is equal, by article 20, to an expression of scalar form.

42. Conversely if, in any investigation conducted on the present principles, we meet with the expression $-\rho\kappa\rho^{-1}$, we may perceive in the way just now mentioned, that it denotes a vector of which the square is equal to that of κ; and that, if κ be subtracted from it, the remainder gives a scalar product when it is multiplied into ρ; so that, if we denote this expression by σ, or establish the equation (17.), the equations (11.) and (12.) will then be satisfied, and the vector σ will have the same length as κ, while the directions of $\sigma - \kappa$ and ρ will be either exactly similar or exactly opposite to each other. We may therefore be thus led to regard, subject to this condition (17.) or (16.), the two vector-symbols σ and κ as denoting, in length and in direction, two radii of one common sphere, such that the chord-line $\sigma - \kappa$ connecting their extremities has the direction of the line ρ, or of that line reversed. Hence also, by the elemetary property of a plane isosceles triangle, we may see that, under the same condition, the inclination of σ to ρ is equal to the inclination of κ to $-\rho$, or of $-\kappa$ to ρ; in such a manner that the bisector of the external vertical angle of the isosceles triangle, or the bisector of the angle at the centre of the sphere between the two radii σ and $-\kappa$, is a new radius parallel to ρ, because it is parallel to the base of the triangle (ACD), or to the chord (AD) just now mentioned. And by conceiving a diameter of the sphere parallel to this chord, or to ρ, and supposing $-\kappa$ to denote that reversed radius which coincides in situation with the radius κ, but is drawn from the surface to the centre (that is, in the recent construction, from A to C), while σ is still drawn from centre to surface (from C to D), we may be led to regard σ, or $-\rho\kappa\rho^{-1}$, as the *reflexion* of $-\kappa$ with respect to the diameter parallel to ρ, or simply with respect to ρ itself, as was remarked in the 38th article; since the vector-symbols ρ, σ, &c. are supposed, in these calculations, to indicate indeed the *lengths and directions, but not the situations*, of the straight lines which they are employed to denote.

43. The same geometrical interpretation of the symbol $-\rho\kappa\rho^{-1}$ may be obtained in several other ways, among which we shall specify the following. Whatever the lengths and directions of the two straight lines denoted by ρ and κ may be, we may always conceive that the latter line, regarded as a vector, is or may be decomposed, by two different projections, into two partial or component vectors, κ' and κ'', of which one is parallel and the other is perpendicular to ρ; so that they satisfy respectively the equations of parallelism and perpendicularity (see article 21), and that we have consequently,

$$\kappa = \kappa' + \kappa''; \quad \mathrm{V} . \kappa'\rho = 0; \quad \mathrm{S} . \kappa''\rho = 0; \tag{18.}$$

where S is the characteristic of the operation of *taking the scalar* of a quaternion. The equation of parallelism gives $\rho\kappa' = \kappa'\rho$, and the equation of perpendicularity gives $\rho\kappa'' = -\kappa''\rho$; hence the proposed expression $-\rho\kappa\rho^{-1}$ resolves itself into the two parts,

$$\left.\begin{aligned} -\rho\kappa'\rho^{-1} &= -\kappa'\rho\rho^{-1} = -\kappa'; \\ -\rho\kappa''\rho^{-1} &= +\kappa''\rho\rho^{-1} = +\kappa''; \end{aligned}\right\} \tag{19.}$$

so that we have, upon the whole,

$$-\rho\kappa\rho^{-1} = -\rho(\kappa' + \kappa'')\rho^{-1} = -\kappa' + \kappa''. \tag{20.}$$

The part $-\kappa'$ of this last expression, which is parallel to ρ, is the same as the corresponding part of $-\kappa$; but the part $+\kappa''$, perpendicular to ρ, is the same with the corresponding part of $+\kappa$, or is opposite to the corresponding part of $-\kappa$; we may therefore be led by this process also to regard the expression (17.) as denoting the reflexion of the vector $-\kappa$, with respect to the vector ρ, regarded as a reflecting line; and we see that the direction of ρ, or that of $-\rho$, is exactly intermediate between the two directions of $-\kappa$ and $-\rho\kappa\rho^{-1}$, or between those of κ and of $\rho\kappa\rho^{-1}$.

44. The equation (9.) of the ellipsoid, in article 38, or the equation (4.) in article 37, may be more fully written thus:

$$(\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) = (\kappa^2 - \iota^2)^2. \tag{21.}$$

And to express that we propose to cut this surface by any diametral plane, we may write the equation

$$\varpi\rho + \rho\varpi = 0, \tag{22.}$$

where ϖ denotes a vector to which that cutting plane is perpendicular: thus, if in particular, we change ϖ to κ, we find, for the corresponding plane through the centre, the equation

$$\kappa\rho + \rho\kappa = 0, \tag{23.}$$

which, when combined with (21.), gives

$$(\kappa^2 - \iota^2)^2 = (\iota - \kappa)\rho \,.\, \rho(\iota - \kappa) = (\iota - \kappa)\rho^2(\iota - \kappa) = (\iota - \kappa)^2\rho^2,$$

that is,

$$\rho^2 = \left(\frac{\kappa^2 - \iota^2}{\iota - \kappa}\right)^2; \tag{24.}$$

but this is the equation of a sphere concentric with the ellipsoid; therefore the diametral plane (23.) cuts the ellipsoid in a *circle*, or the plane itself is a *cyclic plane*. We see also that the vector κ, as being perpendicular to this plane (23.), is one of the *cyclic normals*, or normals to planes of circular section; which agrees with the construction, since we saw, in article 36, that the auxiliary or diacentric sphere, with centre C, touches one cyclic plane at the centre A of the ellipsoid. The same construction shows that the other cyclic plane ought to be perpendicular to the vector ι; and accordingly the equation

$$\iota\rho + \rho\iota = 0 \tag{25.}$$

represents this second cyclic plane; for, when combined with the equation (21.) of the ellipsoid, it gives

$$(\kappa^2 - \iota^2)^2 = \rho(\kappa - \iota) \,.\, (\kappa - \iota)\rho = \rho(\kappa - \iota)^2\rho = (\kappa - \iota)^2\rho^2,$$

and therefore conducts to the same equation (24.) of a concentric sphere as before; which sphere (24.) is thus seen to contain the intersection of the ellipsoid (21.) with the plane (25.), as well as that with the plane (23.). If we use the form (9.), we have only to observe that whether we change $\rho\kappa$ to $-\kappa\rho$, or $\iota\rho$ to $-\rho\iota$, we are conducted in each case to the following expression for the length of the radius vector of the ellipsoid, which agrees with the equation (24.):

$$\mathrm{T}\rho = \frac{\kappa^2 - \iota^2}{\mathrm{T}(\iota - \kappa)}. \tag{26.}$$

And because $\kappa^2 - \iota^2$ denotes the square upon the tangent drawn to the auxiliary sphere from the external point B, while $\mathrm{T}(\iota - \kappa)$ denotes the length of the side BA of the generating triangle, we see by this easy calculation with quaternions, as well as by the more purely geometrical reasoning which was alluded to, and partly stated, in the 36th article, that the common radius of the two diametral and circular sections of the ellipsoid is equal to the straight line which was there called BG, and which had the direction of BA, while terminating, like it, on the surface of the auxiliary sphere; so that the two last lines BA, and BG, were connected with that sphere and with each other, in this or in the opposite order, as the whole secant and the external part. In fact, as the point D, in the construction approaches, in any direction, on the surface of the auxiliary sphere, to A, the point D′ approaches to G; and BD′, and therefore also AE, tends to become equal in length to BG; while the direction of AE, being the same with that of AD, or opposite thereto, tends to become tangential to the sphere, or perpendicular to AC: the line BG is therefore equal to the radius of that diametral and circular section of the ellipsoid which is made by the plane that touches the auxiliary sphere at A. And again, if we conceive the point D′ to revolve on the surface of the sphere from G to G again, in a plane perpendicular to BC, then the lines AD and AE will revolve together in another plane parallel to that last mentioned, and perpendicular likewise to BC; while the length of AE will be still equal to the same constant line BG as before: which line is therefore found to be equal to the common radius of both the diametral and circular sections of the ellipsoid, whether as determined by the geometrical construction which the calculus of quaternions suggested, or immediately by that calculus itself.

45. We may write the equation (21.) of the ellipsoid as follows:

$$f(\rho) = 1, \tag{27.}$$

if we introduce a scalar function f of the variable vector ρ, defined as follows:

$$(\kappa^2 - \iota^2)^2 f(\rho) = (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) = \iota\rho^2\iota + \iota\rho\kappa\rho + \rho\kappa\rho\iota + \rho\kappa^2\rho;$$

or thus, in virtue of article 20,

$$(\kappa^2 - \iota^2)^2 f(\rho) = (\iota^2 + \kappa^2)\rho^2 + 2\mathrm{S}\,.\,\iota\rho\kappa\rho. \tag{28.}$$

Let $\rho + \tau$ denote another vector from the centre to the surface of the same ellipsoid; we shall have, in like manner,

$$f(\rho + \tau) = 1, \tag{29.}$$

where

$$f(\rho + \tau) = f(\rho) + 2\mathrm{S}\,.\,\nu\tau + f(\tau), \tag{30.}$$

if we introduce a new vector symbol ν, defined by the equation

$$(\kappa^2 - \iota^2)^2 \nu = (\iota^2 + \kappa^2)\rho + \iota\rho\kappa + \kappa\rho\iota; \tag{31.}$$

because generally, for any two vectors ρ and τ,

$$(\rho + \tau)^2 = \rho^2 + 2\mathrm{S} \,.\, \rho\tau + \tau^2, \tag{32.}$$

and, for any four vectors, ι, κ, ρ, τ,

$$\mathrm{S} \,.\, \iota\tau\kappa\rho = \mathrm{S} \,.\, \tau\kappa\rho\iota = \mathrm{S} \,.\, \kappa\rho\iota\tau = \mathrm{S} \,.\, \rho\iota\tau\kappa; \tag{33.}$$

which last principle, respecting certain transpositions of vector symbols, as factors of a product under the sign S., shows, when combined with the equations (27.), (28.), and (31.), that we have also this simple relation:

$$\mathrm{S} \,.\, \nu\rho = 1. \tag{34.}$$

Subtracting (27.) from (29.), attending to (30.), changing τ to $\mathrm{T}\tau \,.\, \mathrm{U}\tau$, where U is, as in article 19, the characteristic of the operation of *taking the versor* of a quaternion (or of a vector), and dividing by $\mathrm{T}\tau$, we find:

$$0 = \frac{f(\rho + \tau) - f(\rho)}{\mathrm{T}\tau} = 2\mathrm{S} \,.\, \nu\,\mathrm{U}\tau + \mathrm{T}\tau \,.\, f(\mathrm{U}\tau). \tag{35.}$$

This is a rigorous equation, connecting the *length* or the *tensor* $\mathrm{T}\tau$, of any chord τ of the ellipsoid, drawn from the extremity of the semidiameter ρ, with the *direction* of that chord τ, or with the *versor* $\mathrm{U}\tau$; it is therefore only a new form of the equation of the ellipsoid itself, with the origin of vectors removed from the centre to a point upon the surface. If we now conceive the chord τ to diminish in length, the term $\mathrm{T}\tau \,.\, f(\mathrm{U}\tau)$ of the right-hand member of this equation (35.) tends to become $= 0$, on account of the factor $\mathrm{T}\tau$; and therefore the other term $2\mathrm{S} \,.\, \nu\mathrm{U}\tau$ of the same member must tend to the same limit zero. In this way we arrive easily at an equation expressing the *ultimate law of the directions of the evanescent chords* of the ellipsoid, at the extremity of any given or assumed semidiameter ρ; which equation is $0 = 2\mathrm{S} \,.\, \nu\mathrm{U}\tau$, or simply,

$$0 = \mathrm{S} \,.\, \nu\tau, \tag{36.}$$

if τ be a tangential vector. The vector ν is therefore perpendicular to all such tangents, or infinitesimal chords of the ellipsoid, at the extremity of the semidiameter ρ; and consequently it has the direction of the *normal* to that surface, at the extremity of that semidiameter. The *tangent plane* to the same surface at the same point is represented by the equation (34.), if we treat, therein, the normal vector ν as constant, and if we regard the symbol ρ as denoting, in the same equation (34.), a variable vector, drawn from the centre of the ellipsoid to any point upon that tangent plane. This equation (34.) of the tangent plane may be written as follows:

$$\mathrm{S} \,.\, \nu(\rho - \nu^{-1}) = 0; \tag{37.}$$

and under this form it shows easily that the symbol ν^{-1} represents, in length and in direction, the perpendicular let fall from the origin of the vectors ρ, that is from the centre of the ellipsoid, upon the plane which is thus represented by the equation (34.) or (37.); so that the vector ν itself, as determined by the equation (31.) may be called the *vector of proximity** *of the tangent plane* of the ellipsoid, or of an element of that surface, to the centre, at the end of that semidiameter ρ from which ν is deduced by that equation.

46. Conceive now that at the extremity of an infinitesimal chord $\mathrm{d}\rho$ or τ, we draw another normal to the ellipsoid; the expression for any arbitrary point on the former normal, that is the symbol for the vector of this point, drawn from the centre of the ellipsoid, or from the origin of the vectors ρ, is of the form $\rho + n\nu$, where n is an arbitrary scalar; and in like manner the corresponding expression for an arbitrary point on the latter and infinitely near normal, or for its vector from the same centre of the ellipsoid, is $\rho + \mathrm{d}\rho + (n + \mathrm{d}n)(\nu + \mathrm{d}\nu)$, where $\mathrm{d}n$ is an arbitrary but infinitesimal scalar, and $\mathrm{d}\nu$ is the differential of the vector of proximity ν, which may be found as a function of the differential $\mathrm{d}\rho$ by differentiating the equation (31.), which connects the two vectors ν and ρ themselves. In this manner we find, from (31.),

$$(\kappa^2 - \iota^2)^2\,\mathrm{d}\nu = (\iota^2 + \kappa^2)\mathrm{d}\rho + \iota\,\mathrm{d}\rho\,\kappa + \kappa\,\mathrm{d}\rho\,\iota; \tag{38.}$$

and the condition required for the intersection of the two near normals, or for the existence of a point common to both, is expressed by the formula

$$\rho + \mathrm{d}\rho + (n + \mathrm{d}n)(\nu + \mathrm{d}\nu) = \rho + n\nu; \tag{39.}$$

which may be more concisely written as follows:

$$\mathrm{d}\rho + \mathrm{d}\,.\,n\nu = 0; \tag{40.}$$

or thus:

$$\mathrm{d}\rho + n\,\mathrm{d}\nu + \mathrm{d}n\,\nu = 0. \tag{41.}$$

We can eliminate the two scalar coefficients, n and $\mathrm{d}n$, from this last equation, according to the rules of the calculus of quaternions, by the method exemplified in the 24th article of this paper (Phil. Mag., August 1846), or by operating with the characteristic $\mathrm{S}\,.\,\nu\,\mathrm{d}\nu$, because generally

$$\mathrm{S}\,.\,\nu\mu^2 = 0, \quad \mathrm{S}\,.\,\nu\mu\nu = 0,$$

whatever vectors μ and ν may be; so that here,

$$\mathrm{S}\,.\,\nu\,\mathrm{d}\nu\,n\,\mathrm{d}\nu = 0, \quad \mathrm{S}\,.\,\nu\,\mathrm{d}\nu\,\mathrm{d}n\,\nu = 0.$$

* This name, "vector of proximity," was suggested to the writer by a phraseology of Sir John Herschel's; and the equation (31.), of article 45, which determines this vector for the ellipsoid, was one of a few equations which were designed to have been exhibited to the British Association at its meeting in 1846: but were accidentally forwarded at the last moment to Collingwood, instead of Southampton, and did not come to the hands of the eminent philosopher just mentioned, until it was too late for him to do more than return the paper, with some of those encouraging expressions by which he delights to cheer, as opportunities present themselves, all persons whom he conceives to be labouring usefully for science.

In this manner we find from (41.) the following very simple formula:

$$\mathrm{S} \,.\, \nu \, \mathrm{d}\nu \, \mathrm{d}\rho = 0; \tag{42.}$$

which is easily seen, on the same principles, to hold good, as the *quaternion form of the differential equation of the lines of curvature on a curved surface generally*, if ν be still the *vector of proximity of the superficial element* of the curved surface to the origin of the vectors ρ, which vector ν is determined by the general condition

$$\mathrm{S} \,.\, \nu \, \mathrm{d}\rho = 0, \tag{43.}$$

combined with the equation already written,

$$\mathrm{S} \,.\, \nu\rho = 1 \qquad (34.);$$

or simply if ν be a *normal vector*, satisfying the condition (43.) alone. Substituting, therefore, in the case of the ellipsoid, the expression for $\mathrm{d}\nu$ given by (38.), and observing that $\mathrm{S}.\nu \, \mathrm{d}\rho^2 = 0$, we find that we may write the equation of the lines of curvature for this particular surface as follows:

$$\mathrm{S} \,.\, \nu(\iota \, \mathrm{d}\rho \, \kappa + \kappa \, \mathrm{d}\rho \, \iota) \, \mathrm{d}\rho = 0; \tag{44.}$$

which equation, when treated by the rules of the present calculus, admits of being in many ways symbolically transformed, and may also, with little difficulty, be translated into geometrical enunciations.

47. Thus if we observe that, by article 20, $\iota\tau\kappa - \kappa\tau\iota$ is a *scalar form*, whatever three vectors may be denoted by ι, κ, τ; and if we attend to the equation (43.), which expresses that the normal ν is perpendicular to the linear element, or infinitesimal chord, $\mathrm{d}\rho$; we shall perceive that, for *every* direction of that element, the following equation holds good:

$$\mathrm{S} \,.\, \nu(\iota \, \mathrm{d}\rho \, \kappa - \kappa \, \mathrm{d}\rho \, \iota) \, \mathrm{d}\rho = 0. \tag{45.}$$

We have therefore, from (44.), for those *particular* directions which belong to the lines of curvature, this simplified equation;

$$\mathrm{S} \,.\, \nu\iota \, \mathrm{d}\rho \, \kappa \, \mathrm{d}\rho = 0; \tag{46.}$$

which may be still a little abridged, by writing instead of $\mathrm{d}\rho$ the symbol τ of a tangential vector, already used in (36.); for thus we obtain the formula:

$$\mathrm{S} \,.\, \nu\iota\tau\kappa\tau = 0. \tag{47.}$$

We might also have observed that by the same article 20 (Phil. Mag., July 1846), $\iota\tau\kappa + \kappa\tau\iota$ and therefore $\iota \, \mathrm{d}\rho \, \kappa + \kappa \, \mathrm{d}\rho \, \iota$ is a *vector form*, and that by article 26 (Phil. Mag., August 1846), three vector-factors under the characteristic S may be in any manner transposed, with only a change (at most) in the positive or negative sign of the resulting scalar; from which it would have followed, by a process exactly similar to the foregoing, that the equation (44.) of the lines of curvature on an ellipsoid may be thus written,

$$\mathrm{S} \,.\, \nu \, \mathrm{d}\rho \, \iota \, \mathrm{d}\rho \, \kappa = 0; \tag{48.}$$

or, substituting for the linear element $\mathrm{d}\rho$ the tangential vector τ,

$$\mathrm{S} \,.\, \nu\tau\iota\tau\kappa = 0; \tag{49.}$$

or finally, by the principles of the same 20th article,

$$\nu\tau\iota\tau\kappa - \kappa\tau\iota\tau\nu = 0. \tag{50.}$$

48. Under this last form, it was one of a few equations selected in September 1846, for the purpose of being exhibited to the Mathematical Section of the British Association at Southampton; although it happened* that the paper containing those equations did not reach its destination in time to be so exhibited. The equations here marked (49.) and (50.) were however published before the close of the year in which that meeting was held, as part of the abstract of a communication which had been made to the Royal Irish Academy in the summer of that year. (See the Proceedings of the Academy for July 1846, equations (46.) and (47.).) From the somewhat discursive character of the present series of communications on Quaternions, and from the desire which the author feels to render them, to some extent, complete within themselves, or at least intelligible to those mathematical readers of the Philosophical Magazine who may be disposed to favour him with their attention, to the degree which the novelty of the conceptions and method may require, without its being *necessary* for such readers to refer to other publications of his own, he is induced, and believes himself to be authorized, to copy here a few other equations from that short and hitherto unpublished Southampton paper, and to annex to them another formula which may be found in the Proceedings, already cited, of the Royal Irish Academy: together with a more extensive formula, which he believes to be new.

49. Besides the equation of the ellipsoid,

$$(\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) = (\kappa^2 - \iota^2)^2 \qquad \text{(21.), art. 44;}$$

with the expression derived from it, for the vector of proximity of that surface to its centre,

$$(\kappa^2 - \iota^2)^2\nu = (\iota^2 + \kappa^2)\rho + \iota\rho\kappa + \kappa\rho\iota \qquad \text{(31.), art. 45;}$$

the equation for the lines of curvature on the ellipsoid,

$$\nu\tau\iota\tau\kappa - \kappa\tau\iota\tau\nu = 0 \qquad \text{(50.), art. 47;}$$

and the equation

$$\nu\tau + \tau\nu = 0, \tag{51.}$$

* See the note to article 45.

which is a form of the relation $\mathrm{S}.\nu\tau = 0$, that is of the equation (36.), article 45, of the present series of communications; the author gave, in the paper which has been above referred to, the following symbolic transformation, for the well-known characteristic of operation,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2,$$

which seems to him to open a wide and new field of analytical research, connected with many important and difficult departments of the mathematical study of nature.

A QUATERNION, *symbolically considered*, being (according to the views originally proposed by the author in 1843) an algebraical quadrinomial of the form $w + ix + jy + kz$, where $w\,x\,y\,z$ are any four real numbers (positive or negative or zero), while $i\,j\,k$ are three co-ordinate imaginary units, subject to the fundamental laws of combination (see Phil. Mag. for July 1844):

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = -1; \\ ij = k; \quad jk = i; \quad ki = j; \\ ji = -k; \quad kj = -i; \quad ik = -j; \end{array}\right\} \tag{a.}$$

it results at once from these definitions, or laws of symbolic combination, (a.), that if we introduce a new characteristic of operation, $\triangleleft$, defined with relation to these three symbols $i\,j\,k$, and to the known operation of partial differentiation, performed with respect to three independent but real variables $x\,y\,z$, as follows:

$$\triangleleft = \frac{i\mathrm{d}}{\mathrm{d}x} + \frac{j\mathrm{d}}{\mathrm{d}y} + \frac{k\mathrm{d}}{\mathrm{d}z}; \tag{b.}$$

this new characteristic $\triangleleft$ *will have the negative of its symbolic square expressed by the following formula:*

$$-\triangleleft^2 = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2; \tag{c.}$$

of which it is clear that the applications to analytical physics must be extensive in a high degree. In the paper* designed for Southampton it was remarked, as an illustration, that this result enables us to put the known thermological equation,

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + a\frac{\mathrm{d}v}{\mathrm{d}t} = 0,$$

under the new and more symbolic form,

$$\left(\triangleleft^2 - \frac{a\mathrm{d}}{\mathrm{d}t}\right) v = 0; \tag{d.}$$

while $\triangleleft\, v$ denotes, in quantity and in direction, the *flux* of heat, at the time t and at the point $x\,y\,z$.

* In that paper itself, the characteristic was written ∇; but this more common sign has been so often used with other meanings, that it seems desirable to abstain from appropriating it to the new signification here proposed.

50. In the Proceedings of the Royal Irish Academy for July 1846, it will be found to have been noticed that the same new characteristic $\lhd$ gives also this other general transformation, perhaps not less remarkable, nor having less extensive consequences, and which presents itself under the form of a quaternion:

$$\lhd(it+ju+kv) = -\left(\frac{\mathrm{d}t}{\mathrm{d}x}+\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}v}{\mathrm{d}z}\right)+i\left(\frac{\mathrm{d}v}{\mathrm{d}y}-\frac{\mathrm{d}u}{\mathrm{d}z}\right)+j\left(\frac{\mathrm{d}t}{\mathrm{d}z}-\frac{\mathrm{d}v}{\mathrm{d}x}\right)+k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{\mathrm{d}t}{\mathrm{d}y}\right). \quad \text{(e.)}$$

In fact the equations (a.) give generally (see art. 21 of the present series),

$$(ix+jy+kz)(it+ju+kv) = -(xt+yu+zv)+i(yv-zu)+j(zt-xv)+k(xu-yt), \quad \text{(f.)}$$

if $x\,y\,z\,t\,u\,v$ denote any six real numbers; and the calculations by which this is proved, show, still more generally, that the same transformation must hold good, if each of the three symbols i, j, k, subject still to the equations (a.), be commutative in arrangement, as a symbolic factor, with each of the three other symbols x, y, z; even though the latter symbols, like the former, should not be commutative in that way among themselves; and even if they should denote symbolical instead of numerical multipliers, possessing still the distributive character. We may therefore change the three symbols x, y, z, respectively, to the three characteristics of partial differentiation, $\dfrac{\mathrm{d}}{\mathrm{d}x}$, $\dfrac{\mathrm{d}}{\mathrm{d}y}$, $\dfrac{\mathrm{d}}{\mathrm{d}z}$; and thus the formula (e.) is seen to be included in the formula (f.). And if we then, in like manner, change the three symbols t, u, v, regarded as factors, to $\dfrac{\mathrm{d}}{\mathrm{d}x'}$, $\dfrac{\mathrm{d}}{\mathrm{d}y'}$, $\dfrac{\mathrm{d}}{\mathrm{d}z'}$, that is, to the characteristics of three partial differentiations performed with respect to three new and independent variables x', y', z', we shall thereby change $\dfrac{\mathrm{d}t}{\mathrm{d}x}$ to $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\mathrm{d}}{\mathrm{d}x'}$, and so obtain the formula:

$$\left.\begin{aligned}\left(i\frac{\mathrm{d}}{\mathrm{d}x}+j\frac{\mathrm{d}}{\mathrm{d}y}+k\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(i\frac{\mathrm{d}}{\mathrm{d}x'}+j\frac{\mathrm{d}}{\mathrm{d}y'}+k\frac{\mathrm{d}}{\mathrm{d}z'}\right) &= -\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x'}+\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}y'}+\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}z'}\right)\\ +i\left(\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}z'}-\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}y'}\right)+j\left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}x'}-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}z'}\right)&+k\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y'}-\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x'}\right);\end{aligned}\right\} \quad \text{(g.)}$$

which includes the formula (c.), and is now for the first time published.

This formula (g.) is, however, seen to be a very easy and immediate consequence from the author's fundamental equations of 1843, or from the relations (a.) of the foregoing article, which admit of being concisely summed up in the following continued equation:

$$i^2 = j^2 = k^2 = ijk = -1. \quad \text{(h.)}$$

The geometrical interpretation of the equation $\mathrm{S}\,.\,\nu\tau\iota\tau\kappa = 0$ of the lines of curvature on the ellipsoid, with some other applications of quaternions to that important surface, must be reserved for future articles of the present series, of which some will probably appear in an early number of this Magazine.

51. It has been shown* that if the two symbols ι, κ denote certain constant vectors, perpendicular to the two cyclic planes of an ellipsoid, and if ν, τ denote two other and variable vectors, of which the former is normal to the ellipsoid at any proposed point upon its surface, while the latter is tangential to a line of curvature at that point, then the *directions* of these four vectors ι, κ, ν, τ are so related to each other as to satisfy the condition

$$\mathrm{S} \,.\, \nu\tau\iota\tau\kappa = 0 \qquad \text{(49.), article 47;}$$

S being the characteristic of the operation of taking the scalar part of a quaternion. And because the two latter of these four directions, namely the directions of the normal and tangential vectors ν and τ, are always perpendicular to each other, this additional equation has been seen to hold good:

$$\mathrm{S} \,.\, \nu\tau = 0 \qquad \text{(36.), article 45.}$$

Retaining the same significations of the symbols, and carrying forward for convenience the recent numbering of the formulæ, it is now proposed to point out some of the modes of combining, transforming, and interpreting the system of these two equations, consistently with the principles and rules of the Calculus of Quaternions, from which the equations themselves have been derived.

52. Whatever two vectors may be denoted by ι and τ, the ternary product $\tau\iota\tau$ is always a *vector form*, because (by article 20) its scalar part is zero; and on the other hand the square τ^2 is a pure scalar: therefore we may always write

$$\tau\iota\tau = \mu\tau^2, \quad \tau\iota = \mu\tau, \tag{52.}$$

where μ is a new vector, expressible in terms of ι and τ as follows:

$$\mu = \tau\iota\tau^{-1}; \tag{53.}$$

so that it is, in general, by the principles of articles 40, 41, 42, 43, the *reflexion* of the vector ι with respect to the vector τ, and that thus the direction of τ is exactly intermediate between the directions of ι and μ. In the present question, this new vector μ, defined by the equation (53.) may therefore represent the reflexion of the first cyclic normal ι, with respect to any reflecting line which is parallel to the vector τ, which latter vector is tangential to one of the curves of curvature on the ellipsoid. Substituting for $\tau\iota\tau$ its value (52.) in the lately cited equation (49.), and suppressing the scalar factor τ^2, we find this new equation:

$$\mathrm{S} \,.\, \nu\mu\kappa = 0; \tag{54.}$$

which, in virtue of the general *equation of coplanarity* assigned in the 21st article (Phil. Mag. for July 1846), expresses that the reflected vector μ, the normal vector ν, and the second

* See the Philosophical Magazine for October 1847; or Proceedings of the Royal Irish Academy for July 1846.

cyclic normal κ, are parallel to one common plane. This result gives already a characteristical geometric property of the lines of curvature on an ellipsoid, from which the directions of those curved lines, or of their tangents (τ), can generally be assigned, at any given point upon the surface, when the direction of the normal (ν) at that point, and those of the two cyclic normals (ι and κ), are known. For it shows that if a straight line μ be found, in any plane parallel to the given lines ν and κ, such that the bisector τ of the angle between this line μ and a line parallel to the other given line ι shall be perpendicular to the given line ν, then this bisecting line τ will have the sought direction of a tangent to a line of curvature. But it is possible to deduce a geometrical determination, or construction, more simple and direct than this, by carrying the calculation a little further.

53. The equation (52.) gives

$$(\mu + \iota)\tau = \tau\iota + \iota\tau = \mathrm{V}^{-1}0, \tag{55.}$$

this last symbol $\mathrm{V}^{-1}0$ denoting generally any quaternion of which the vector part vanishes; that is any pure scalar, or in other words any real number, whether positive or negative or null. Hence $\mu + \iota$ and τ denote, in the present question, two coincident or parallel vectors, of which the directions are either exactly similar or else exactly opposite to each other; since if they were inclined at any actual angle, whether acute or obtuse, their product would be a quaternion, of which the vector part would not be equal to zero. Accordingly the expression (53.) gives this equation between tensors,

$$\mathrm{T}\mu = \mathrm{T}\iota; \tag{56.}$$

so that the symbols μ and ι denote here two equally long straight lines; and therefore one diagonal of the equilateral parallelogram (or rhombus) which is constructed with those lines for two adjacent sides bisects the angle between them. But by the last article, this bisector has the direction of τ (or of $-\tau$); and by one of those fundamental principles of the geometrical interpretation of symbols, which are *common* to the calculus of quaternions and to several earlier and some later systems, the symbol $\mu + \iota$ denotes generally the intermediate diagonal of a parallelogram constructed with the lines denoted by μ and ι for two adjacent sides: we might therefore in this way also have seen that the vector $\mu + \iota$ has, in the present question, the direction of $\pm\tau$. This vector $\mu + \iota$ is therefore perpendicular to ν, and we have the equation

$$0 = \mathrm{S}\,.\,\nu(\mu + \iota), \quad \text{or} \quad \mathrm{S}\,.\,\nu\mu = -\mathrm{S}\,.\,\nu\iota. \tag{57.}$$

But by (56.), and by the general rule for the tensor of a product (see art. 20), we have also

$$\mathrm{T}\,.\,\nu\mu = \mathrm{T}\,.\,\nu\iota; \tag{58.}$$

and in general (by art. 19), the square of the tensor of a quaternion is equal to the square of the scalar part, minus the square of the vector part of that quaternion; or in symbols (Phil. Mag., July 1846),

$$(\mathrm{TQ})^2 = (\mathrm{SQ})^2 - (\mathrm{VQ})^2.$$

Hence the two quaternions $\nu\mu$ and $\nu\iota$, since they have equal tensors and opposite scalar parts, must have the squares of their vector parts equal, and those vector parts themselves must have their tensors equal to each other; that is, we may write

$$(\mathrm{V}\,.\,\nu\mu)^2 = (\mathrm{V}\,.\,\nu\iota)^2, \quad \mathrm{TV}\,.\,\nu\mu = \mathrm{TV}\,.\,\nu\iota: \tag{59.}$$

and may regard these two vector parts of these two quaternions, or of the products $\nu\mu$ and $\nu\iota$, as denoting two equally long straight lines. Consequently the vector $\pm\nu\tau$, which has the direction of the line represented by the pure vector product $\nu(\mu + \iota)$, or by the sum $\mathrm{V}\,.\,\nu\mu + \mathrm{V}\,.\,\nu\iota$ of two equally long vectors, has at the same time a direction of the sum of the two corresponding *versors* of those vectors, or that of the sum of their *vector-units*; so that we may write the equation

$$t\nu\tau = \mathrm{UV}\,.\,\nu\mu + \mathrm{UV}\,.\,\nu\iota, \tag{60.}$$

where U is (as in art. 19) the characteristic of the operation of taking the versor of a quaternion, or of a vector; and t is a scalar coefficient. Again, the equation $0 = \mathrm{S}\,.\,\nu\mu\kappa$, (54.), which expresses that the three vectors ν, μ, κ are coplanar, shows also that the two vectors $\mathrm{V}\,.\,\nu\mu$ and $\mathrm{V}\,.\,\nu\kappa$ are parallel to each other, as being both perpendicular to that common plane to which ν, μ and κ are parallel; hence we have the following equation between two versors of vectors, or between two vector-units,

$$\mathrm{UV}\,.\,\nu\mu = \pm\mathrm{UV}\,.\,\nu\kappa; \tag{61.}$$

and therefore instead of the formula (60.) we may write

$$t\tau = \nu^{-1}\,\mathrm{UV}\,.\,\nu\iota \pm \nu^{-1}\,\mathrm{UV}\,.\,\nu\kappa. \tag{62.}$$

In this expression for a vector touching a line of curvature, or parallel to such a tangent, the two terms connected by the sign $\pm$ are easily seen to denote (on the principles of the present calculus) two equally long vectors, in the directions respectively of the projections of the two cyclic normals ι and κ on a plane perpendicular to ν; that is, on the tangent plane to the ellipsoid at the proposed point, or on any plane parallel thereto. If then we draw two straight lines through the point of contact, bisecting the acute and obtuse angles which will in general be formed at that point by the projections on the tangent plane of two indefinite lines drawn through the same point in the directions of the two cyclic normals, or in directions perpendicular to the two planes of circular section of the surface, *the two rectangular bisectors of angles, so obtained, will be the tangents to the two lines of curvature*: which very simple construction agrees perfectly with known geometrical results, as will be more clearly seen, when it is slightly transformed as follows.

54. If we multiply either of the two tangential vectors τ by the normal vector ν, the product of these two rectangular vectors will be, by one of the fundamental and *peculiar**

* See the author's Letter of October 17, 1843, to John T. Graves, Esq., printed in the Supplementary Number of the Philosophical Magazine for December 1844: in which Letter, the three fundamental symbols, i, j, k were what it has been since proposed to name *direction-units.*

principles of the calculus of quaternions, a third vector rectangular to both; we shall therefore only pass by this multiplication, so far as *directions* are concerned, from one to the other of the tangents of the two lines of curvature: consequently we may omit the factor ν^{-1} in the second member of (62.), at least if we change (for greater facility of comparison of the results among themselves) the ambiguous sign $\pm$ to its opposite. We may also suppress the scalar coefficient t, if we only wish to form an expression for a line τ which shall have the required *direction* of a tangent, without obliging the *length* of this line τ to take any previously chosen value. The formula for the system of the two tangents to the two lines of curvature thus takes the simplified form:

$$\tau = \mathrm{UV}\,.\,\nu\iota \mp \mathrm{UV}\,.\,\nu\kappa; \tag{63.}$$

in which the two terms connected by the sign $\mp$ are two vector-units, in the respective directions of the traces of the two cyclic planes upon the tangent plane. The tangents to the two lines of curvature at any point of the surface of an ellipsoid (and the same result holds good also for other surfaces of the second order), are therefore parallel to the two rectangular straight lines which bisect the angles between those traces; or they are themselves the bisectors of the angles made at the point of contact by the traces of planes parallel to the two cyclic planes. The discovery of this remarkable geometrical theorem appears to be due to M. Chasles. It is only brought forward here for the sake of the *process* by which it has been above deduced (and by which the writer was in fact led to perceive the theorem before he was aware that it was already known), through an application of the method of quaternions, and as a corollary from the geometrical construction of the ellipsoid itself to which that method conducted him.* For that new geometrical *construction* has been shown (in a recent Number of this Magazine) to admit of being easily *retranslated* into that quaternion form of the *equation*† of the ellipsoid, namely

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad \text{equation (9.), art. (38.),}$$

as an *interpretation* of which equation it had been assigned by the present writer; and then a *general* method for investigating by quaternions the directions of the lines of curvature on *any* curved surface whatever, conducts, as has been shown (in articles 46 and 47), to the equation of those lines for the ellipsoid,

$$\mathrm{S}\,.\,\nu\tau\iota\tau\kappa = 0 \qquad (49.);$$

from which, when combined with the general equation $\mathrm{S}\,.\,\nu\tau = 0$, the formula (63.) has been deduced and geometrically interpreted as above.

55. Another mode of investigating generally the directions of those tangential vectors τ which satisfy the system of the two conditions in art. 51, may be derived from observing that

* See the Numbers of the Philosophical Magazine for June, September, and October 1847; or the Proceedings of the Royal Irish Academy for July 1846.

† Another very simple construction, derived from the same quaternion equation, and serving to generate, by a moving sphere, a system of *two* reciprocal ellipsoids, will be given in an early Number of this Magazine.

those conditions fail to distinguish one such tangential vector from another in each of the two cases where the variable normal ν coincides in direction with either of the two fixed cyclic normals, ι and κ; that is, at the four *umbilical points* of the ellipsoid, as might have been expected from the known properties of that surface. In fact if we suppose

$$\nu = m\iota, \quad \mathrm{S} \,.\, \iota\tau = 0, \tag{64.}$$

where m is a scalar coefficient, that is if we attend to either of those two opposite *umbilics* at which ν has the direction of ι, we find the value

$$\nu\tau\iota\tau\kappa = m(\iota\tau)^2\kappa, \tag{65.}$$

which is here a vector-form, because by (64.) the product $\iota\tau$ denotes in this case a *pure vector*, so that *its square (like that of every other vector in this theory) will be a negative scalar*, by one of the fundamental and *peculiar** principles of the present calculus; the scalar part of the product $\nu\tau\iota\tau\kappa$ therefore vanishes, or the condition (49.) is satisfied by the suppositions (64.). Again, if we suppose

$$\nu = m'\kappa, \tag{66.}$$

m' being another scalar coefficient, that is, if we consider either of those two other opposite umbilics at which ν has the direction of κ, we are conducted to this other expression,

$$\nu\tau\iota\tau\kappa = m'\kappa\tau\iota\tau\kappa; \tag{67.}$$

which also is a vector-form, by the principles of the 20th article. In this manner we may be led to see that if in general we decompose, by orthogonal projections, each of the two cyclic normals, ι and κ, into two partial or component vectors, ι', ι'', and κ', κ'', of which ι' and κ' shall be tangential to the surface, or perpendicular to the variable normal ν, but ι'' and κ'' parallel to that normal, in such a manner as to satisfy the two sets of equations,

$$\left.\begin{aligned} \iota = \iota' + \iota''; \quad \mathrm{S} \,.\, \iota'\nu = 0; \quad \mathrm{V} \,.\, \iota''\nu = 0; \\ \kappa = \kappa' + \kappa''; \quad \mathrm{S} \,.\, \kappa'\nu = 0; \quad \mathrm{V} \,.\, \kappa''\nu = 0; \end{aligned}\right\} \tag{68.}$$

then, on substituting these values for ι and κ in the condition (49.) or in the equation $0 = \mathrm{S} \,.\, \nu\tau\iota\tau\kappa$, the terms involving ι'' and κ'' will vanish of themselves, and the equation to be satisfied will become

$$0 = \mathrm{S} \,.\, \nu\tau\iota'\tau\kappa'; \tag{69.}$$

which is thus far a simplified form of the equation (49.), that three of the four directions to be compared (namely those of ι', κ', and τ) are now parallel to one common plane, namely to the plane which touches the ellipsoid at the proposed point, and to which the fourth direction (that of ν) is perpendicular. Decomposing the two quaternion products, $\tau\iota'$ and $\tau\kappa'$, into their respective scalar and vector parts, by the general formulæ,

$$\left.\begin{aligned} \tau\iota' = \mathrm{S} \,.\, \tau\iota' + \mathrm{V} \,.\, \tau\iota'; \\ \tau\kappa' = \mathrm{S} \,.\, \tau\kappa' + \mathrm{V} \,.\, \tau\kappa'; \end{aligned}\right\} \tag{70.}$$

* See the author's letter of October 17, 1843, already cited in a note to article 54.

and observing that the vectors $\mathrm{V}\,.\,\tau\iota'$ and $\mathrm{V}\,.\,\tau\kappa'$ both represent lines parallel to ν, because ν is perpendicular to the common plane of τ, ι', κ'; so that the three following binary products $\mathrm{V}\,.\,\tau\iota'\,.\,\mathrm{V}\,.\,\tau\kappa'$, $\nu\,\mathrm{V}\,.\,\tau\iota'$, $\nu\,\mathrm{V}\,.\,\tau\kappa'$, are in the present question scalars; we find that we may write

$$\mathrm{S}\,.\,\nu\tau\iota'\tau\kappa' = \nu\,\mathrm{S}\,.\,\tau\iota'\,.\,\mathrm{V}\,.\,\tau\kappa' + \nu\,\mathrm{V}\,.\,\tau\iota'\,.\,\mathrm{S}\,.\,\tau\kappa'. \tag{71.}$$

Hence the equation (69.) or (49.) reduces itself, after being multiplied by ν^{-1}, to the form

$$\mathrm{S}\,.\,\tau\iota'\,.\,\mathrm{V}\,.\,\tau\kappa' + \mathrm{V}\,.\,\tau\iota'\,.\,\mathrm{S}\,.\,\tau\kappa' = 0; \tag{72.}$$

which gives, in general, by the rules of the present calculus,

$$\frac{\mathrm{V}\,.\,\iota'\tau}{\mathrm{S}\,.\,\iota'\tau} = \frac{\mathrm{V}\,.\,\tau\kappa'}{\mathrm{S}\,.\,\tau\kappa'}; \tag{73.}$$

and by another transformation,

$$\frac{\mathrm{V}\,.\,\iota'\tau^{-1}}{\mathrm{S}\,.\,\iota'\tau^{-1}} = -\frac{\mathrm{V}\,.\,\kappa'\tau^{-1}}{\mathrm{S}\,.\,\kappa'\tau^{-1}}; \tag{74.}$$

which may perhaps be not inconveniently written also thus:

$$\frac{\mathrm{V}}{\mathrm{S}}\,.\,\frac{\iota'}{\tau} = -\frac{\mathrm{V}}{\mathrm{S}}\,.\,\frac{\kappa'}{\tau}; \tag{75.}$$

in using which abridged notation, we must be careful to remember, respecting the characteristic $\dfrac{\mathrm{V}}{\mathrm{S}}$, of which the effect is to form or to denote the *quotient of the vector part divided by the scalar part* of any quaternion expression to which it is prefixed, that *this new characteristic of operation is not* (like S and V themselves) *distributive relatively to the operand.* The vector denoted by the first member of (74.) or (75.) is a line perpendicular to the plane of ι' and τ, that is to the tangent plane of the ellipsoid; and its length is the trigonometric tangent of the angle of rotation in that plane from the direction of the line τ to that of the line ι'; while a similar interpretation applies to the second member of either of the same two equations, the sign $-$ in that second member signifying here that the two equally long angular motions, or rotations, from τ to ι', and from τ to κ' are performed in opposite directions. Thus the vector τ, which touches a line of curvature, coincides in direction with the bisector of the angle in the tangent plane between the projections, ι' and κ', of the cyclic normals thereupon; or with that other line, at right angles to this last bisector, which bisects in like manner the other and supplementary angle in the same tangent plane, between the directions of ι' and $-\kappa'$: since κ' may be changed to $-\kappa'$, without altering essentially any one of the four last equations between τ, ι', κ'. Those two rectangular and known directions of the tangents to the lines of curvature at any point of an ellipsoid, which were obtained by the process of article 53, are therefore obtained also by the process of the present article; which conducts, by the help of the geometrical reasoning above indicated, to the following expression for the system of those two tangents τ, as the symbolical solution (in the language of the present calculus) of any one of the four last equations (72.) … (75.):

$$\tau = t'(\mathrm{U}\iota' \pm \mathrm{U}\kappa'); \tag{76.}$$

where t' is a scalar coefficient.

The agreement of this symbolical result with that marked (62.) may be made evident observing that the equations (68.) give

$$\iota' = \nu^{-1}\,\mathrm{V}\,.\,\nu\iota; \quad \kappa' = \nu^{-1}\,\mathrm{V}\,.\,\nu\kappa; \tag{77.}$$

so that if we establish, as we may, the relation

$$tt' = (\mathrm{T}\nu)^{-1}, \tag{78.}$$

between the arbitrary scalar coefficients t and t', which enter into the formulæ (62.) and (76.), those formulæ will coincide with each other. And to show, without introducing geometrical considerations, that (for example) the form (73.) of the recent condition relatively to τ is symbolically satisfied by the expression (76.), we may remark that this expression, when operated upon according to the *general rules* of this calculus, gives

$$\left.\begin{aligned} &\mathrm{T}\kappa'\,.\,\mathrm{V}\,.\,\iota'\tau = \pm t'\,\mathrm{V}\,.\,\iota'\kappa'; \quad \mathrm{T}\kappa'\,.\,\mathrm{S}\,.\,\iota'\tau = t'(-\mathrm{T}\,.\,\iota'\kappa' \pm \mathrm{S}\,.\,\iota'\kappa');\\ &\mathrm{T}\iota'\,.\,\mathrm{V}\,.\,\tau\kappa' = t'\,\mathrm{V}\,.\,\iota'\kappa'; \qquad \mathrm{T}\iota'\,.\,\mathrm{S}\,.\,\tau\kappa' = t'(\mathrm{S}\,.\,\iota'\kappa' \mp \mathrm{T}\,.\,\iota'\kappa'); \end{aligned}\right\} \tag{79.}$$

and that therefore the two members of (73.) do in fact receive, in virtue of (76.) one common symbolical value, namely one or other of the two which are included in the ambiguous form

$$\frac{\mathrm{V}\,.\,\iota'\kappa'}{\mathrm{S}\,.\,\iota'\kappa' \mp \mathrm{T}\,.\,\iota'\kappa'}; \tag{80.}$$

respecting which form it may not be useless to remark that the product of its two values is unity.

56. If we denote by b the length of the common radius of the two diametral and circular sections, or the mean semiaxis of the ellipsoid, which is also the radius of that concentric sphere of which the equation (24.) was assigned in art. 44, we shall have, by the formula (26.) of that article, the following expression for this radius, or semiaxis:

$$b = \frac{\kappa^2 - \iota^2}{\mathrm{T}(\iota - \kappa)}. \tag{81.}$$

And hence, on account of the general formula,

$$\iota\rho + \rho\kappa = (\iota - \kappa)\left(\rho + \frac{\kappa\rho + \rho\kappa}{\iota - \kappa}\right), \tag{82.}$$

which holds good for *any* three vectors, ι, κ, ρ, the quaternion equation of the ellipsoid may be changed from a form already assigned, namely

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad \text{(9.), art. 38,}$$

to the following equivalent form:

$$\mathrm{T}\left(\rho + \frac{\kappa\rho + \rho\kappa}{\iota - \kappa}\right) = b. \tag{83.}$$

If then we introduce a new vector-symbol λ, denoting a line of variable length, but one drawn in the fixed direction of $\iota - \kappa$, or in the exactly opposite direction of $\kappa - \iota$, and determined by the condition

$$\lambda(\kappa - \iota) = \kappa\rho + \rho\kappa, \tag{84.}$$

we shall have also

$$\mathrm{T}(\rho - \lambda) = b; \tag{85.}$$

and thus the equation (83.) of the ellipsoid may be regarded as the result of the elimination of the auxiliary vector-symbol λ between the two last equations (84.) and (85.). But if we suppose that this symbol λ receives any *given* and constant value, of the form

$$\lambda = h(\iota - \kappa), \tag{86.}$$

where h is a scalar coefficient, which we here suppose to be constant and given, and if we still conceive the symbol ρ to denote a variable vector, drawn from the centre of the ellipsoid as an origin, the equation (84.) will then express that this vector ρ terminates in a point which is contained on a *given plane* parallel to that one of the two cyclic planes of the ellipsoid which has for its equation

$$\kappa\rho + \rho\kappa = 0, \qquad \text{(23.), art. 44;}$$

while the equation (85.) will express that the same vector ρ terminates also on a given spheric surface, of which the vector of the centre (drawn from the same centre of the ellipsoid) is λ, and of which the radius is $= b$. The *system of the two equations*, (84.) and (85.), expresses therefore that, for any given value of the auxiliary vector λ, or for any given value of the scalar coefficient h in the formula (86.), the termination of the vector ρ is contained on the circumference of a *given circle*, which is the mutual intersection of the plane (84.) and of the sphere (85.). And the equation (83.) of the ellipsoid, as being derived, or at least derivable, by elimination of λ, from that system of equations (84.) and (85.), is thus seen to express the known theorem, that the surface of an ellipsoid may be regarded as the locus of a certain *system of circular circumferences*, of which the planes are parallel to a fixed plane of diametral and circular section.

57. *One set* of the known circular sections of the ellipsoid, in planes parallel to *one* of the two cyclic planes, may therefore be assigned in this manner, as the result of a very simple calculation; and the *other set* of such known circular sections, parallel to the *other* cyclic plane, may be symbolically determined, with equal facility, as the result of an entirely similar process of calculation with quaternions. For if, instead of (82.), we employ this other general formula, which likewise holds good for any three vectors,

$$\iota\rho + \rho\kappa = \left(\rho + \frac{\iota\rho + \rho\iota}{\kappa - \iota}\right)(\kappa - \iota), \tag{87.}$$

we shall thereby transform the lately cited equation (9.) of the ellipsoid into this other form,

$$\mathrm{T}\left(\rho + \frac{\iota\rho + \rho\iota}{\kappa - \iota}\right) = b; \qquad (88.)$$

which is analogous to the form (83.), and from which similar inferences may be drawn. Thus, we may treat this equation (88.) as the result of elimination of a new auxiliary vector symbol μ between the two equations,

$$\mu(\iota - \kappa) = \iota\rho + \rho\iota; \qquad (89.)$$

$$\mathrm{T}(\rho - \mu) = b; \qquad (90.)$$

of which the former, namely the equation (89.), is, relatively to ρ, the equation of a *new plane*, parallel to that other cyclic plane of the ellipsoid for which we have seen that

$$\iota\rho + \rho\iota = 0, \qquad (25.), \text{ art. } 44;$$

while the latter equation, namely (90.), is that of a *new sphere*, with the same radius b as before, but with μ for the vector of its centre: which sphere (90.), determines, by its intersection with the plane (89.), a *new circle* as the locus of the termination of ρ, when μ receives any given value of the form

$$\mu = h'(\kappa - \iota), \qquad (91.)$$

where h' is a new scalar coefficient. The ellipsoid (9.) is therefore the locus of all the circles of this second system also, answering to the equations (89.), (90.), as it was seen to be the locus of all those of the first system, represented by the equations (84.), (85.); which agrees with the known properties of the surface.

58. For any three vectors, ι, κ, ρ, we have (because ρ^2, κ^2, and $\kappa\rho + \rho\kappa$ are scalars) the general transformations,

$$\left.\begin{aligned}(\iota\rho + \rho\iota)(\kappa\rho + \rho\kappa) &= \iota(\kappa\rho + \rho\kappa)\rho + \rho(\kappa\rho + \rho\kappa)\iota \\ &= (\iota\kappa + \kappa\iota)\rho^2 + \iota\rho\kappa\rho + \rho\kappa\rho\iota \\ &= -(\iota - \kappa)^2\rho^2 + (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho);\end{aligned}\right\} \qquad (92.)$$

and therefore, with the recent significations of the symbols b, λ, μ, expressed by the formulæ (81.), (84.), (89.), the equation of the ellipsoid assigned in a foregoing article, namely

$$(\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) = (\kappa^2 - \iota^2)^2, \qquad (21.), \text{ art. } 44,$$

takes easily this shorter form,

$$\rho^2 + b^2 = \lambda\mu. \qquad (93.)$$

If now we cut this surface by the system of two planes, parallel respectively to the two cyclic planes (23.) and (25.), and included in the joint equation

$$\{\lambda - h(\iota - \kappa)\}\{\mu - h'(\kappa - \iota)\} = 0, \qquad (94.)$$

which is derived by multiplication from the equations (86.) and (91.), we are conducted to this other equation,

$$\rho^2 + b^2 = h(\iota\rho + \rho\iota) + h'(\kappa\rho + \rho\kappa) + hh'(\iota - \kappa)^2; \tag{95.}$$

which may be put under the form

$$-b^2 = (\rho - h\iota - h'\kappa)^2 - (h + h')(h\iota^2 + h'\kappa^2); \tag{96.}$$

or under this other form,

$$\mathrm{T}(\rho - \xi) = r, \tag{97.}$$

if we write, for abridgment,

$$\xi = h\iota + h'\kappa, \tag{98.}$$

and

$$r = \sqrt{\{b^2 - (h + h')(h\iota^2 + h'\kappa^2)\}}. \tag{99.}$$

Any two circular sections of the ellipsoid, parallel to two different cyclic planes, or belonging to two *different* systems, are therefore contained upon one *common* sphere (97.), of which the radius r, and the vector of the centre ξ, are assigned by these last formulæ: which again agrees with the known properties of surfaces of the second order. And the equation of the *mean sphere* which contains the two *diametral* and circular sections, is seen to reduce itself, in this system of algebraical geometry, to the very simple form*

$$\rho^2 + b^2 = 0. \tag{100.}$$

59. The expressions (86.), (91.), (98.), for λ, μ, ξ, give

$$\frac{\xi - \lambda}{\kappa} = \frac{\xi - \mu}{\iota} = \frac{\lambda - \mu}{\iota - \kappa} = h + h'; \tag{101.}$$

if then we regard λ, μ, ξ as the vectors of the three corners L, M, N of a plane triangle, and observe that 0, $\iota - \kappa$, and $-\kappa$ were seen to be the vectors of the three corners A, B, C, of the *generating triangle* described in our construction of the ellipsoid, we see that the new triangle LMN is similar to that generating triangle ABC, and similarly situated in one common plane therewith, namely in the plane of the greatest and least axes of the ellipsoid; the sides LM, MN, NL of the one triangle being parallel and proportional to the sides AB, BC, CA of the other, while the points L and M are situated on the same indefinite straight line as A, B; that is, on the axis of that circumscribed cylinder of revolution which has been considered in former articles. The vectors of the points D, E, in the same construction of the ellipsoid, (if drawn from its centre as their origin,) having been seen to be respectively $\sigma - \kappa$ and ρ, (compare article 40,) the equation

$$\sigma\rho + \rho\kappa = 0, \qquad \text{(16.), art. 41,}$$

* Compare article 21, in the Phil. Mag. for July 1846.

combined with (84.) and (86.), gives for their product the expressions:

$$(\sigma - \kappa)\rho = \lambda(\iota - \kappa) = h(\iota - \kappa)^2; \tag{102.}$$

and in general if two pairs of co-initial vectors, as here $\sigma - \kappa$, ρ and λ, $\iota - \kappa$, give, when respectively multiplied, one common scalar product, they terminate on four concircular points: the four points D, E, L, B are therefore contained on the circumference of one common circle: and consequently the point L may be found by an elementary construction, derived from this simple calculation with quaternions, namely as the second point of intersection of the circle BDE with the straight line AB (which is situated in the plane of the circle). Again, the equations (85.) and (90.) give

$$\mathrm{T}(\rho - \lambda) = \mathrm{T}(\rho - \mu); \tag{103.}$$

therefore the point E of the ellipsoid is the vertex of an isosceles triangle, constructed on LM as base; and the point M may thus be found as the intersection of the same straight line AB or AL, with a circle described round the point E as centre, and having its radius $= \overline{\mathrm{EL}} = b =$ the mean semiaxis of the ellipsoid. When the two points L and M have thus been found, the third point N can then be deduced from them, in an equally simple geometrical manner, by drawing parallels LN, MN to the sides AC, BC of the generating triangle ABC, from which the ellipsoid itself has been constructed; these sides LN, MN, of the new and variable triangle LMN, will thus be parallel to the two cyclic normals of the ellipsoid; and the foregoing analysis shows that they will be portions of the axes of the two circles, which are contained upon the surface of that ellipsoid, and pass through the point E on that surface: while the point N, of intersection of those two axes, is the centre of that common sphere (97.), which contains both those two circular sections. It is evident that this common sphere must *touch* the ellipsoid at E, since it is itself touched at that point by the two distinct tangents to the two circular sections of the surface; and hence we might infer that the semidiameter NE or $\xi - \rho$ of the sphere, of which the length r has been assigned in the formula (99.), and which is terminated at the point N by the plane of the generating triangle, must coincide in direction with the *normal* ν to the ellipsoid: of which latter normal the direction may thus be found by a simple geometrical construction, and an expression for it be obtained without the employment of differentials. But to show that this geometrical result agrees with the symbolical expression already found for ν, by means of differentials and quaternions, we have only to substitute, on the one hand, in the expression (98.) for ξ, the following values for h and h', derived from (84.), (86.), and from (89.), (91.):

$$h = \frac{\kappa\rho + \rho\kappa}{-(\iota - \kappa)^2}; \quad h' = \frac{\iota\rho + \rho\iota}{-(\iota - \kappa)^2}; \tag{104.}$$

and to observe, on the other hand, that the equation (31.), which has served to determine the normal vector of proximity ν, may be thus written:

$$(\kappa^2 - \iota^2)^2\nu = (\iota - \kappa)^2\rho + \iota(\kappa\rho + \rho\kappa) + \kappa(\iota\rho + \rho\iota); \tag{105.}$$

for thus we are conducted, by means of (81.), to the formula:

$$\xi - \rho = b^2\nu; \tag{106.}$$

which expresses the agreement of the recent construction with the results that had been previously obtained.

60. If we introduce two new constant vectors ι' and κ', connected with the two former constant vectors ι, κ, by the equations

$$\iota\kappa' = \iota'\kappa = \mathrm{T} \,.\, \iota\kappa, \tag{107.}$$

which give

$$\iota'^2 = \iota^2, \quad \kappa'^2 = \kappa^2, \quad \iota'\kappa' = \kappa\iota, \tag{108.}$$

then one of the lately cited forms of the equation of the ellipsoid, namely the equation

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2 \qquad (9.), \text{ art. } 38,$$

takes easily, by the rules of this calculus, the new but analogous form:

$$\mathrm{T}(\iota'\rho + \rho\kappa') = \kappa'^2 - \iota'^2. \tag{109.}$$

The perfect similarity of these two forms, (9.) and (109.), renders it evident that all the conclusions which have been deduced from the one form can, with suitable and easy modifications, be deduced from the other also. Thus if we still regard the centre A as the origin of vectors, and treat $\iota' - \kappa'$ and $-\kappa'$ as the vectors of two new fixed points B′ and C′, we may consider AB′C′ as a *new generating triangle*, and may derive from it the *same ellipsoid* as before, by a geometrical process of generation or construction, which is similar in all respects to the process already assigned. (See the Numbers of the Philosophical Magazine for June, September, and October, 1847; or the Proceedings of the Royal Irish Academy for July, 1846.) Hence the two new sides B′C′ and C′A, which indeed are parallel by (107.) to the two old sides AC and CB, or to κ and ι, must have the directions of the two cyclic normals; and the third new side, AB′, or $\iota' - \kappa'$, must be the axis of a *second cylinder of revolution*, circumscribed round the same ellipsoid. If we determine on this new axis two new points, L′ and M′, as the extremities of two new vectors λ' and μ', analogous to the recently considered vectors λ and μ, and assigned by equations similar to (84.) and (89.), namely

$$\lambda'(\kappa' - \iota') = \kappa'\rho + \rho\kappa', \quad \mu'(\iota' - \kappa') = \iota'\rho + \rho\iota', \tag{110.}$$

we shall have results analogous to (85.) and (90.), namely

$$\mathrm{T}(\rho - \lambda') = b; \quad \mathrm{T}(\rho - \mu') = b; \tag{111.}$$

with others similar to (101.), namely

$$\frac{\xi - \lambda'}{\kappa'} = \frac{\xi - \mu'}{\iota'} = \frac{\lambda' - \mu'}{\iota' - \kappa'}; \tag{112.}$$

the common value of these three quotients being a new scalar, but ξ being still the same vector as before, namely that vector which terminates in the point N, where the normal to the surface at E meets the common plane of the new and old generating triangles, or the plane of the greatest and least axes of the ellipsoid. It is easy hence to infer that the new variable triangle L′M′N is similar to the new generating traingle AB′C′, and similarly situated in the same fixed plane therewith; and that the sides L′N, M′N, having respectively the same directions as AC′, B′C′ have likewise the same directions as BC, AC, and therefore also as MN, LN, or else directions opposite to these; in such a manner that the two straight lines, L′M, M′L, must cross each other in the point N. But these two lines may be regarded as the diagonals of a certain quadrilateral inscribed in a circle, namely the plane quadrilateral L′M′ML; of which the four corners are, by (85.), (90.), and (111.), at one common and constant distance $= b$, from the variable point E of the ellipsoid. If then we assume it as known that the vector $b^2\nu$, which is in direction opposite and is in length reciprocal to the perpendicular let fall from the centre A on the tangent plane at E, must terminate in a point F on the surface of *another ellipsoid, reciprocal* (in a well-known sense) to that former ellipsoid which contains the point E itself, or the termination of the vector ρ; we may combine the recent results, so as to obtain the following geometrical construction,* which serves *to generate a system of two reciprocal ellipsoids, by means of a moving sphere.*

61. Let then a sphere of constant magnitude, with centre E, move so that it always intersects two fixed and mutually intersecting straight lines, AB, AB′, in four points L, M, L′, M′, of which L and M are on AB, while L′ and M′ are on AB′; and let one diagonal LM′, of the inscribed quadrilateral LMM′L′, be constantly parallel to a third fixed line AC, which will oblige the other diagonal ML′ of the same quadrilateral to move parallel to a fourth fixed line AC′. Let N be the point in which the diagonals intersect, and draw AF equal and parallel to EN; so that AENF is a parallelogram: then *the locus of the centre* E *of the moving sphere is one ellipsoid, and the locus of the opposite corner* F *of the parallelogram is another ellipsoid reciprocal thereto.* These two ellipsoids have a common centre A, and a common mean axis, which is equal to the diameter of the moving sphere, and is a mean proportional between the greatest axis of either ellipsoid and the least axis of the other, of which two last-mentioned axes the directions coincide. Two sides, AE, AF, of the parallelogram AENF, are thus two semidiameters which may be regarded as mutually *reciprocal*, one of the one ellipsoid, and the other of the other: but because they fall at *opposite* sides of the *principal plane* (containing the four fixed lines and the greatest and least axes of the ellipsoids), it may be proper to call them, more fully, *opposite reciprocal semidiameters*; and to call the points E and F, in which they terminate, *opposite reciprocal points.* The two other sides EN, FN, of the same variable parallelogram, are the *normals* to the two ellipsoids, meeting each other in the point N, upon the same principal plane. In that plane, the two former fixed lines, AB, AB′, are the *axes of two cylinders of revolution*, circumscribed about the first ellipsoid; and the two latter fixed lines, AC, AC′, are the *two cyclic normals* of the same first ellipsoid: while the diagonals, LM′, ML′, of the inscribed quadrilateral in the construction, are the *axes of the two circles* on the surface of that first ellipsoid, which circles pass through the point E, that is through the centre of the moving sphere; and the intersection N of those two diagonals is the centre of another sphere, which cuts the first ellipsoid in the system of those two circles: all which is easily adapted, by suitable interchanges, to the other or reciprocal ellipsoid, and flows with facility from the quaternion equations above given.

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* This is the construction referred to in a note to article 54. It was communicated by the author to the Royal Irish Academy, at the meeting of November 30, 1847. See the Proceedings of that date.

62. The equations (85.), (90.), and (111.), of articles 56, 57, and 60, give

$$\mathrm{T}(\rho - \lambda) = \mathrm{T}(\rho - \mu') = b; \tag{113.}$$

and

$$\mathrm{T}(\rho - \mu) = \mathrm{T}(\rho - \lambda') = b; \tag{114.}$$

whence, by the meanings of the signs employed, the two following mutually connected constructions may be derived, for *geometrically generating an ellipsoid from a rhombus of constant perimeter*, or for geometrically describing an arbitrary curve on the surface of such an ellipsoid by the motion of a corner of such a rhombus, which the writer supposes to be new.

1st Generation. Let a rhombus LEM′E′, of which each side preserves constantly a fixed length $= b$, but of which the angles vary, move so that the two opposite corners L, M′ traverse two fixed and mutually intersecting straight lines AB, AB′, (the point L moving along the line AB, and the point M′ along AB′,) while the diagonal LM′, connecting these two opposite corners of the rhombus, remains constantly parallel to a third fixed right line AC (in the plane of the two former right lines); then, *according to whatever arbitrary law the plane of the rhombus may turn*, during its motion, *its two remaining corners* E, E′ *will describe curves upon the surface of a fixed ellipsoid*; which surface is thus the *locus of all the pairs of curves* that can be described by this first mode of generation.

2nd Generation. Let now *another rhombus*, L′E″ME‴, with the *same constant perimeter* $= 4b$, move so that its opposite corners L′, M traverse the *same two fixed lines* AB, AB′, as before, but in such a manner that the diagonal L′M, connecting these two corners, remains parallel (not to the third fixed line AC, but) to a *fourth fixed line* AC′; then, whatever may be the arbitrary law according to which the plane of this new rhombus turns, provided that the angles BAB′, CAC′, between the first and second, and between the third and fourth fixed lines, have one *common bisector*, the *two remaining corners* E″, E‴ *of this second rhombus will describe curves upon the surface of the same fixed ellipsoid*, as that determined by the former generation: which surface is thus the *locus of all the new pairs of curves*, described in this second mode, as it was just now seen to be the locus of all the old pairs of curves, obtained in the first mode of description.

63. The ellipsoid (with three unequal axes), thus generated, is therefore the *common locus of the four curves*, described by the four points E E′ E″ E‴; of which four curves, the first and third may be made to coincide with *any arbitrary curves on that ellipsoid*; but the second and fourth become determined, when the first and third have been chosen. And in this new *system of two connected constructions for generating an ellipsoid*, as well as in that other construction* which was given in article 61 for a *system of two reciprocal ellipsoids*, the two former fixed lines, AB, AB′, are the *axes of two cylinders of revolution*, circumscribed about the ellipsoid which is the locus of the point E; while the two latter fixed lines, AC, AC′, are the *two cyclic normals* (or the normals to the two planes of circular section) of that ellipsoid. The common (internal and external) bisectors, at the centre A, of the angles BAB′, CAC′, made by the first and second, and by the third and fourth fixed lines, coincide

* See Phil. Mag. for May 1848; or Proceedings of Royal Irish Academy for November 1847.

in direction with the *greatest and least axes* of the ellipsoid; and the constant length b, of the side of either rhombus, is the length of the *mean semiaxis.* The diagonal LM' of the first rhombus is the *axis of a first circle* on the ellipsoid, of which circle a diameter coincides with the second diagonal EE' of the same rhombus; and, in like manner, the diagonal $\text{L}'\text{M}$ of the second rhombus is the *axis of a second circle* on the same ellipsoid, belonging to the second (or *subcontrary*) system of circular sections of that surface: while the other diagonal $\text{E}''\text{E}'''$, of the same second rhombus, is a diameter of the same second circle. In the quaternion analysis employed, the first of these two circular sections of the ellipsoid corresponds to the equations (113.); and the second circular section is represented by the equations (114.), of the foregoing article.

64. We may also present the interpretation of those quaternion equations, or the recent double construction of the ellipsoid, in the following other way, which also appears to be new; although the writer is aware that there would be no difficulty in proving its correctness, or in deducing it anew, either by the method of co-ordinates, or in a more purely geometrical mode. *Conceive two equal spheres to slide within two cylinders* (of revolution, whose axes intersect each other, and of which each touches its own sphere along a great circle of contact), *in such a manner that the right line joining the centres of the spheres shall be parallel to a fixed right line*; then *the locus of the varying circle in which the two spheres intersect each other will be an ellipsoid, inscribed at once in both the cylinders*, so as to touch one cylinder along one ellipse of contact, and the other cylinder along another such ellipse. And the *same* ellipsoid may be generated as the locus of *another* varying circle, which shall be the intersection of *two other equal spheres sliding within the same two cylinders of revolution*, but with a connecting line of centres which now moves parallel to *another fixed right line*; provided that the angle between these two fixed lines, and the angle between the axes of the two cylinders, have both one common pair of (internal and external) bisectors, which will then coincide in direction with the greatest and least axes of the ellipsoid, while the diameter of each of the *four sliding spheres* is equal to the mean axis. In fact, we have only to conceive (with the recent significations of the letters), that four spheres, with the same common radius $= b$, are described about the points L, M', and L', M, as centres; for then the first pair of spheres will cross each other in that circular section of the ellipsoid which has EE' for a diameter; and the second pair of spheres will cross in the circle of which the diameter is $\text{E}''\text{E}'''$; after which the other conclusions above stated will follow, from principles already laid down.

65. If we make

$$\rho - \lambda = \lambda_{\prime}; \quad \rho - \mu = \mu_{\prime}; \quad \rho - \lambda' = \lambda'_{\prime}; \quad \rho - \mu' = \mu'_{\prime}; \tag{115.}$$

and in like manner, (see (106.),)

$$\rho - \xi = -b^2\nu = \xi_{\prime}; \tag{116.}$$

and if we regard these five new vectors, $\lambda_{\prime}$, $\mu_{\prime}$, $\lambda'_{\prime}$, $\mu'_{\prime}$, and $\xi_{\prime}$, as lines which, being drawn from the centre A, terminate respectively in five new points, $\text{L}_{\prime}$, $\text{M}_{\prime}$, $\text{L}'_{\prime}$, $\text{M}'_{\prime}$, and H; while the vector ρ, drawn from the same centre A, still terminates in the point E, upon the surface of the ellipsoid; then the equations (113.), (114.), of art. 62, will give:

$$\mathrm{T}\lambda_{\prime} = \mathrm{T}\mu_{\prime} = \mathrm{T}\lambda'_{\prime} = \mathrm{T}\mu'_{\prime} = b; \tag{117.}$$

while the equations (101.) will enable us to write

$$\frac{\lambda_{\prime} - \xi_{\prime}}{\kappa} = \frac{\mu_{\prime} - \xi_{\prime}}{\iota} = \frac{\mu_{\prime} - \lambda_{\prime}}{\iota - \kappa} = \mathrm{V}^{-1}0; \tag{118.}$$

and in like manner, (see (112.),)

$$\frac{\lambda'_{\prime} - \xi_{\prime}}{\kappa'} = \frac{\mu'_{\prime} - \xi_{\prime}}{\iota'} = \frac{\mu'_{\prime} - \lambda'_{\prime}}{\iota' - \kappa'} = \mathrm{V}^{-1}0; \tag{119.}$$

this symbol $\mathrm{V}^{-1}0$ denoting (as already explained) a *scalar*. We shall have also, by (84.), (89.),

$$\frac{\rho - \lambda_{\prime}}{\iota - \kappa} = \frac{\lambda}{\iota - \kappa} = \mathrm{V}^{-1}0; \quad \frac{\rho - \mu_{\prime}}{\kappa - \iota} = \frac{\mu}{\kappa - \iota} = \mathrm{V}^{-1}0; \tag{120.}$$

the scalars denoted by the symbol $\mathrm{V}^{-1}0$ being not generally obliged to be equal to each other, and being, in these last equations (120.), respectively equal, by (86.), (91.), to those which have been denoted by h and h'. In like manner, by (110.),

$$\frac{\rho - \lambda'_{\prime}}{\iota' - \kappa'} = \frac{\lambda'}{\iota' - \kappa'} = \mathrm{V}^{-1}0; \quad \frac{\rho - \mu'_{\prime}}{\kappa' - \iota'} = \frac{\mu'}{\kappa' - \iota'} = \mathrm{V}^{-1}0. \tag{121.}$$

And because, by (107.), ι' has a scalar ratio to κ, and κ' has a scalar ratio to ι, we may infer, from (118.), (119.), the existence of the two following other scalar ratios:

$$\frac{\mu'_{\prime} - \xi_{\prime}}{\lambda_{\prime} - \xi_{\prime}} = \mathrm{V}^{-1}0; \quad \frac{\lambda'_{\prime} - \xi_{\prime}}{\mu_{\prime} - \xi_{\prime}} = \mathrm{V}^{-1}0. \tag{122.}$$

Finally we may observe that, by (120.), (121.), there exist scalar ratios between certain others also of the foregoing vector-differences, and especially the following:

$$\frac{\rho - \lambda_{\prime}}{\rho - \mu_{\prime}} = \mathrm{V}^{-1}0; \quad \frac{\rho - \lambda'_{\prime}}{\rho - \mu'_{\prime}} = \mathrm{V}^{-1}0. \tag{123.}$$

66. Proceeding now to consider the geometrical signification of the equations in the last article, we see first, from the equations (117.), that the four new points, $\mathrm{L}_{\prime}$, $\mathrm{M}_{\prime}$, $\mathrm{L}'_{\prime}$ $\mathrm{M}'_{\prime}$, are all situated upon the surface of that *mean sphere*, which is described on the mean axis of the ellipsoid as a diameter; because the equation of that mean sphere has been already seen to

be*

$$\rho^2 + b^2 = 0 \qquad \text{equation (100.), article 58;}$$

which may also be thus written, by the principles and notations of the calculus of quaternions:

$$\mathrm{T}\rho = b. \tag{124.}$$

From the relations (122.) it follows that the two chords $\mathrm{L}_{\prime}\mathrm{M}'_{\prime}$ and $\mathrm{L}'_{\prime}\mathrm{M}_{\prime}$, of this mean sphere, both pass through the point H, of which the vector $\xi_{\prime}$ is assigned by the formula (116.); for the first equation (122.) shows that the three vectors $\lambda_{\prime}$, $\mu'_{\prime}$, $\xi_{\prime}$, which are all drawn from one common point, namely the centre A of the ellipsoid, all terminate on one straight line; since otherwise the quotient of their differences, $\mu'_{\prime}-\xi_{\prime}$ and $\lambda_{\prime}-\xi_{\prime}$, would be a *quaternion*,† of which the vector part would not be equal to zero: and in like manner, the second equation (122.) expresses that the three lines $\lambda'_{\prime}$, $\mu_{\prime}$, $\xi_{\prime}$, all terminate on another straight line. The four-sided figure $\mathrm{L}_{\prime}\,\mathrm{M}_{\prime}\,\mathrm{L}'_{\prime}\,\mathrm{M}'_{\prime}$ is therefore a *plane quadrilateral, inscribed* (generally) *in a small circle of the mean sphere*, and having the point H for the intersection of its second and fourth sides, $\mathrm{M}_{\prime}\mathrm{L}'_{\prime}$ and $\mathrm{M}'_{\prime}\mathrm{L}_{\prime}$, or of those two sides prolonged. And these two sides, having respectively the directions of $\mathrm{HM}_{\prime}$ and $\mathrm{HL}_{\prime}$, or of the vector-differences $\mu_{\prime} - \xi_{\prime}$ and $\lambda_{\prime} - \xi_{\prime}$, are respectively parallel, by (118.), to the two fixed vectors, ι and κ; or (by what was shown in former articles), to the two cyclic normals, AC' and AC, of the original ellipsoid. The plane of the quadrilateral inscribed in the mean sphere is therefore constantly parallel to the *principal plane* CAC' of that ellipsoid, namely to the plane of the greatest and least axes, which contains those two cyclic normals. The first and third sides, $\mathrm{L}_{\prime}\mathrm{M}_{\prime}$ and $\mathrm{L}'_{\prime}\mathrm{M}'_{\prime}$, of the same inscribed quadrilateral, being in the directions of $\mu_{\prime} - \lambda_{\prime}$ and $\mu'_{\prime} - \lambda'_{\prime}$, are parallel, by (118.), (119.), to two other constant vectors, namely $\iota - \kappa$ and $\iota' - \kappa'$, or to the axes AB, AB', of the two cylinders of revolution which can be circumscribed about the same ellipsoid. And the point of intersection of this other pair of opposite sides of the same inscribed quadrilateral is, by (123.), the extremity of the vector ρ, or the point E on the surface of the original ellipsoid; while the point H, which has been already seen to be the intersection of the former pair of opposite sides of

* This *form* of the equation of the *sphere* was published in the Philosophical Magazine for July 1846; and it is an immediate and a very easy consequence of that fundamental formula of the whole theory of Quaternions, namely

$$i^2 = j^2 = k^2 = ijk = -1,$$

which was communicated under a slightly more developed form, to the Royal Irish Academy, on the 13th of November 1843. (See Phil. Mag. for July 1844.)

It may perhaps be thought not unworthy of curious notice hereafter, that *after* the publication of this form of the equation of the *sphere*, there should have been found in England, and in 1846, a person with any mathematical character to lose, who could profess publicly his inability to distinguish the method of *quaternions* from that of *couples*; and who could thus confound the system of the present writer with those of Argand and of Français, of Mourey and of Warren.

† A Quaternion, *geometrically* considered, is the *product, or the quotient, of any two directed lines in space.*

the quadrilateral, since it has, by (116.), its vector $\xi_{\prime} = -b^2\nu$, is the *reciprocal point*, on the surface of that *other* and *reciprocal ellipsoid*, which was considered in article 61; namely the point which is, on that reciprocal ellipsoid, diametrically *opposite* to the point which was named F in that article, and had its vector $= b^2\nu$.

67. Conversely it is easy to see, that the foregoing analysis by quaternions conducts to the following mode of *constructing,* or generating, geometrically*, and by a *graphic* rather than by a *metric* process, *a system of two reciprocal ellipsoids, derived from one fixed sphere*; and of determining, also *graphically*, for each point on either ellipsoid, the *reciprocal point* on the other.

Inscribe in the fixed sphere a plane quadrilateral ($\mathrm{L}_{\prime}\mathrm{M}_{\prime}\mathrm{L}'_{\prime}\mathrm{M}'_{\prime}$), of which the four sides ($\mathrm{L}_{\prime}\mathrm{M}_{\prime}$, $\mathrm{M}_{\prime}\mathrm{L}'_{\prime}$, $\mathrm{L}'_{\prime}\mathrm{M}'_{\prime}$, $\mathrm{M}'_{\prime}\mathrm{L}_{\prime}$) shall be respectively parallel to four fixed right lines (AB, AC′, AB′, AC), diverging from the centre (A) of the sphere; and prolong (if necessary) the first and third sides of this inscribed quadrilateral, till they meet in a point E; and the second and fourth sides of the same quadrilateral, till they intersect in another point H. Then *these two points, of intersection* E and H, *thus found from two pairs of opposite sides of this inscribed quadrilateral, will be two reciprocal points on two reciprocal ellipsoids*; which ellipsoids will have a common mean axis, namely that diameter of the fixed sphere which is perpendicular to the plane of the four fixed lines: and those lines, AB, AC′, AB′, AC, will be related to the two ellipsoids which are thus the loci of the two points E and H, according to the laws enunciated in article 61, in connexion with a different construction of a system of two reciprocal ellipsoids (derived there from one common *moving sphere*); which former construction *also* was obtained by the aid of the calculus of quaternions. Thus the lines AC, AC′ will be the two cyclic normals of the ellipsoid which is the locus of E, but will be the axes of circumscribed cylinders of revolution, for that reciprocal ellipsoid which is the locus of H; and conversely, the lines AB, AB′ will be the axes of the two cylinders of revolution circumscribed about the ellipsoid (E), but will be the cyclic normals, or the perpendiculars to the cyclic planes, for the reciprocal ellipsoid (H).

68. The equation of the ellipsoid (see Philosophical Magazine for October 1847, or Proceedings of the Royal Irish Academy for July 1846),

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad \text{eq. (9.), art. 38,}$$

which has so often presented itself in these researches, may be anew transformed as follows. Writing it thus,

$$\mathrm{T}\frac{(\iota\rho + \rho\kappa)(\iota - \kappa)}{\kappa^2 - \iota^2} = \mathrm{T}(\iota - \kappa), \tag{125.}$$

which we are allowed to do, because the tensor of a product is equal to the product of the tensors, we may observe that while the denominator of the fraction in the first member is a

* This construction, of two reciprocal ellipsoids from one sphere, was communicated to the Royal Irish Academy in June 1848; together with an extension of it to a mode of generating two reciprocal cones of the second degree from one rectangular cone of revolution; and also to a construction of two reciprocal hyperboloids, whether of one sheet, or of two sheets, from one equilateral hyperboloid of revolution, of one or of two sheets.

pure scalar, the numerator is a pure vector; for the identity,

$$\iota\rho + \rho\kappa = \mathrm{S} \,.\, (\iota + \kappa)\rho + \mathrm{V} \,.\, (\iota - \kappa)\rho, \tag{126.}$$

gives

$$\mathrm{S} \,.\, (\iota\rho + \rho\kappa)(\iota - \kappa) = 0 : \tag{127.}$$

the fraction itself is therefore a pure vector, and the sign T, of the operation of taking the tensor of a quaternion, may be changed to the sign TV, of the generally distinct but in this case equivalent operation, of taking the tensor of the vector part. But, under the sign V, we may reverse the order of any *odd* number of vector factors (see article 20 in the Philosophical Magazine for July 1846); and thus may change, in the numerator of the fraction in (125.), the partial product $\iota\rho(\iota - \kappa)$ to $(\iota - \kappa)\rho\iota$. Again, it is always allowed to *divide* (though *not*, generally, in this calculus, to *multiply*) *both* the numerator and denominator of a quaternion fraction, *by* any *common* quaternion, or by any common vector; that is, to multiply both numerator and denominator *into the reciprocal* of such common quaternion or vector: namely by writing the symbol of this new factor to the *right* (but not generally to the left) of both the symbols of numerator and denominator, above and below the fractional bar. *Dividing* therefore thus above and below *by* ι, or *multiplying into* ι^{-1}, after that permitted transposition of factors which was just now specified, and after the change of T to TV, we find that the equation (125.) of the ellipsoid assumes the following form:

$$\mathrm{TV}\frac{(\iota - \kappa)\rho + \rho(\kappa - \kappa^2\iota^{-1})}{(\iota - \kappa) + (\kappa - \kappa^2\iota^{-1})} = \mathrm{T}(\iota - \kappa); \tag{128.}$$

the new denominator first presenting itself under the form $\kappa^2\iota^{-1} - \iota$, but being changed for greater symmetry to that written in (128.), which it is allowed to do, because, under the sign T, or under the sign TV, we may multiply by negative unity.

69. In the last equation of the ellipsoid, since

$$\kappa - \kappa^2\iota^{-1} = \kappa(\iota - \kappa)\iota^{-1},$$

we have

$$\mathrm{T}(\kappa - \kappa^2\iota^{-1}) = \mathrm{T}\kappa\,\mathrm{T}(\iota - \kappa)\,\mathrm{T}\iota^{-1}; \tag{129.}$$

and under the characteristic U, of the operation of taking the versor of a quaternion, we may multiply by any positive scalar, such as $-\kappa^2$ is, because κ^2 and κ^{-2} are negative* scalars;

* By this, which is one of the earliest and most fundamental principles of the whole quaternion theory (see the author's letter to John T. Graves, Esq., of October 17th, 1843, printed in the Supplementary Number of the Philosophical Magazine for December 1844), namely by the principle that *the square of* EVERY VECTOR (or directed straight line in tridimensional space) is to be regarded as a NEGATIVE NUMBER, this theory is not merely *distinguished from*, but sharply CONTRASTED *with, every other system* of algebraic geometry of which the

whereas to multiply by a negative scalar, under the same sign U, is equivalent to multiplying the versor itself by -1: hence,

$$\mathrm{U}(\kappa - \kappa^2\iota^{-1}) = -\mathrm{U}(\kappa^2\iota^{-1} - \kappa) = -\mathrm{U}(\kappa^{-1} - \iota^{-1}). \qquad (130.)$$

If then we introduce two new fixed vectors, η and θ, defined by the equations,

$$\eta = \mathrm{T}\iota\,\mathrm{U}(\iota - \kappa); \quad \theta = \mathrm{T}\kappa\,\mathrm{U}(\kappa^{-1} - \iota^{-1}); \qquad (131.)$$

and if we remember that any quaternion is equal to the product of its own tensor and versor (Phil. Mag. for July 1846); we shall obtain the transformations,

$$\iota - \kappa = \eta\,\mathrm{T}\frac{\iota - \kappa}{\iota}; \quad \kappa - \kappa^2\iota^{-1} = -\theta\,\mathrm{T}\frac{\iota - \kappa}{\iota}; \qquad (132.)$$

which will change the equation of the ellipsoid (128.) to the following:

$$\mathrm{TV}\frac{\eta\rho - \rho\theta}{\eta - \theta} = \mathrm{T}(\iota - \kappa). \qquad (133.)$$

70. To complete the elimination of the two old fixed vectors, ι, κ, and the introduction, in their stead, of the two new fixed vectors, η, θ, we may observe that the two equations (132.) give, by addition,

$$\iota - \kappa^2\iota^{-1} = (\eta - \theta)\mathrm{T}\frac{\iota - \kappa}{\iota}; \qquad (134.)$$

taking then the tensors of both members, dividing by $\mathrm{T}\frac{\iota - \kappa}{\iota}$, and attending to the expression (81.) in article 56, (Phil. Mag. for May 1848,) for the mean semiaxis b of the ellipsoid, we find this new expression for that semiaxis:

$$\mathrm{T}(\eta - \theta) = \frac{\kappa^2 - \iota^2}{\mathrm{T}(\iota - \kappa)} = b. \qquad (135.)$$

But also, by (131.), or by (132.),

$$\mathrm{T}\eta = \mathrm{T}\iota; \quad \mathrm{T}\theta = \mathrm{T}\kappa; \qquad (136.)$$

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writer has hitherto acquired any knowledge, or received any intimation. In saying this, he hopes that he will not be supposed to desire to depreciate the labours of any other past or present inquirer into the properties of that important and precious Symbol in Geometry, $\sqrt{-1}$. And he gladly takes occasion to repeat the expression of his sense of the assistance which he received, in the progress of his own speculations, from the study of Mr. Warren's work, before he was able to examine any of those earlier essays referred to in Dr. Peacock's Report: however *distinct*, and even *contrasted*, on several *fundamental* points, may be (as was above observed) the methods of the Calculus of Quaternions from those of what Professor De Morgan has happily named Double Algebra.

and therefore,

$$\theta^2 - \eta^2 = \kappa^2 - \iota^2. \tag{137.}$$

Hence, by (135.), we obtain the expression,

$$\mathrm{T}(\iota - \kappa) = \frac{\theta^2 - \eta^2}{\mathrm{T}(\eta - \theta)}; \tag{138.}$$

which may be substituted for the second member of the equation (133.), so as to complete the required elimination of ι and κ. And if we then multiply on both sides by $\mathrm{T}(\eta - \theta)$, we obtain this new form* of the equation of the ellipsoid:

$$\mathrm{TV}\frac{\eta\rho - \rho\theta}{\mathrm{U}(\eta - \theta)} = \theta^2 - \eta^2; \tag{139.}$$

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* This form was communicated to the Royal Irish Academy, at the stated meeting of that body on March 16th, 1849, in a note addressed by the present writer to the Rev. Charles Graves. [The Proceedings of the Royal Irish Academy show that this communication was in fact made at the meeting on April 9th, 1849.—ED.] It was remarked, in that note, that the *directions* of the two fixed vectors, η, θ, are those of the two *asymptotes* to the focal hyperbola; while their *lengths* are such that the two extreme *semiaxes* of the ellipsoid may be expressed as follows:

$$a = \mathrm{T}\eta + \mathrm{T}\theta; \quad c = \mathrm{T}\eta - \mathrm{T}\theta;$$

the *mean* semiaxis being, at the same time, expressed (as in the text of the present paper) by the formula

$$b = \mathrm{T}(\eta - \theta).$$

It was observed, further, that $\eta - \theta$ has the direction of *one cyclic normal* of the ellipsoid, and that $\eta^{-1} - \theta^{-1}$ has the direction of the *other* cyclic normal; that $\eta + \theta$ is the vector of *one umbilic*, and that $\eta^{-1} + \theta^{-1}$ has the direction of *another* umbilicar vector, or umbilicar semidiameter of the ellipsoid; that the *focal ellipse* is represented by the system of the two equations

$$\mathrm{S} \,.\, \rho\,\mathrm{U}\eta = \mathrm{S} \,.\, \rho\,\mathrm{U}\theta,$$

and

$$\mathrm{TV} \,.\, \rho\,\mathrm{U}\eta = 2\mathrm{S}\sqrt{\eta\theta},$$

of which the first represents its *plane*, while the second, which (it was remarked) might also be thus written,

$$\mathrm{TV} \,.\, \rho\,\mathrm{U}\theta = 2\mathrm{S}\sqrt{\eta\theta},$$

represents a *cylinder of revolution* (or, under the latter form, a *second* cylinder of the same kind), whereon the focal ellipse is situated; and that the *focal hyperbola* is adequately expressed or represented by the *single* equation,

$$\mathrm{V} \,.\, \eta\rho \,.\, \rho\theta = (\mathrm{V} \,.\, \eta\theta)^2.$$

To which it may be added, that by changing the two fixed vectors η and θ to others of the forms $t^{-1}\eta$ and $t\theta$, we pass to a *confocal* surface.

which will be found to include several interesting geometrical significations.

71. Before entering on any discussion of this new form of the equation of the ellipsoid, namely the form

$$\mathrm{TV}\frac{\eta\rho - \rho\theta}{\mathrm{U}(\eta - \theta)} = \theta^2 - \eta^2, \qquad \text{eq. (139.), art. 70,}$$

it may be useful to point out another manner of arriving at the same equation of the ellipsoid, by a different process of calculation, from that construction or generation of the surface, as the locus of the circle which is the mutual intersection of a pair of equal spheres, sliding within two fixed cylinders of revolution whose axes intersect each other; while the right line, connecting the centres of the two sliding spheres, moves parallel to itself, or remains constantly parallel to a fixed right line in the plane of the fixed axes of the cylinders: which mode of generating the ellipsoid was published in the Philosophical Magazine for July 1848 (having also been communicated to the Royal Irish Academy in the preceding May), as a deduction from the Calculus of Quaternions. And whereas the fixed right line, through the centre of the ellipsoid, to which the line connecting the centres of the two sliding spheres is parallel, may have either of two positions, since it may coincide with either of the two cyclic normals, we shall here suppose it to have the direction of the cyclic normal ι, or shall consider the second pair of sliding spheres mentioned in article 64, of which the quaternion equations are, by article 62 (Phil. Mag. for July 1848),

$$\mathrm{T}(\rho - \mu) = \mathrm{T}(\rho - \lambda') = b. \qquad (114.).$$

72. Here (see Phil. Mag. for May 1848), we have for μ the value,

$$\mu = h'(\kappa - \iota), \qquad \text{eq. (91.), art. 57;}$$

and

$$\lambda'(\kappa' - \iota') = \kappa'\rho + \rho\kappa', \qquad \text{eq. (110.), art. 60;}$$

also

$$\iota\kappa' = \iota'\kappa = \mathrm{T}\,.\,\iota\kappa, \qquad \text{eq. (107.), same article;}$$

whence we derive for λ' the expression,

$$\lambda' = \frac{\iota^{-1}\rho + \rho\iota^{-1}}{\iota^{-1} - \kappa^{-1}} = \frac{\iota\rho + \rho\iota}{\iota - \iota^2\kappa^{-1}}. \tag{140.}$$

But

$$(\iota - \iota^2\kappa^{-1})^{-1} = \{\iota(\kappa - \iota)\kappa^{-1}\}^{-1} = \kappa(\kappa - \iota)^{-1}\iota^{-1}; \tag{141.}$$

and by (104.),

$$\iota\rho + \rho\iota = -h'(\kappa - \iota)^2; \tag{142.}$$

therefore

$$\lambda' = -h'\kappa(\kappa - \iota)\iota^{-1} = h'(\kappa - \kappa^2\iota^{-1}). \tag{143.}$$

If then we make, for abridgment,

$$g = -h' \, \mathrm{T}\frac{\iota - \kappa}{\iota}, \tag{144.}$$

and employ the two new fixed vectors η and θ, defined by the equations (see Phil. Mag. for May 1849),

$$\eta = \mathrm{T}\iota \, \mathrm{U}(\iota - \kappa), \quad \theta = \mathrm{T}\kappa \, \mathrm{U}(\kappa^{-1} - \iota^{-1}), \tag{131.}$$

which have been found to give

$$\iota - \kappa = \eta \, \mathrm{T}\frac{\iota - \kappa}{\iota}, \quad \kappa - \kappa^2\iota^{-1} = -\theta \, \mathrm{T}\frac{\iota - \kappa}{\iota}, \tag{132.}$$

we shall have the values,

$$\mu = g\eta; \quad \lambda' = g\theta; \tag{145.}$$

and the lately cited equations (114.) of the two sliding spheres will become,

$$\mathrm{T}(\rho - g\eta) = b; \quad \mathrm{T}(\rho - g\theta) = b; \tag{146.}$$

between which it remains to eliminate the scalar coefficient g, in order to find the equation of the ellipsoid, regarded as the locus of the circle in which the two spheres intersect each other.

73. Squaring the equations (146.), we find (by the general rules of this Calculus) for the two sliding spheres the two following more developed equations:

$$\left.\begin{aligned} 0 &= b^2 + \rho^2 - 2g\mathrm{S} \,.\, \eta\rho + g^2\eta^2; \\ 0 &= b^2 + \rho^2 - 2g\mathrm{S} \,.\, \theta\rho + g^2\theta^2. \end{aligned}\right\} \tag{147.}$$

Taking then the difference, and dividing by g, we find the equation

$$g(\theta^2 - \eta^2) = 2\mathrm{S} \,.\, (\theta - \eta)\rho; \tag{148.}$$

which, relatively to ρ, is linear, and may be considered as the equation of the plane of the varying circle of intersection of the two sliding spheres; any one position of that plane being distinguished from any other by the value of the coefficient g. Eliminating therefore that coefficient g, by substituting in (146.) its value as given by (148.), we find that the equation of the ellipsoid, regarded as the locus of the varying circle, may be presented under either of the two following new forms:

$$\mathrm{T}\left(\rho - \frac{2\eta \, \mathrm{S} \,.\, (\theta - \eta)\rho}{\theta^2 - \eta^2}\right) = b; \tag{149.}$$

$$\mathrm{T}\left(\rho - \frac{2\theta \, \mathrm{S} \,.\, (\eta - \theta)\rho}{\eta^2 - \theta^2}\right) = b; \tag{150.}$$

respecting which two forms it deserves to be noticed, that either may be obtained from the other, by interchanging η and θ. And we may verify that these two last equations of the ellipsoid are consistent with each other, by observing that the seimisum of the two vectors under the sign T is perpendicular to their semidifference (as it ought to be, in order to allow of those two vectors themselves having any common length, such as b); or that the condition of rectangularity,

$$\rho - \frac{(\theta+\eta)\,\mathrm{S}\,.\,(\theta-\eta)\rho}{\theta^2-\eta^2} \perp \theta - \eta, \tag{151.}$$

is satisfied: which may be proved by showing (see Phil. Mag. for July 1846) that the scalar of the product of these two last vectors vanishes, as in fact it does, since the identity

$$(\theta-\eta)(\theta+\eta) = \theta^2 + \theta\eta - \eta\theta - \eta^2,$$

resolves itself into the two following formulæ:

$$\left.\begin{aligned} \mathrm{S}\,.\,(\theta-\eta)(\theta+\eta) &= \theta^2 - \eta^2; \\ \mathrm{V}\,.\,(\theta-\eta)(\theta+\eta) &= \theta\eta - \eta\theta; \end{aligned}\right\} \tag{152.}$$

of which the first is sufficient for our purpose. We may also verify the recent equations (149.), (150.) of the ellipsoid, by observing that they concur in giving the mean semiaxis b as the length $\mathrm{T}\rho$ of the radius of that diametral and circular section, which is made by the cyclic plane having for equation

$$\mathrm{S}\,.\,(\theta-\eta)\rho = 0; \tag{153.}$$

this plane being found by the consideration that $\eta-\theta$ has the direction of the cyclic normal ι, or by making the coefficient $g = 0$, in the formula (148.).

74. The equation (149.) of the ellipsoid may be successively transformed as follows:

$$\begin{aligned} b(\theta^2-\eta^2) &= \mathrm{T}\{(\theta^2-\eta^2)\rho - 2\eta\,\mathrm{S}\,.\,(\theta-\eta)\rho\} \\ &= \mathrm{T}\{(\theta^2-\eta^2)\rho - \eta(\theta-\eta)\rho - \eta\rho(\theta-\eta)\} \\ &= \mathrm{T}\{\theta^2\rho - \eta(\theta\rho+\rho\theta) + \eta\rho\eta\} \\ &= \mathrm{TV}\{(\theta-\eta)\theta\rho - \eta\rho(\theta-\eta)\} \\ &= \mathrm{TV}\,.\,(\rho\theta-\eta\rho)(\theta-\eta) \\ &= \mathrm{TV}\,.\,(\eta\rho-\rho\theta)(\eta-\theta); \end{aligned} \tag{154.}$$

and by a similar series of transformations, performed on the equation (150.), we find also (remembering that $\theta^2-\eta^2$, being equal to $\kappa^2-\iota^2$, is positive),

$$b(\theta^2-\eta^2) = \mathrm{TV}\,.\,(\rho\eta-\theta\rho)(\eta-\theta). \tag{155.}$$

The same result (155.) may also be obtained by interchanging η and θ in either of the two last transformed expressions (154.), for the positive product $b(\theta^2-\eta^2)$; and we may otherwise establish the agreement of these recent results, by observing that, in general, if Q and Q′

be any two *conjugate quaternions* (see Phil. Mag. for July 1846), such as here $\eta\rho - \rho\theta$ and $\rho\eta - \theta\rho$, and if α be any vector, then

$$\mathrm{TV} \,.\, \mathrm{Q}\alpha = \mathrm{TV} \,.\, \mathrm{Q}'\alpha; \tag{156.}$$

for

$$\left.\begin{aligned} \mathrm{V} \,.\, \mathrm{Q}\alpha &= \alpha\,\mathrm{SQ} - \mathrm{V} \,.\, \alpha\,\mathrm{VQ}, \\ \mathrm{V} \,.\, \mathrm{Q}'\alpha &= \alpha\,\mathrm{SQ} + \mathrm{V} \,.\, \alpha\,\mathrm{VQ}; \end{aligned}\right\} \tag{157.}$$

and because

$$0 = \mathrm{S} \,.\, \alpha\mathrm{V} \,.\, \alpha\,\mathrm{VQ}, \tag{158.}$$

the common value of the two members of the formula (156.) is

$$\mathrm{TV} \,.\, \mathrm{Q}\alpha = \sqrt{\{(\mathrm{TV} \,.\, \alpha\,\mathrm{VQ})^2 + (\mathrm{T}\alpha \,.\, \mathrm{SQ})^2\}}. \tag{159.}$$

If then we substitute for b its value,

$$b = \mathrm{T}(\eta - \theta), \qquad \text{eq. (135.), art. 70,}$$

and divide both sides by this value of b, we see, from (154.), (155.), that the equation of the ellipsoid may be put under either of these two other forms:

$$\mathrm{TV} \,.\, (\eta\rho - \rho\theta)\,\mathrm{U}(\eta - \theta) = \theta^2 - \eta^2, \tag{160.}$$

$$\mathrm{TV} \,.\, (\rho\eta - \theta\rho)\,\mathrm{U}(\eta - \theta) = \theta^2 - \eta^2. \tag{161.}$$

But the versor of *every* vector is, in this calculus, a square root of negative unity; we have therefore in particular,

$$(\mathrm{U}(\eta - \theta))^2 = -1; \tag{162.}$$

and under the sign TV, as under the sign T, it is allowed to divide by -1, without affecting the value of the tensor: it is therefore permitted to write the equation (160.) under the form

$$\mathrm{TV} \,.\, \frac{\eta\rho - \rho\theta}{\mathrm{U}(\eta - \theta)} = \theta^2 - \eta^2, \qquad (139.)$$

which form is thus demonstrated anew.

75. A few connected transformations may conveniently be noticed here. Since, for any quaternion Q,

$$(\mathrm{TVQ})^2 = -(\mathrm{VQ})^2 = (\mathrm{TQ})^2 - (\mathrm{SQ})^2, \tag{163.}$$

while the tensor of a product is the product of the tensors, and the tensor of a versor is unity; and since

$$\mathrm{S} \,.\, (\rho\eta - \theta\rho)(\eta - \theta) = \mathrm{S}(\rho\eta^2 - \rho\eta\theta - \theta\rho\eta + \theta\rho\theta) = -2\mathrm{S} \,.\, \eta\theta\rho, \tag{164.}$$

because

$$0 = \mathrm{S} \,.\, \rho\eta^2 = \mathrm{S} \,.\, \theta\rho\theta, \quad \text{and} \quad \mathrm{S} \,.\, \rho\eta\theta = \mathrm{S} \,.\, \theta\rho\eta = \mathrm{S} \,.\, \eta\theta\rho; \tag{165.}$$

we have therefore, *generally*,

$$\left.\begin{aligned} \mathrm{T}\,.\,(\rho\eta - \theta\rho)\,\mathrm{U}(\eta - \theta) &= \mathrm{T}(\rho\eta - \theta\rho); \\ \mathrm{S}\,.\,(\rho\eta - \theta\rho)\,\mathrm{U}(\eta - \theta) &= -2\mathrm{T}(\eta - \theta)^{-1}\,\mathrm{S}\,.\,\eta\theta\rho; \end{aligned}\right\} \tag{166.}$$

and there results the equation,

$$\mathrm{TV}\,.\,(\rho\eta - \theta\rho)\,\mathrm{U}(\eta - \theta) = \sqrt{\{\mathrm{T}(\rho\eta - \theta\rho)^2 - 4\mathrm{T}(\eta - \theta)^{-2}(\mathrm{S}\,.\,\eta\theta\rho)^2\}}, \tag{167.}$$

as a general formula of transformation, valid for *any three vectors*, η, θ, ρ. We may also, by the general rules of the present calculus, write the last result as follows,

$$\mathrm{TV}\,.\,(\rho\eta - \theta\rho)\,\mathrm{U}(\eta - \theta) = \sqrt{\{(\rho\eta - \theta\rho)(\eta\rho - \rho\theta) + (\eta - \theta)^{-2}(\eta\theta\rho - \rho\theta\eta)^2\}}; \tag{168.}$$

the signs S and T thus disappearing from the expression of the radical. For the ellipsoid, this radical, being thus equal to the left-hand member of the formula (167.), or to that of (168.), must, by (161.), receive the constant value $\theta^2 - \eta^2$; so that, by squaring on both sides, we find as a new form of the equation (161.) of the ellipsoid, the following:

$$(\theta^2 - \eta^2)^2 = (\rho\eta - \theta\rho)(\eta\rho - \rho\theta) + (\eta - \theta)^{-2}(\eta\theta\rho - \rho\theta\eta)^2. \tag{169.}$$

Or, by a partial reintroduction of the signs S and T, we find this somewhat shorter form:

$$\mathrm{T}(\rho\eta - \theta\rho)^2 + 4(\eta - \theta)^{-2}(\mathrm{S}\,.\,\eta\theta\rho)^2 = (\theta^2 - \eta^2)^2. \tag{170.}$$

And instead of the square of the tensor of the quaternion $\rho\eta - \theta\rho$, we may write any one of several general expressions for that square, which will easily suggest themselves to those who have studied the transformations (already printed in this Magazine), of the earlier and in some respects simpler equation of the ellipsoid, proposed by the present writer, namely the equation

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad \text{eq. (9.), art. 38.}$$

For instance, we may employ any of the following general equalities, which all flow with little difficulty from the principles of the present calculus:

$$\begin{aligned} \mathrm{T}(\rho\eta - \theta\rho)^2 &= \mathrm{T}(\eta\rho - \rho\theta)^2 \\ &= (\rho\eta - \theta\rho)(\eta\rho - \rho\theta) = (\eta\rho - \rho\theta)(\rho\eta - \theta\rho) \\ &= (\eta^2 + \theta^2)\rho^2 - \rho\eta\rho\theta - \theta\rho\eta\rho \\ &= (\eta^2 + \theta^2)\rho^2 - \eta\rho\theta\rho - \rho\theta\rho\eta \\ &= (\eta + \theta)^2\rho^2 - (\eta\rho + \rho\eta)(\theta\rho + \rho\theta) \\ &= (\eta^2 + \theta^2)\rho^2 - 2\mathrm{S}\,.\,\eta\rho\theta\rho \\ &= (\eta + \theta)^2\rho^2 - 4\mathrm{S}\,.\,\eta\rho\,.\,\mathrm{S}\,.\,\theta\rho \\ &= (\eta - \theta)^2\rho^2 + 4\mathrm{S}(\mathrm{V}\,.\,\eta\rho\,.\,\mathrm{V}\,.\,\rho\theta); \end{aligned} \tag{171.}$$

and which all hold good, independently of any relation between the three vectors η, θ, ρ.

76. As bearing on the last of these transformations it seems not useless to remark, that a general formula published in the Philosophical Magazine of August 1846, for any three vectors α, α', α'', namely the formula

$$\alpha \,\mathrm{S} \,.\, \alpha'\alpha'' - \alpha' \,\mathrm{S} \,.\, \alpha''\alpha = \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \alpha''), \qquad \text{eq. (12.) of art. 22,}$$

which is found to be extensively useful, and indeed of constant recurrence in the applications of the calculus of quaternions, may be proved symbolically in the following way, which is shorter than that employed in the 23rd article:

$$\begin{aligned} \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \alpha'') &= \tfrac{1}{2}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \alpha'' - \alpha'' \,\mathrm{V} \,.\, \alpha\alpha') = \tfrac{1}{2}(\alpha\alpha' \,.\, \alpha'' - \alpha'' \,.\, \alpha\alpha') \\ &= \tfrac{1}{2}\alpha(\alpha'\alpha'' + \alpha''\alpha') - \tfrac{1}{2}(\alpha\alpha'' + \alpha''\alpha)\alpha' = \alpha \,\mathrm{S} \,.\, \alpha'\alpha'' - \alpha' \,\mathrm{S} \,.\, \alpha''\alpha. \quad (172.) \end{aligned}$$

The formula may be also written thus:

$$\mathrm{V} \,.\, \alpha'' \,\mathrm{V} \,.\, \alpha'\alpha = \alpha \,\mathrm{S} \,.\, \alpha'\alpha'' - \alpha' \,\mathrm{S} \,.\, \alpha\alpha''; \tag{173.}$$

whence easily flows this other general and useful transformation, for the vector part of the product of any three vectors, α, α', α'':

$$\mathrm{V} \,.\, \alpha''\alpha'\alpha = \alpha \,\mathrm{S} \,.\, \alpha'\alpha'' - \alpha' \,\mathrm{S} \,.\, \alpha''\alpha + \alpha'' \,\mathrm{S} \,.\, \alpha\alpha'. \tag{174.}$$

Operating on this by $\mathrm{S} \,.\, \alpha'''$, we find, for the scalar part of the product of any *four vectors*, the expression:

$$\mathrm{S} \,.\, \alpha'''\alpha''\alpha'\alpha = \mathrm{S} \,.\, \alpha'''\alpha \,.\, \mathrm{S} \,.\, \alpha'\alpha'' - \mathrm{S} \,.\, \alpha'''\alpha' \,.\, \mathrm{S} \,.\, \alpha''\alpha + \mathrm{S} \,.\, \alpha'''\alpha'' \,.\, \mathrm{S} \,.\, \alpha\alpha'. \tag{175.}$$

But a quaternion, such as is $\alpha'\alpha$, or $\alpha'''\alpha''$, is always equal to the sum of its own scalar and vector parts; and the product of a scalar and a vector is a vector, while the scalar of a vector is zero; therefore

$$\alpha'\alpha = \mathrm{S} \,.\, \alpha'\alpha + \mathrm{V} \,.\, \alpha'\alpha, \quad \alpha'''\alpha'' = \mathrm{S} \,.\, \alpha'''\alpha'' + \mathrm{V} \,.\, \alpha'''\alpha'', \tag{176.}$$

and

$$\mathrm{S} \,.\, \alpha'''\alpha''\alpha'\alpha = \mathrm{S} \,.\, \alpha'''\alpha'' \,.\, \mathrm{S} \,.\, \alpha'\alpha + \mathrm{S}(\mathrm{V} \,.\, \alpha'''\alpha'' \,.\, \mathrm{V} \,.\, \alpha'\alpha). \tag{177.}$$

Comparing then (175.) and (177.), and observing that

$$\mathrm{S} \,.\, \alpha\alpha' = +\mathrm{S} \,.\, \alpha'\alpha, \quad \mathrm{V} \,.\, \alpha\alpha' = -\mathrm{V} \,.\, \alpha'\alpha, \tag{178.}$$

we obtain the following general expression for the scalar part of the product of the vectors of any two binary products of vectors:

$$\mathrm{S}(\mathrm{V} \,.\, \alpha'''\alpha'' \,.\, \mathrm{V} \,.\, \alpha'\alpha) = \mathrm{S} \,.\, \alpha'''\alpha \,.\, \mathrm{S} \,.\, \alpha'\alpha'' - \mathrm{S} \,.\, \alpha'''\alpha' \,.\, \mathrm{S} \,.\, \alpha''\alpha; \tag{179.}$$

while the vector part of the same product of vectors is easily found, by similar processes, to admit of being expressed in either of the two following ways (compare equation (3.) of article 24):

$$\begin{aligned} \mathrm{V}(\mathrm{V}\,.\,\alpha'''\alpha''\,.\,\mathrm{V}\,.\,\alpha'\alpha) &= \alpha'''\,\mathrm{S}\,.\,\alpha''\alpha'\alpha - \alpha''\,\mathrm{S}\,.\,\alpha'''\alpha'\alpha \\ &= \alpha\,\mathrm{S}\,.\,\alpha'''\alpha''\alpha' - \alpha'\,\mathrm{S}\,.\,\alpha'''\alpha''\alpha; \end{aligned} \tag{180.}$$

of which the combination conducts to the following general expression for any fourth vector α''', or ρ, in terms of any three given vectors α, α', α'', which are not parallel to any one common plane (compare equation (4.) of article 26):

$$\rho\,\mathrm{S}\,.\,\alpha''\alpha'\alpha = \alpha\,\mathrm{S}\,.\,\alpha''\alpha'\rho + \alpha'\mathrm{S}\,.\,\alpha''\rho\alpha + \alpha''\mathrm{S}\,.\,\rho\alpha'\alpha. \tag{181.}$$

If we further suppose that

$$\alpha'' = \mathrm{V}\,.\,\alpha'\alpha, \tag{182.}$$

we shall have

$$\mathrm{S}\,.\,\alpha''\alpha'\alpha = (\mathrm{V}\,.\,\alpha'\alpha)^2 = \alpha''^2; \tag{183.}$$

and after dividing by α''^2, the recent equation (181.) will become

$$\rho = \alpha\,\mathrm{S}\frac{\alpha'\rho}{\alpha''} + \alpha'\,\mathrm{S}\frac{\rho\alpha}{\alpha''} + \frac{\mathrm{S}\,.\,\alpha''\rho}{\alpha''}; \tag{184.}$$

whereby an arbitrary vector ρ may be expressed, in terms of any two given vectors α, α', which are not parallel to any common line, and of a third vector α'', perpendicular to both of them. And if, on the other hand, we change α, α', α'', α''' to θ, ρ, ρ, η, in the general formula (179.), we find that generally, for any three vectors η, θ, ρ, the following identity holds good:

$$\mathrm{S}(\mathrm{V}\,.\,\eta\rho\,.\,\mathrm{V}\,.\,\rho\theta) = \rho^2\,\mathrm{S}\,.\,\eta\theta - \mathrm{S}\,.\,\eta\rho\,.\,\mathrm{S}\,.\,\rho\theta; \tag{185.}$$

which serves to connect the two last of the expressions (171.), and enables us to transform either into the other.

77. To show the geometrical meaning of the equation (185.), let us divide it on both sides by $\mathrm{T}\,.\,\rho^2\eta\theta$; it then becomes, after transposing,

$$-\mathrm{SU}\,.\,\eta\theta = \mathrm{SU}\,.\,\eta\rho\,.\,\mathrm{SU}\,.\,\rho\theta + \mathrm{S}(\mathrm{VU}\,.\,\eta\rho\,.\,\mathrm{VU}\,.\,\rho\theta). \tag{186.}$$

Here, by the general principles of the geometrical interpretation of the symbols employed in this calculus (see the remarks in the Philosophical Magazine for July 1846), the symbol $\mathrm{SU}\,.\,\eta\theta$ is an expression for the cosine of the supplement of the angle between the two arbitrary vectors η and θ; and therefore the symbol $-\mathrm{SU}\,.\,\eta\theta$ is an expression for the cosine of that angle itself. In like manner, $-\mathrm{SU}\,.\,\eta\rho$ and $-\mathrm{SU}\,.\,\rho\theta$ are expressions for the cosines of the respective inclinations of those two vectors η and θ to a third arbitrary vector ρ; and at the same time $\mathrm{VU}\,.\,\eta\rho$ and $\mathrm{VU}\,.\,\rho\theta$ are vectors, of which the lengths represent the sines of the same two inclinations last mentioned, while they are directed towards the poles of the two positive rotations corresponding; namely the rotations from η to ρ, and from ρ to θ, respectively. The vectors $\mathrm{VU} \,.\, \eta\rho$ and $\mathrm{VU} \,.\, \rho\theta$ are therefore inclined to each other at an angle which is the supplement of the dihedral or spherical angle, subtended at the unit-vector $\mathrm{U}\rho$, or at its extremity on the unit-sphere, by the two other unit-vectors $\mathrm{U}\eta$ and $\mathrm{U}\theta$, or by the arc between their extremities: so that the scalar part of their product, in the formula (186.), represents the cosine of this spherical angle itself (and not of its supplement), multiplied into the product of the sines of the two sides or arcs upon the sphere, between which that angle is included. If then we denote the three sides of the spherical triangle, formed by the extremities of the three unit-vectors $\mathrm{U}\eta$, $\mathrm{U}\theta$, $\mathrm{U}\rho$, by the symbols, $\overset{\frown}{\eta\theta}$, $\overset{\frown}{\eta\rho}$, $\overset{\frown}{\rho\theta}$, and the spherical angle opposite to the first of them by the symbol $\widehat{\eta\rho\theta}$, the equation (186.) will take the form

$$\cos \overset{\frown}{\eta\theta} = \cos \overset{\frown}{\eta\rho} \cos \overset{\frown}{\rho\theta} + \sin \overset{\frown}{\eta\rho} \sin \overset{\frown}{\rho\theta} \cos \widehat{\eta\rho\theta}; \tag{187.}$$

which obviously coincides with the well-known and fundamental formula of spherical trigonometry, and is brought forward here merely as a verification of the consistency of the results of this calculus, and as an example of their geometrical interpretability.

A more interesting example of the same kind is furnished by the general formula (179.) for *four* vectors, which, when divided by the tensor of their product, becomes

$$\mathrm{S}(\mathrm{VU} \,.\, \alpha'''\alpha'' \,.\, \mathrm{VU} \,.\, \alpha'\alpha) = \mathrm{SU} \,.\, \alpha'''\alpha \,.\, \mathrm{SU} \,.\, \alpha'\alpha'' - \mathrm{SU} \,.\, \alpha'''\alpha' \,.\, \mathrm{SU} \,.\, \alpha''\alpha; \tag{188.}$$

and signifies, when interpreted on the same principles, that

$$\sin \overset{\frown}{\alpha\alpha'} \,.\, \sin \overset{\frown}{\alpha''\alpha'''} \,.\, \cos(\overset{\frown}{\alpha\alpha'} \frown \overset{\frown}{\alpha''\alpha'''}) = \cos \overset{\frown}{\alpha\alpha''} \,.\, \cos \overset{\frown}{\alpha'\alpha'''} - \cos \overset{\frown}{\alpha\alpha'''} \,.\, \cos \overset{\frown}{\alpha'\alpha''}; \tag{189.}$$

where the spherical angle between the two arcs from α to α' and from α'' to α''' may be replaced by the interval between the poles of the two positive rotations corresponding. The same result may be otherwise stated as follows: If L, L', L'', L''', denote any four points upon the surface of an unit-sphere, and A the angle which the arcs LL, $\mathrm{L}''\mathrm{L}'''$ form where they meet each other, (the arcs which include this angle being measured in the directions of the progressions from L to L', and from L'' to L''' respectively,) then the following equation will hold good:

$$\cos \mathrm{LL}'' \,.\, \cos \mathrm{L}'\mathrm{L}''' - \cos \mathrm{LL}''' \,.\, \cos \mathrm{L}'\mathrm{L}'' = \sin \mathrm{LL}' \,.\, \sin \mathrm{L}''\mathrm{L}''' \,.\, \cos A. \tag{190.}$$

Accordingly this last equation has been incidentally given, as an auxiliary theorem or lemma, at the commencement of those profound and beautiful researches, entitled *Disquisitiones Generales circa Superficies Curvas*, which were published by Gauss at Göttingen in 1828. That great mathematician and philosopher was content to prove the last written equation by the usual formulæ of spherical and plane trigonometry; but, however simple and elegant may be the demonstration thereby afforded, it appears to the present writer that something is gained by our being able to present the result (190.) or (189.), under the form (188.) or (179.), as an identity in the quaternion calculus. In general, all the results of plane and spherical trigonometry take the form of *identities* in this calculus; and their expressions, when so obtained, are associated with a reference to *vectors*, which is usually suggestive of *graphic* as well as *metric* relations.

78. Since

$$\rho\eta - \theta\rho = \mathrm{S} \,.\, \rho(\eta - \theta) + \mathrm{V} \,.\, \rho(\eta + \theta), \tag{191.}$$

the quaternion $\rho\eta - \theta\rho$ gives a pure vector as a product, or as a quotient, if it be multiplied or divided by the vector $\eta + \theta$ (compare article 68); we may therefore write

$$\rho\eta - \theta\rho = \lambda_1(\eta + \theta), \tag{192.}$$

λ_1 being a new vector-symbol, of which the value may be thus expressed:

$$\lambda_1 = \rho - 2(\eta + \theta)^{-1}\mathrm{S} \,.\, \theta\rho. \tag{193.}$$

The equation (192.) will then give,

$$\left.\begin{aligned} \mathrm{T}(\rho\eta - \theta\rho) &= \mathrm{T}\lambda_1 \,.\, \mathrm{T}(\eta + \theta); \\ \mathrm{T}(\rho\eta - \theta\rho)^2 &= \lambda_1^2(\eta + \theta)^2. \end{aligned}\right\} \tag{194.}$$

We have also the identity,

$$(\theta^2 - \eta^2)^2 = (\eta - \theta)^2(\eta + \theta)^2 + (\eta\theta - \theta\eta)^2; \tag{195.}$$

which may be shown to be such, by observing that

$$\begin{aligned} (\eta - \theta)^2(\eta + \theta)^2 &= (\eta^2 + \theta^2 - 2\mathrm{S} \,.\, \eta\theta)(\eta^2 + \theta^2 + 2\mathrm{S} \,.\, \eta\theta) \\ &= (\eta^2 + \theta^2)^2 - 4(\mathrm{S} \,.\, \eta\theta)^2 = (\eta^2 - \theta^2)^2 + 4(\mathrm{T} \,.\, \eta\theta)^2 - 4(\mathrm{S} \,.\, \eta\theta)^2 \\ &= (\eta^2 - \theta^2)^2 - 4(\mathrm{V} \,.\, \eta\theta)^2 = (\theta^2 - \eta^2)^2 - (\eta\theta - \theta\eta)^2; \end{aligned} \tag{196.}$$

or by remarking that (see equations (152.), (163.)),

$$\left.\begin{aligned} \eta^2 - \theta^2 = \mathrm{S} \,.\, (\eta - \theta)(\eta + \theta), \quad \eta\theta - \theta\eta = \mathrm{V} \,.\, (\eta - \theta)(\eta + \theta), \\ \text{and} \quad (\eta - \theta)^2(\eta + \theta)^2 = (\mathrm{T} \,.\, (\eta - \theta)(\eta + \theta))^2; \end{aligned}\right\} \tag{197.}$$

or in several other ways. Introducing then a new vector ϵ, such that

$$\eta\theta - \theta\eta = \epsilon\,\mathrm{T}(\eta + \theta), \quad \text{or,} \quad \epsilon = 2\mathrm{V} \,.\, \eta\theta \,.\, \mathrm{T}(\eta + \theta)^{-1}; \tag{198.}$$

and that therefore

$$(\eta\theta - \theta\eta)^2 = -\epsilon^2(\eta + \theta)^2, \tag{199.}$$

and

$$2\mathrm{S} \,.\, \eta\theta\rho = \mathrm{S} \,.\, \epsilon\rho \,.\, \mathrm{T}(\eta + \theta), \quad 4(\mathrm{S} \,.\, \eta\theta\rho)^2 = -(\mathrm{S} \,.\, \epsilon\rho)^2(\eta + \theta)^2; \tag{200.}$$

while, by (135.),

$$\mathrm{T}(\eta - \theta) = b, \quad (\eta - \theta)^2 = -b^2; \tag{201.}$$

we find that the equation (170.) of the ellipsoid, after being divided by $(\eta+\theta)^2$, assumes the following form:

$$\lambda_1^2 + b^{-2}(\mathrm{S}\,.\,\epsilon\rho)^2 + b^2 + \epsilon^2 = 0. \tag{202.}$$

But also, by (193.), (198.),

$$\mathrm{S}\,.\,\epsilon\lambda_1 = \mathrm{S}\,.\,\epsilon\rho; \tag{203.}$$

the equation (202.) may therefore be also written thus:

$$0 = (\lambda_1 - \epsilon)^2 + (b + b^{-1}\,\mathrm{S}\,.\,\epsilon\rho)^2; \tag{204.}$$

and the scalar $b + b^{-1}\mathrm{S}\,.\,\epsilon\rho$ is positive, even at an extremity of the mean axis of the ellipsoid, because, by (195.), (199.), (201.), we have

$$(\theta^2 - \eta^2)^2 = -(b^2 + \epsilon^2)(\eta + \theta)^2 = (b^2 - \mathrm{T}\epsilon^2)\,\mathrm{T}(\eta + \theta)^2, \tag{205.}$$

and therefore

$$\mathrm{T}\epsilon < b. \tag{206.}$$

We have then this new form of the equation of the ellipsoid, deduced by transposition and extraction of square roots (according to the rules of the present calculus), from the form (204.):

$$\mathrm{T}(\lambda_1 - \epsilon) = b + b^{-1}\,\mathrm{S}\,.\,\epsilon\rho. \tag{207.}$$

By a process exactly similar to the foregoing, we find also the form

$$\mathrm{T}(\lambda_1 + \epsilon) = b - b^{-1}\,\mathrm{S}\,.\,\epsilon\rho. \tag{208.}$$

which differs from the equation last found, only by a change of sign of the auxiliary and constant vector ϵ: and hence, by addition of the two last equations, we find still another form, namely,

$$\mathrm{T}(\lambda_1 - \epsilon) + \mathrm{T}(\lambda_1 + \epsilon) = 2b; \tag{209.}$$

or substituting for λ_1, ϵ, and b their values in terms of η, θ, and ρ, and multiplying into $\mathrm{T}(\eta + \theta)$,

$$\mathrm{T}\left(\frac{\rho\eta - \theta\rho}{\mathrm{U}(\eta + \theta)} - 2\mathrm{V}\,.\,\eta\theta\right) + \mathrm{T}\left(\frac{\rho\eta - \theta\rho}{\mathrm{U}(\eta + \theta)} + 2\mathrm{V}\,.\,\eta\theta\right) = 2\mathrm{T}\,.\,(\eta - \theta)(\eta + \theta). \tag{210.}$$

79. The locus of the termination of the auxiliary and variable vector λ_1, which is *derived* from the vector ρ of the original ellipsoid by the *linear* formula (193.), is expressed or represented by the equation (209.); it is therefore evidently a certain *new* ellipsoid, namely an *ellipsoid of revolution*, which has the mean axis $2b$ of the old or given ellipsoid for its major axis, or for its axis of revolution, while the vectors of its two *foci* are denoted by the symbols $+\epsilon$ and $-\epsilon$. If a denote the greatest, and c the least semiaxis, of the original ellipsoid, while b still denotes its mean semiaxis, then, by what has been shown in former articles, we have the values,

$$\mathrm{T}\eta = \mathrm{T}\iota = \tfrac{1}{2}(a + c); \quad \mathrm{T}\theta = \mathrm{T}\kappa = \tfrac{1}{2}(a - c); \tag{211.}$$

and consequently (compare the note to art. 70),

$$a = \mathrm{T}\eta + \mathrm{T}\theta; \quad c = \mathrm{T}\eta - \mathrm{T}\theta; \tag{212.}$$

therefore

$$ac = \mathrm{T}\eta^2 - \mathrm{T}\theta^2 = \theta^2 - \eta^2; \tag{213.}$$

also

$$\begin{aligned} \mathrm{T}(\eta + \theta)^2 + b^2 &= -(\eta + \theta)^2 - (\eta - \theta)^2 = -2\eta^2 - 2\theta^2 \\ &= 2\mathrm{T}\eta^2 + 2\mathrm{T}\theta^2 = (\mathrm{T}\eta + \mathrm{T}\theta)^2 + (\mathrm{T}\eta - \mathrm{T}\theta)^2, \end{aligned} \tag{214.}$$

and

$$\mathrm{T}(\eta + \theta)^2 = a^2 - b^2 + c^2; \tag{215.}$$

whence, by (205.),

$$\mathrm{T}\epsilon^2 = b^2 - \frac{a^2c^2}{a^2 - b^2 + c^2} = \frac{(a^2 - b^2)(b^2 - c^2)}{a^2 - b^2 + c^2}. \tag{216.}$$

Such, then, is the expression for the square of the distance of either focus of the new or derived ellipsoid of revolution, which has λ_1 for its varying vector, from the common centre of the new and old ellipsoids, which centre is also the common origin of the vectors λ_1 and ρ: while these two foci of the new ellipsoid are situated upon the mean axis of the old one. There exist also other remarkable relations, between the original ellipsoid with three unequal semiaxes a, b, c, and the new ellipsoid of revolution, of which some will be brought into view, by pursuing the quaternion analysis in a way which we shall proceed to point out.

80. The geometrical construction already mentioned (in articles 64, 71, &c.), of the original ellipsoid as the locus of the circle in which two sliding spheres intersect, shows easily (see art. 72) that the scalar coefficient g, in the equations (146.) of that pair of sliding spheres, becomes equal to the number 2, at one of those limiting positions of the pair, for which, after cutting, they *touch*, before they cease to meet each other. In fact, if we thus make

$$g = 2, \tag{217.}$$

the values (145.) of the vectors of the centres will give, for the interval between those two centres of the two sliding spheres, the expression

$$\mathrm{T}(\mu - \lambda') = g\,\mathrm{T}(\eta - \theta) = 2b; \tag{218.}$$

this interval will therefore be in this case equal to the diameter of either sliding sphere, because it will be equal to the mean axis of the ellipsoid: and the two spheres will touch one another. Had we assumed a value for g, less by a very little than the number 2, the two spheres would have cut each other in a very small circle, of which the circumference would have been (by the construction) entirely contained upon the surface of the ellipsoid; and the plane of this little circle would have been parallel and very near to that other plane, which was the common tangent plane of the two spheres, and also of the ellipsoid, when g received the value 2 itself. It is clear, then, that this value 2 of g corresponds to an *umbilicar point* on the ellipsoid; and that the equation

$$\mathrm{S} \,.\, (\theta - \eta)\rho = \theta^2 - \eta^2, \tag{219.}$$

which is obtained from the more general equation (148.) of the plane of a circle on the ellipsoid, by changing g to 2, represents an *umbilicar tangent plane*, at which the normal has the direction of the vector $\eta - \theta$. Accordingly it has been seen that this last vector has the direction of the cyclic normal ι: in fact, the expressions (131.), for η and θ in terms of ι and κ, give conversely these other expressions for the latter vectors in terms of the former,

$$\iota = \mathrm{T}\eta\,\mathrm{U}(\eta - \theta); \quad \kappa = \mathrm{T}\theta\,\mathrm{U}(\theta^{-1} - \eta^{-1}) : \tag{220.}$$

whence (it may here be noted) follow the two parallelisms,

$$\mathrm{U}\iota - \mathrm{U}\kappa = \mathrm{U}(\eta - \theta) + \mathrm{U}(\eta^{-1} - \theta^{-1}) \parallel \mathrm{U}\eta + \mathrm{U}\theta; \tag{221.}$$

$$\mathrm{U}\iota + \mathrm{U}\kappa = \mathrm{U}(\eta - \theta) - \mathrm{U}(\eta^{-1} - \theta^{-1}) \parallel \mathrm{U}\eta - \mathrm{U}\theta; \tag{222.}$$

the members of (221.) having each the direction of the greatest axis of the ellipsoid, and the members of (222.) having each the direction of the least axis; as may easily be proved, for the first members of these formulæ, by the construction with the *diacentric sphere*, which was communicated by the writer to the Royal Irish Academy in 1846, and was published in the present Magazine in the course of the following year. The equation (219.) may be verified by observing that it gives, for the length of the perpendicular let fall from the centre of the ellipsoid on an umbilicar tangent plane, the expression

$$p = (\theta^2 - \eta^2)\mathrm{T}(\eta - \theta)^{-1} = acb^{-1}; \tag{223.}$$

agreeing with known results. And the vector ω of the umbilicar point itself must be the semisum of the vectors of the centres of the two equal and sliding spheres, in that limiting position of the pair in which (as above) they touch each other; this *umbilicar vector* ω is therefore expressed as follows:

$$\omega = \eta + \theta; \tag{224.}$$

because this is the semisum of μ and λ' in (145.), or of $g\eta$ and $g\theta$ when $g = 2$. (Compare the note to article 70.) As a verification, we may observe that this expression (224.) gives, by (215.), the following known value for the length of an umbilicar semidiameter of the ellipsoid,

$$u = \mathrm{T}\omega = \mathrm{T}(\eta + \theta) = \sqrt{(a^2 - b^2 + c^2)}. \tag{225.}$$

By similar reasonings it may be shown that the expression

$$\omega' = \mathrm{T}\eta\,\mathrm{U}\theta + \mathrm{T}\theta\,\mathrm{U}\eta, \tag{226.}$$

which may also be thus written, (see same note to art. 70,)

$$\omega' = -\mathrm{T} \,.\, \eta\theta \,.\, (\eta^{-1} + \theta^{-1}), \tag{227.}$$

represents *another* umbilicar vector; in fact, we have, by (224.) and (226.),

$$\mathrm{T}\omega' = \mathrm{T}\omega, \tag{228.}$$

and

$$\left.\begin{aligned} \omega + \omega' &= (\mathrm{T}\eta + \mathrm{T}\theta)(\mathrm{U}\eta + \mathrm{U}\theta), \\ \omega - \omega' &= (\mathrm{T}\eta - \mathrm{T}\theta)(\mathrm{U}\eta - \mathrm{U}\theta); \end{aligned}\right\} \tag{229.}$$

so that the vectors ω ω' are equally long, and the angle between them is bisected by $\mathrm{U}\eta + \mathrm{U}\theta$, or (see (221.)) by the axis major of the ellipsoid; while the supplementary angle between ω and $-\omega'$ is bisected by $\mathrm{U}\eta - \mathrm{U}\theta$, or (as is shown by (222.)) by the axis minor. It is evident that $-\omega$ and $-\omega'$ are also umbilicar vectors; and it is clear, from what has been shown in former articles, that the vectors η and θ have the directions of the axes of the two cylinders of revolution, which can be circumscribed about that given or original ellipsoid, to which all the remarks of the present article relate.

81. These remarks being premised, let us now resume the consideration of the variable vector λ_1, of art. 78, which has been seen to terminate on the surface of a certain derived ellipsoid of revolution. Writing, under a slightly altered form, the expression (193.) for that vector λ_1, and combining with it three other analogous expressions, for three other vectors, λ_2, λ_3, λ_4, as follows,

$$\lambda_1 = \frac{\rho\eta - \theta\rho}{\eta + \theta}; \quad \lambda_2 = \frac{\rho\theta - \eta\rho}{\eta + \theta}; \quad \lambda_3 = \frac{\rho\theta^{-1} - \eta^{-1}\rho}{\eta^{-1} + \theta^{-1}}; \quad \lambda_4 = \frac{\rho\eta^{-1} - \theta^{-1}\rho}{\eta^{-1} + \theta^{-1}}; \tag{230.}$$

it is easy to prove that

$$\mathrm{T}\lambda_1 = \mathrm{T}\lambda_2 = \mathrm{T}\lambda_3 = \mathrm{T}\lambda_4; \tag{231.}$$

and that

$$\mathrm{S}\,.\,\eta\theta\lambda_1 = \mathrm{S}\,.\,\eta\theta\lambda_2 = \mathrm{S}\,.\,\eta\theta\lambda_3 = \mathrm{S}\,.\,\eta\theta\lambda_4 = \mathrm{S}\,.\,\eta\theta\rho; \tag{232.}$$

whence it follows that the four vectors, λ_1, λ_2, λ_3, λ_4, being supposed to be all drawn from the centre A of the original ellipsoid, terminate in four points, L_1, L_2, L_3, L_4, which are the corners of a quadrilateral inscribed in a circle of the derived ellipsoid of revolution; the plane of this circle being parallel to the plane of the greatest and least axes of the original ellipsoid, and passing through the point E of that ellipsoid, which is the termination of the vector ρ. We shall have also the equations,

$$\frac{\lambda_2 - \rho}{\lambda_1 - \rho} = \frac{\mathrm{S}\,.\,\eta\rho}{\mathrm{S}\,.\,\theta\rho} = \mathrm{V}^{-1}0; \quad \frac{\lambda_3 - \rho}{\lambda_4 - \rho} = \frac{\mathrm{S}\,.\,\eta^{-1}\rho}{\mathrm{S}\,.\,\theta^{-1}\rho} = \mathrm{V}^{-1}0; \tag{233.}$$

which show that the two opposite sides $\mathrm{L}_1\mathrm{L}_2$, $\mathrm{L}_3\mathrm{L}_4$, of this inscribed quadrilateral, being prolonged, if necessary, intersect in the lately-mentioned point E of the original ellipsoid. And because the expressions (230.) give also

$$\mathrm{V}\frac{\lambda_2 - \lambda_1}{\eta + \theta} = 0, \quad \mathrm{V}\frac{\lambda_4 - \lambda_3}{\eta^{-1} + \theta^{-1}} = 0, \tag{234.}$$

these opposite sides L_1L_2, L_3L_4, of the plane quadrilateral thus inscribed in a circle of the derived ellipsoid of revolution, are parallel respectively to the vectors $\eta + \theta$, $\eta^{-1} + \theta^{-1}$, or to the two umbilicar vectors ω, ω', of the original ellipsoid, with the semiaxes $a\,b\,c$. At the same time, the equations

$$\mathrm{V}\frac{\lambda_3 - \lambda_2}{\eta} = 0, \quad \mathrm{V}\frac{\lambda_1 - \lambda_4}{\theta} = 0, \tag{235.}$$

hold good, and show that the two other mutually opposite sides of the same inscribed quadrilateral, namely the sides L_2L_3, L_4L_1, are respectively parallel to the two vectors η, θ, or to the axes of the two cylinders of revolution which can be circumscribed about the same original ellipsoid. Hence it is easy to infer the following theorem, which the author supposes to be new:—*If on the mean axis* $2b$ *of a given ellipsoid, abc, as the major axis, and with two foci,* F_1, F_2, *of which the common distance from the centre* A *is*

$$\overline{AF_1} = \overline{AF_2} = e = \frac{\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}}{\sqrt{(a^2 - b^2 + c^2)}}, \tag{236.}$$

we construct an ellipsoid of revolution; and if, in any circular section of this new ellipsoid, we inscribe a quadrilateral, $L_1L_2L_3L_4$, *of which the two opposite sides* L_1L_2, L_3L_4, *are respectively parallel to the two umbilicar diameters of the given ellipsoid; while the two other and mutually opposite sides,* L_2L_3, L_4L_1, *of the same inscribed quadrilateral, are respectively parallel to the axes of the two cylinders of revolution which can be circumscribed about the same given ellipsoid; then the point of intersection* E *of the first pair of opposite sides* (namely of those parallel to the umbilicar diameters), *will be a point upon that given ellipsoid.* It seems to the present writer that, in consequence of this remarkable relation between these two ellipsoids, the two foci F_1, F_2 of the above described ellipsoid of revolution, which have been seen to be situated upon the mean axis of the original ellipsoid, of which the three unequal semiaxes are denoted by a, b, c, may not inconveniently be called the TWO MEDIAL FOCI of that original ellipsoid: but he gladly submits the question of the propriety of such a designation, to the judgement of other and better geometers. Meanwhile it may be noticed that the two ellipsoids intersect each other in a system of two ellipses, of which the planes are perpendicular to the axes of the two cylinders of revolution above mentioned; and that those four common tangent planes of the two ellipsoids, which are parallel to their common axis, that is to the mean axis of the original ellipsoid $a\,b\,c$, are parallel also to its umbilicar diameters.

82. This seems to be a proper place for inserting some notices of investigations and results, respecting the inscription of rectilinear (but not generally plane) polygons, in spheres, and other surfaces of the second degree.

Let ρ and σ be any two unit-vectors, or directed radii of an unit-sphere; so that, according to a fundamental principle of the present Calculus, we may write

$$\rho^2 = \sigma^2 = -1. \tag{237.}$$

We shall then have also,

$$0 = \sigma^2 - \rho^2 = \sigma(\sigma - \rho) + (\sigma - \rho)\rho, \tag{238.}$$

and consequently

$$\sigma = -(\sigma - \rho)\rho(\sigma - \rho)^{-1} = -\lambda\rho\lambda^{-1}, \tag{239.}$$

if λ be the directed chord $\sigma - \rho$ itself, or any portion or prolongation thereof, or any vector parallel thereto. If then $\rho, \rho_1, \rho_2, \ldots \rho_n$, be any series or succession of unit-vectors, while $\lambda_1, \lambda_2, \ldots \lambda_n$ are any vectors respectively coincident with, or parallel to, the successive and rectilinear chords of the unit-sphere, connecting the successive points where the vectors $\rho \ldots \rho_n$ terminate; and if we introduce the quaternions,

$$q_1 = \lambda_1; \quad q_2 = \lambda_2\lambda_1; \quad q_3 = \lambda_3\lambda_2\lambda_1; \quad \&\text{c.}, \tag{240.}$$

we shall have the expressions,

$$\rho_1 = -q_1\rho q_1^{-1}; \quad \rho_2 = +q_2\rho q_2^{-1}; \quad \rho_3 = -q_3\rho q_3^{-1}; \quad \&\text{c.} \tag{241.}$$

Hence if we write the equation

$$\rho_n = \rho, \tag{242.}$$

to express the conception of a *closed* polygon of n sides, inscribed in the sphere, we shall have the general formula,

$$\rho q_n = (-1)^n q_n\rho; \tag{243.}$$

which is immediately seen to decompose itself into the two following principal cases, according as the number n of the sides is even or odd:

$$\rho q_{2m} = +q_{2m}\rho; \tag{244.}$$

$$\rho q_{2m+1} = -q_{2m+1}\rho. \tag{245.}$$

The equation (244.) admits also of being written thus, by the general rules of quaternions,

$$0 = \mathrm{V} \,.\, \rho\,\mathrm{V}q_{2m}; \tag{246.}$$

and the equation (245.) resolves itself, by the same general rules, into the two equations following:

$$0 = \mathrm{S}q_{2m+1}; \quad 0 = \mathrm{S} \,.\, q_{2m+1}\rho. \tag{247.}$$

We shall now proceed to consider some of the consequences which follow from the formulæ thus obtained.

83. An immediate consequence of the equations (247.), or rather a translation of those equations into words, is the following quaternion theorem:—*If any rectilinear polygon, with any odd number of sides, be inscribed in a sphere, the continued product of those sides is a vector, tangential to the sphere at the first corner of the polygon.* It is understood that, in forming this continued product of sides, their *directions* and *order* are attended to: the first side being multiplied as a vector by the second, so as to form a certain quaternion product; and this product being afterwards multiplied, in succession, by the third side, then by the fourth, the fifth, &c., so as to form a series of quaternions, of which the *last* will (by the

theorem) have its *scalar* part equal to zero; while the *vector* part, or the product itself, will be constructed by a right line with a certain definite direction, which will (by the same theorem) be that of a certain rectilinear tangent to the sphere, at the point or corner where the first side of the inscribed polygon begins. [The *tensor* of the resulting vector, or the *length* of the product line, will of course represent, at the same time, by the general law of tensors, the product of the lengths of the factor lines, with the usual reference to some assumed unit of length.] And conversely, whenever it happens that an odd number of successive right lines in space, being multiplied together successively by the rules of the present Calculus, give a *line* as their continued product, that is to say, when the scalar of the quaternion obtained by this multiplication vanishes, then those right lines may be inferred to have the *directions* of the successive sides of a polygon inscribed in a sphere.

84. Already, even as applied to the case of an inscribed gauche *pentagon*, the theorem of the last article expresses a *characteristic* property of the *sphere*, which may be regarded as being of a *graphic* rather than of a *metric* character; inasmuch as it concerns immediately *directions* rather than *magnitudes*, although there is no difficulty in deducing from it metric relations also: as will at once appear by considering the formula which expresses it, namely the following,

$$0 = \mathrm{S} \,.\, (\rho - \rho_4)(\rho_4 - \rho_3)(\rho_3 - \rho_2)(\rho_2 - \rho_1)(\rho_1 - \rho). \tag{248.}$$

(See the Proceedings of the Royal Irish Academy for July 1846, where this quaternion theorem for the case of the inscribed pentagon was given.) For the theorem assigns, and in a simple manner expresses, to those who accept the language of this Calculus, a relation between the *five* successive directions of the sides of a gauche pentagon inscribed in a sphere, which appears to the present writer to be *analogous* to (although necessarily more complex than) the angular relation established in the third book of Euclid's Elements, between the *four* directions of the sides of a plane quadrilateral inscribed in a circle. Indeed, it will be found to be easy to deduce the property of the plane inscribed quadrilateral, from the theorem respecting the inscribed gauche pentagon. For, by conceiving the fifth side $\mathrm{P}_4\mathrm{P}$ of the pentagon $\mathrm{P} \ldots \mathrm{P}_4$ to tend to vanish, and therefore to become tangential at the first corner P, it is seen that the vector part of the quaternion which is the continued product of the four first sides must tend, at the same time, to become normal to the sphere at P; in order that, when multiplied into an arbitrary tangential vector there, it may give a vector as the product. Hence the vector part of the product of the four successive sides of an inscribed gauche quadrilateral $\mathrm{PP}_1\mathrm{P}_2\mathrm{P}_3$, is constructed by a right line which is normal to the sphere at the first corner; and more generally, either by the same geometrical reasoning applied to the theorem of art. 83, or by considering the signification of the formula (246.), we may deduce this other theorem, that *the vector of the continued product of the successive sides of an inscribed gauche polygon* $\mathrm{P} \ldots \mathrm{P}_{2m-1}$, *of any even number of sides, is normal to the sphere at the first corner* P. Suppose now the inscribed quadrilateral, or more generally the polygon of $2m$ sides, to flatten into a *plane* figure; it will thus come to be inscribed in a *circle*, and consequently in infinitely many spheres *at once*; and the only way to escape a resulting indeterminateness in the value for the vector of the product, is by that vector vanishing: which accordingly it may be otherwise proved to do, although the present mode of proof will appear sufficient to those who examine its principles with care. And thus we shall find ourselves conducted to the well-known graphic property of the quadrilateral inscribed in the circle, and more generally to a corresponding theorem respecting inscribed hexagons, octagons, &c., under the form of the following proposition in quaternions, which expresses a characteristic property of the *circle*:—*The vector part of the product of the successive sides of any polygon, with any even number of sides, inscribed in a circle, vanishes*; or, in other words, the product thus obtained, instead of being a *complete* quaternion, reduces itself simply to a positive or negative *number*. On the other hand, it is easy to see, from what precedes, that *the product of the successive sides of a triangle, pentagon, or other polygon of any odd number of sides, inscribed in a circle, is a vector, which touches the circle at the first corner of the polygon*, or is parallel to such a tangent.

85. Although the *precise law* of the relation between the directions of the sides of an inscribed gauche pentagon, heptagon, &c., expressed by the first formulæ (247.), is *peculiar* to the *sphere*; yet it is easy to abstract from that relation a *part*, which shall hold good, as a law of a *more general* character, for *other* surfaces of the second order. For we may easily infer, from that formula, especially when combined with the other equations of art. 82, that *if the first* $2m$ *sides of an inscribed polygon of* $2m+1$ *sides,* $\mathrm{P}'\mathrm{P}'_1\mathrm{P}'_2 \ldots \mathrm{P}'_{2m}$*, be respectively parallel to the successive sides of another polygon of* $2m$ *sides,* $\mathrm{P}\mathrm{P}_1 \ldots \mathrm{P}_{2m-1}$*, inscribed in the same surface, then the last side,* $\mathrm{P}'_{2m}\mathrm{P}'$*, of the former polygon, will be parallel to the plane which touches the surface at the first corner* P *of the latter polygon*: and under *this* form of enunciation, it is obvious that the theorem must admit of being extended, by deformation, to ellipsoids, and other surfaces of the second degree. We may then enunciate also this other theorem, respecting the inscription of rectilinear polygons in such surfaces (which theorem was communicated to the Royal Irish Academy in March 1849):—*If, after inscribing, in a surface of the second degree, any gauche polygon of* $2m$ *sides,* $\mathrm{P}\mathrm{P}_1 \ldots \mathrm{P}_{2m-1}$*, we then inscribe in the same surface another gauche polygon, of* $4m+1$ *sides,* $\mathrm{P}'\mathrm{P}'_1 \ldots \mathrm{P}'_{4m}$*, under the following* $4m$ *conditions of parallelism:*

$$\mathrm{P}'\mathrm{P}'_1 \parallel \mathrm{P}\mathrm{P}_1; \quad \mathrm{P}'_1\mathrm{P}'_2 \parallel \mathrm{P}_1\mathrm{P}_2; \quad \ldots \quad \mathrm{P}'_{2m-1}\mathrm{P}'_{2m} \parallel \mathrm{P}_{2m-1}\mathrm{P}; \tag{249.}$$

and

$$\mathrm{P}'_{2m}\mathrm{P}'_{2m+1} \parallel \mathrm{P}\mathrm{P}_1; \quad \mathrm{P}'_{2m+1}\mathrm{P}'_{2m+2} \parallel \mathrm{P}_1\mathrm{P}_2; \quad \ldots \quad \mathrm{P}'_{4m-1}\mathrm{P}'_{4m} \parallel \mathrm{P}_{2m-1}\mathrm{P}; \tag{250.}$$

(the first corner P' of the second polygon being *assumed* at pleasure on the surface, and the other corners P'_1, &c., of that polygon, being successively *derived* from this one, by drawing two series of parallels as here directed;) *then the diagonal plane* $\mathrm{P}'\mathrm{P}'_{2m}\mathrm{P}'_{4m}$*, which contains the first, middle, and last corners of the polygon with* $4m+1$ *sides, will be parallel to the plane which touches the surface at the first corner* P *of the polygon with* $2m$ *sides.* In fact, the two rectilinear diagonals, $\mathrm{P}'\mathrm{P}'_{2m}$ and $\mathrm{P}'_{2m}\mathrm{P}'_{4m}$, will, by a former theorem of the present article, be parallel to that tangent plane. For example, if the first, second, third and fourth sides, of a gauche *quadrilateral* inscribed in a surface of the second order, be parallel to the first, second, third, and fourth, and *also* to the fifth, sixth, seventh, and eighth sides respectively, of a gauche *enneagon* inscribed in the same surface; than that diagonal plane of the enneagon which contains the first, fifth and ninth corners thereof, will be parallel to the plane which touches the surface at the first corner of the quadrilateral.

86. The same sort of quaternion analysis, proceeding from the formulæ in art. 82, and from others analogous to them, has conducted the author to many other geometrical theorems, respecting the inscription of gauche polygons in surfaces of the second degree. An outline of some of these was given to the Royal Irish Academy in June 1849; and some of them may be mentioned here. To avoid, at first, imaginary* deformations, in passing from an original sphere, the surface in which the polygons are inscribed shall be supposed, for the present, to be an *ellipsoid*. Results of the same *general* character, but with *some* important modifications (connected with the *ordinary* square root of negative unity,) hold good for the inscription of such polygons in *other* surfaces of the same order, as the writer may afterwards point out. He is aware, indeed, that the corresponding class of questions, respecting the inscription of *plane* polygons in *conics*, has attained sufficient celebrity; and feels that his own acquaintance with what has been already done in that department of geometrical science is inferior to the knowledge of its history possessed by several of his contemporaries, for instance, by Professor Davies. He knows also that some of the published methods for inscribing in a circle, or plane conic, a polygon whose sides shall pass through the same number of given points, can be adapted to the case of a polygon formed by *arcs* of great circles on the surface of a sphere, and inscribed in a *spherical* conic; and he has, by quaternions, been conducted to some such methods himself, for the solution of the latter problem. But he acknowledges that he shall feel some little surprise, though perhaps not entitled to do so, if it shall turn out that the results of which he proceeds to give an outline, respecting the *inscription of rectilinear but gauche polygons in an ellipsoid*, have been wholly (or even partially) anticipated. They have certainly been, in his own case, results of the application of the quaternion calculus: but whatever geometrical truth has been attained by any one *general mathematical method* (such as the Quaternions claim to be), may also be found, or at least *proved*, by any *other* method equally general. And those who shall take the pains of *proving* for themselves, by the

* While acknowledging, as the author is bound to do, the great courtesy towards himself that has been shown by several recent and able writers, on subjects having some general connexion or resemblance with those on which he has been engaged, he hopes that he may be allowed to say,—yet rather as requesting a favour than as claiming a right,—that he will be happy if the inventor of the *Pluquaternions* shall consent to his adopting or rather retaining a *word*, namely "biquaternion," which the Rev. Mr. Kirkman has indeed *employed*, with reference to the *octaves* of Mr. J. T. Graves and Mr. Cayley, but does not appear to *want*, for any of his *own* purposes: whereas Sir W. Rowan Hamilton has for years been accustomed to use this word BIQUATERNION,—though perhaps hitherto without printed publication,—and indeed could not, without sensible inconvenience, have dispensed with it, to denote an expression *entirely distinct from those octaves*, namely one of the form

$$\mathrm{Q} + \sqrt{-1}\mathrm{Q}';$$

where $\sqrt{-1}$ is the old and *ordinary imaginary of algebra* (and is therefore quite *distinct* from i, j, k), while Q and Q′ are abridged symbols for *two different quaternions* of the kind $w+ix+jy+kz$, introduced into analysis in 1843. *Biquaternions* of *this* sort have repeatedly *forced* themselves on the attention of Sir W. R. H., in questions respecting *geometrical impossibility, ideal intersections, imaginary deformations*, and the like.

Cartesian Coordinates, or by some less algebraical and more purely geometrical method, the following theorems, (if not already known), which have thus been *found* by the Quaternions, will doubtless be led to perceive *many* new truths, connected with them, which have escaped the present writer; although he too has arrived at other connected results, which he must suppress in the following notice.

87. I. An *ellipsoid* (E) being given, and also a system of any *even* number of points of space, $A_1, A_2, \dots A_{2m}$, of which points it is here supposed that none are situated on the surface of the ellipsoid; it is, in general, possible to *inscribe* in this ellipsoid, *two*, and *only two*, distinct and *real polygons of* $2m$ *sides*, $BB_1 \dots B_{2m-1}$ and $B'B'_1 \dots B'_{2m-1}$, such that the sides of each of these two polygons (B) (B′) shall pass, respectively and successively, through the $2m$ given points; or in other words, so that $BA_1B_1, B_1A_2B_2, \dots B_{2m-1}A_{2m}B$, and also $B'A_1B'_1, B'_1A_2B'_2, \dots B'_{2m-1}A_{2m}B'$, shall be straight lines; while $B, B_1, \dots B_{2m-1}$, and also $B', B'_1, \dots B'_{2m-1}$, shall be points upon the surface of the ellipsoid.

[It should be noted that there are also, in general, what may, by the use of a known phraseology, be called *two other*, but *geometrically imaginary*, modes of inscribing a polygon, under the same conditions, in an *ellipsoid*: which modes *may become real, by imaginary deformation*, in passing to *another* surface of the second order.]

II. If we now take any *other* and *variable point* P on the ellipsoid (E) *instead* of B or B′, and make *it* the *first corner* of an inscribed polygon of $2m + 1$ sides, of which the *first* $2m$ *sides* shall pass, respectively and successively, through the $2m$ given points (A); in such a manner that $PA_1P_1, P_1A_2P_2, \dots P_{2m-1}A_{2m}P_{2m}$, shall be straight lines, while $P, P_1, P_2, \dots P_{2m}$ shall all be points on the surface of the ellipsoid: then the *last side*, or *closing chord*, $P_{2m}P$, of this new and *variable polygon* (P), thus inscribed in the ellipsoid (E), shall *touch*, in all its positions, a certain *other ellipsoid* (E′).

III. This *new* ellipsoid (E′) is itself *inscribed* in the given ellipsoid (E), having *double contact* therewith, but being elsewhere interior thereto.

IV. The *two points of contact* of these two ellipsoids are the points B and B′; that is, they are the *first corners of the two inscribed polygons of* $2m$ *sides*, (B) and (B′), which were considered in I.

[So far, the results are evidently analogous to known theorems, respecting polygons in conics; what follows is more peculiar to space.]

V. If the two ellipsoids, (E) and (E′), be cut by any plane parallel to either of their two common tangent planes, the sections will be *two similar and similarly situated ellipses.*

[For example, if the *original* ellipsoid reduce itself to a *sphere*, then the two points of contact, B and B′, become two of the four *umbilics on the inscribed ellipsoid.*]

VI. The closing chords PP_{2m} are also tangents to a certain series or *system of curves* (C′), not generally plane, on the surface of the inscribed ellipsoid (E′); and therefore may be arranged into a *system of developable surfaces*, (D′), of which these curves (C′) are the *arêtes de rebroussement.*

VII. The same closing chords may also be arranged into a *second system of developable surfaces*, (D$''$), which *envelope the inscribed ellipsoid* (E$'$) and have *their arêtes de rebroussement* (C$''$) all situated on a certain *other surface* (E$''$), which is, in its turn, enveloped by the *first* set of developable surfaces (D$'$); so that *the closing chords* PP_{2m} *are all tangents to a second set of curves,* (C$''$)*, and to a second surface,* (E$''$).

VIII. This second surface (E$''$) is a *hyperboloid of two sheets*, having *double contact* with the given ellipsoid (E), and *also* with the inscribed ellipsoid (E$'$), at the points B and B$'$; one sheet having external contact with each ellipsoid at one of those two points, and the other at the other.

IX. If either sheet of this hyperboloid (E$''$) be cut by a plane parallel to either of the two common tangent planes, *the elliptic section of the sheet is similar to a parallel section of either ellipsoid, and is similarly situated therewith.*

[For example, the points of contact B and B$'$ are two of the *umbilics of the hyperboloid* (E$''$), when the given surface (E) is a *sphere*.]

X. The *centres* of the three surfaces, (E) (E$'$) (E$''$), are situated *on one straight line.*

XI. The two systems of developable surfaces, cut the original ellipsoid, (E), in *two new series of curves*, (F$'$), (F$''$), not generally plane, which everywhere so cross each other on (E), that at any one such point of crossing, P, *the tangents to the two curves* (F$'$) (F$''$) *are parallel to two conjugate semidiameters* of the surface (E) on which the curves are contained.

[For example, if the original surface (E) be a *sphere*, then these two sets of curves (F$'$) (F$''$) cross each other everywhere *at right angles*, upon that spheric surface.]

XII. *Each closing chord* PP_m *is cut harmonically*, at the two points, C$'$, C$''$, where it touches the inscribed ellipsoid (E$'$), and the exscribed hyperboloid (E$''$); or *where it touches the curves* (C$'$) and (C$''$).

XIII. The closing chords, or *the positions of the last side of the variable polygon* (P)*, are not, in general, all cut perpendicularly by any one common surface* (notwithstanding the analogy of their arrangement, or distribution in space, in many respects, to that of the normals to a surface). In fact, the two systems of developable surfaces, (D$'$) and (D$''$), are *not* generally rectangular to each other, in the arrangement *here* considered, though they *are* so for any system of normals.

XIV. *Through any given point of space*, A_{2m+1}, which is at once *exterior* to the inscribed ellipsoid (E$'$), and to *both* sheets of the exscribed hyperboloid (E$''$), it is in general possible to draw *two*, and *only two*, distinct and *real straight lines*, $\mathrm{P'P'_{2m}}$ and $\mathrm{P''P''_{2m}}$, of which *each shall touch at once a curve* (C$'$) on (E$'$), *and a curve* (C$''$) on (E$''$), and of which *each shall coincide with one of the positions of the closing chord*, PP_{2m}; in such a manner as to be *the last side of a rectilinear polygon of* $2m+1$ *sides*, $\mathrm{P'P'_1P'_2} \ldots \mathrm{P'_{2m}}$, or $\mathrm{P''P''_1P''_2} \ldots \mathrm{P''_{2m}}$, *inscribed in the given ellipsoid* (E)*, under the condition that its sides shall pass, respectively and successively, through the* $2m+1$ *given points*, $\mathrm{A_1A_2} \ldots \mathrm{A}_{2m+1}$. But if the last of these points were given *on either of the two enveloped surfaces*, (E$'$), (E$''$), the problem of such inscription would in general admit of *only one distinct solution*, obtained by drawing through the given point the tangent to the particular curve (C') or (C''), on which that point was situated. And if the last given point A_{2m+1} were situated *within* the inscribed ellipsoid (E'), or *within either sheet* of the exscribed hyperboloid (E''), the problem of the inscription of the polygon of $2m+1$ sides would then become *geometrically impossible*: though it might still be said to admit, in that case, of *two imaginary modes of solution.*

88. The writer desires to put on record, in this place, the following enunciations of one or two other theorems, out of many to which the quaternion analysis has conducted him, respecting the inscription of gauche polygons in surfaces of the second order; without *yet* entering on any fuller account of that analysis itself, than what is given or suggested in some of the preceding articles. See the Numbers of the Philosophical Magazine for August and September 1849. And in the first place he will here transcribe the memorandum of a communication, hitherto unprinted, which was sent to him, in the month last mentioned, to the Mathematical and Physical Section of the Meeting of the British Association at Birmingham.

89. Conceive that any rectilinear (but generally gauche) polygon of n sides, $\text{BB}_1\text{B}_2 \dots \text{B}_{n-1}$, has been inscribed in any surface of the second order; and that n fixed points, $\text{A}_1, \text{A}_2, \dots \text{A}_n$, *not* on that surface, have been assumed on its n successive sides, namely A_1 on BB_1, A_2 on B_1B_2, &c. Take then at pleasure any point P upon the same surface, and draw the chords $\text{PA}_1\text{P}_1, \text{P}_1\text{A}_2\text{P}_2, \dots \text{P}_{n-1}\text{A}_n\text{P}_n$, passing respectively through the n fixed points (A). Again, begin with P_n, and draw, through the same n points (A), n other successive chords, $\text{P}_n\text{A}_1\text{P}_{n+1}, \text{P}_{n+1}\text{A}_2\text{P}_{n+2}, \dots \text{P}_{2n-1}\text{A}_n\text{P}_{2n}$. Again begin with P_{2n}, and draw in like manner the n chords, $\text{P}_{2n}\text{A}_1\text{P}_{2n+1}, \text{P}_{2n+1}\text{A}_2\text{P}_{2n+2}, \dots \text{P}_{3n-1}\text{A}_n\text{P}_{3n}$. Then one or other of the two following Theorems will hold good, according as the number n is *odd* or *even.*

Theorem I. If n be *odd*, and if we draw *two tangent planes* to the surface at the points P_n, P_{2n}, meeting the two new chords, PP_{2n}, P_nP_{3n}, respectively, in two new points, R, R'; then *the three points* BRR' *shall be situated on one straight line.*

Theorem II. If n be *even*, and if we describe *two pairs of plane conics on the surface*, each conic being determined by the condition of passing through *three points* thereon, as follows: the first pair of conics passing through BPP_{2n}, and $\text{P}_n\text{P}_{2n}\text{P}_{3n}$; and the second pair through BP_nP_{3n}, and PP_nP_{2n}; it will then be possible to trace, *on the same surface, two other plane conics*, of which *the first shall touch the two conics of the first pair, at the two points* B and P_n; while *the second new conic shall touch the two conics of the second pair, at the two points* B and P_{2n}.

90. With respect to the *first* of the two theorems thus communicated, it may be noticed now, that it gives an easy mode of resolving the following *Problem*, analogous to a celebrated problem in plane conics:—To find the *two* (real or imaginary) polygons, $\text{BB}_1\text{B}_2 \dots \text{B}_{n-1}$ and $\text{B}'\text{B}'_1\text{B}'_2 \dots \text{B}'_{n-1}$, with any given *odd* number n of sides, which can be inscribed in a given *surface* of the second order, so that their n successive sides, namely $\text{BB}_1, \text{B}_1\text{B}_2, \dots$ for one polygon, and $\text{B}'\text{B}'_1, \text{B}'_1\text{B}'_2, \dots$ for the other polygon thus inscribed, shall pass respectively through n given points $\text{A}_1, \text{A}_2, \dots \text{A}_n$, which are not themselves situated upon the surface. For we have only to assume at pleasure *any* point P upon that surface, and to deduce thence the *two* non-superficial points lately called R and R', by the construction assigned in the theorem;

since by then joining the two points thus found, *the joining line* RR′ *will cut the given surface of the second order in the two* (real or imaginary) *points*, B, B′, which are adapted to be, respectively, *the first corners of the two polygons required.*—That there *are* (in general) *two* such (real or imaginary) polygons, *when the number of sides is odd*, had been previously inferred by the writer, from the quaternion analysis which he employed. Indeed, it may have been perceived to be, through geometrical deformation, a consequence of what was stated in § XIV. of article 87 of this series of papers on Quaternions, for the particular case of the ellipsoid, in the Philosophical Magazine for September 1849. See also the account, in the Proceedings of the Royal Irish Academy, of the author's communication to that body, at the meeting of June 25th, 1849; in which account, indeed, will be found (among many others) both the theorems of the preceding article 89; the second of those theorems being however there enunciated under a *metric*, rather than under a *graphic* form.

[*To be continued.*]