ON "GAUCHE" CURVES OF THE THIRD DEGREE

By

William Rowan Hamilton

(Proceedings of the Royal Irish Academy, 8 (1864), pp. 331–334.)

Edited by David R. Wilkins

2000

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[Communicated April 27th, 1863.]

[*Proceedings of the Royal Irish Academy*, vol. viii (1864), pp. 331–334.]

The following letter, addressed to the President, by Sir W. R. Hamilton, was read:—

Observatory, 27 April 1863.

My dear Mr. President,—I have been wishing for your permission to report, through you, to the Royal Irish Academy, some of the results to which I have lately arrived, while extending the applications of Quaternions, in connexion with my forthcoming *Elements*.

I. One set of such results relates to those *gauche curves of the third degree*, which appear to have been first discovered, described, and to some extent applied, by Professor Möbius, in the *Barycentric Calculus* (1827), and afterwards independently by M. Chasles, in a Note to his *Aperçu Historique sur l'Origine et le Developpement des Méthodes en Géométrie* (1837); and for which our countryman, Dr. Salmon, who has done so much for the Classification of Curves in Space, has proposed the short but expressive name of *Twisted Cubics*.

II. A particular curve of that class presented itself to me in an investigation more than ten years ago, and some account of it was given in my *Lectures*, and (I think) to the Academy also, in connexion with the problem of *Inscription of Polygons* in surfaces of the second order. I gave its vector equation, which was short, but was not sufficiently *general*, to represent *all* curves in space of the third degree: nor had I, at the time, any aim at such representation. But I have lately perceived, and printed (in the *Elements*), the strikingly simple, and yet complete equation,

$$\mathrm{V}\alpha\rho + \mathrm{V}\rho\phi\rho = 0,$$

which represents *all twisted cubics*, if only a point of the curve be taken, for convenience, as the origin: $\phi\rho$ denoting that *linear and vector function* of a vector, which has formed the subject of many former studies of mine, and α being a constant vector, while ρ is a variable one.

III. It is known that a twisted cubic can in general be so chosen, as to pass through *any six points of space*. It is therefore natural to inquire, what is the *Osculating Twisted Cubic* to a given curve of double curvature, or the one which has, at any given place, a *six-point contact* with the curve. Yet I have not hitherto been able to learn, from any book or friend, that even the *conception* of the problem of the determination of *such* an osculatrix, had occurred to any one before me. But it presented itself naturally to me lately, in the course of writing out a section on the application of quaternions to curves; and I conceive that I have completely resolved it, in *three* distinct ways, of which *two* seem to admit of being geometrically described, so as to be understood without diagrams or calculation.

IV. It is known that the *cone of chords* of a twisted cubic, having its vertex at any one point of that curve, is a *cone of the second order*, or what Dr. Salmon calls briefly a *quadric cone*. If, then, a point P of a *given curve* in space be made the vertex of a cone of chords of *that curve*, the quadric cone which has its vertex at P, and has *five-side contact* with *that cone*, must *contain* the osculating cubic sought. I have accordingly determined, by my own methods, the *cone* which is thus *one locus* for the cubic: and may mention that I find *fifth differentials* to enter into its equation, *only* through the *second differential* of the *second curvature*, of the given curve in space. *This* may perhaps have not been previously perceived, although I am aware that Mr. Cayley and Dr. Salmon, and probably others, have investigated the problem of *five-point contact* of a *plane conic* with a plane curve.

V. It is known also that *three quadric cylinders* can be described, having their generating lines parallel to the three (real or imaginary) *asymptotes* of a twisted cubic, and wholly *containing* that gauche curve. My *first method*, then, consisted in seeking the (necessarily real) *direction* of *one* such *asymptote*, for the purpose of determining a *cylinder* which, as a *second locus*, should contain the osculating cubic sought. And I found a *cubic cone*, as the locus for the generating line (or edge) of such a cylinder, through the given point P of osculation: and proved that of the *six right lines*, common to the quadric and the cubic cones, *three* were *absorbed* in the *tangent* to the given curve at P.

VI. In fact, I found that this tangent, say PT, was a *nodal side* (or ray) of the cubic cone; and that *one* of the *two tangent planes* to that cone, along that side, was the *osculating plane* to the curve, which plane also touched the quadric cone along that *common side*: while the same side was to be *counted* a *third time*, as being a line of *intersection*, namely, of the quadric cone with the *second branch* of the cubic cone, the tangent plane to which branch was found to cut the first branch, or the quadric cone, or the osculating plane to the curve, at an angle of which the trigonometric *cotangent* was equal to *half the differential of the radius of second curvature, divided by the differential of the arc* of the same given curve.

VII. It might then have been thus expected that a *cubic equation* could be assigned, of an algebraical *form*, but involving fifth differentials in its *coefficients*, which should determine the *three planes*, tangential to the curve, which are parallel to the three asymptotes of the sought twisted cubic: and then, with the help of what had been previously done, should assign the *three quadric cylinders* which wholly *contain* that cubic.

VIII. Accordingly, I succeeded, by quaternions, in forming such a cubic equation, for *curves in space* generally: and its correctness was tested, by application to the case of the *helix*, the fact of the *six-point contact* of my osculating cubic with which well-known curve admitted of a very easy and elementary verification. I had the honour of communicating an outline of my results, so far, to Dr. Hart, a few weeks ago, with a permission, or rather a request, which was acted on, that he should submit them to the inspection of Dr. Salmon.

IX. Such, then, may be said briefly to have been my *first general method* of resolving this new problem, of the determination of the twisted cubic which *osculates*, at a given point, to a given curve of double curvature. Of my *second method* it may be sufficient here to say, that it was suggested by a recollection of the expressions given by Professor Möbius, and led again to a *cubic equation*, but this time for the determination of a *coefficient*, in a development of a comparatively *algebraical kind.* For the moment I only add, that the *second method* of solution, above indicated, bore also the test of verification by the *helix*; and gave me generally *fractional expressions* for the co-ordinates of the osculating twisted cubic, which admitted, in the case of the helix, of elementary verifications.

X. Of my *third general method*, it may be sufficient at this stage of my letter to you to say, that it consists in assigning the *locus of the vertices* of all the *quadric cones*, which have *six-point contact* with a given curve in space, at a given point thereof. I find this locus to be a *ruled cubic surface*, on which the tangent PT to the curve is a *singular line*, counting as a *double line* in the intersection of the surface with any plane drawn *through it*; and such that if the same surface by cut by a plane drawn *across it*, the *plane cubic* which is the section has generally a *node*, at the point where the plane crosses that line: although this node degenerates into a *cusp*, when the cutting plane passes through the point P itself.

XI. And I find, what perhaps is a new *sort* of result in these questions, that the *intersection* of this *new cubic surface* with the former *quadric cone*, consists only of the *right line* PT itself, and of the *osculating twisted cubic* to the proposed curve in space.

XII. These are only *specimens* of *one set* (as above hinted) of recent results obtained through quaternions; but at least they may serve to mark, in some small degree, the respect and affection, to the Academy, and to yourself, with which I remain,

My dear Mr. President,
Faithfully yours,
William Rowan Hamilton

The Very Rev. Charles Graves, D. D., P.R.I.A.,
Dean of the Chapel Royal, &c.