ON FLUCTUATING FUNCTIONS

By

William Rowan Hamilton

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NOTE ON THE TEXT

The paper *On Fluctuating Functions*, by Sir William Rowan Hamilton, appeared in volume 19 of the *Transactions of the Royal Irish Academy*, published in 1843.

The following obvious typographical errors have been corrected:—

in article 8, an upper limit of integration of ∞ has been added to the integral which in the original publication was printed as $\displaystyle\int_0 d\alpha\,\frac{\sin\beta\alpha}{\alpha(1+\alpha^2)}$;

in article 13, the right hand side of equation (d$'''$) was printed in the original publication as $(\alpha - x)^{-1}\psi_{k_{-1(\alpha-x)}}$;

a full stop (period) has been inserted after equation (d^{IX}).

David R. Wilkins
Dublin, June 1999

On Fluctuating Functions. By Sir William Rowan Hamilton, *LL. D., P. R. I. A., F. R. A. S., Fellow of the American Society of Arts and Sciences, and of the Royal Northern Society of Antiquaries at Copenhagen; Honorary or Corresponding Member of the Royal Societies of Edinburgh and Dublin, of the Academies of St. Petersburgh, Berlin, and Turin, and of other Scientific Societies at home and abroad; Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

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The paper now submitted to the Royal Irish Academy is designed chiefly to invite attention to some consequences of a very fertile principle, of which indications may be found in Fourier's Theory of Heat, but which appears to have hitherto attracted little notice, and in particular seems to have been overlooked by Poisson. This principle, which may be called the *Principle of Fluctuation*, asserts (when put under its simplest form) the evanescence of the integral, taken between any finite limits, of the product formed by multiplying together any two finite functions, of which one, like the sine or cosine of an infinite multiple of an arc, changes sign infinitely often within a finite extent of the variable on which it depends, and has for its mean value zero; from which it follows, that if the other function, instead of being always finite, becomes infinite for some particular values of its variable, the integral of the product is to be found by attending only to the immediate neighbourhood of those particular values. The writer is of opinion that it is only requisite to develope the foregoing principle, in order to give a new clearness, and even a new extension, to the existing theory of the transformations of arbitrary functions through functions of determined forms. Such is, at least, the object aimed at in the following pages; to which will be found appended a few general observations on this interesting part of our knowledge.

[1.] The theorem, discovered by Fourier, that between any finite limits, a and b, of any real variable x, any arbitrary but finite and determinate function of that variable, of which the value varies gradually, may be represented thus,

$$fx = \frac{1}{\pi} \int_a^b d\alpha \int_0^\infty d\beta \, \cos(\beta\alpha - \beta x) \, f\alpha, \tag{a}$$

with many other analogous theorems, is included in the following form:

$$fx = \int_a^b d\alpha \int_0^\infty d\beta \, \phi(x, \alpha, \beta) \, f\alpha; \tag{b}$$

the function ϕ being, in each case, suitably chosen. We propose to consider some of the conditions under which a transformation of the kind (b) is valid.

[2.] If we make, for abridgment,

$$\psi(x, \alpha, \beta) = \int_0^\beta d\beta \, \phi(x, \alpha, \beta), \tag{c}$$

the equation (b) may be thus written:

$$fx = \int_a^b d\alpha \, \psi(x, \alpha, \infty) \, f\alpha. \tag{d}$$

This equation, if true, will hold good, after the change of $f\alpha$, in the second member, to $f\alpha + \mathrm{F}\alpha$; provided that, for the particular value $\alpha = x$, the additional function $\mathrm{F}\alpha$ vanishes; being also, for other values of α, between the limits a and b, determined and finite, and gradually varying in value. Let then this function F vanish, from $\alpha = a$ to $\alpha = \lambda$, and from $\alpha = \mu$ to $\alpha = b$; λ and μ being included, either between a and x, or between x and b; so that x is not included between λ and μ, though it is included between a and b. We shall have, under these conditions,

$$0 = \int_\lambda^\mu d\alpha \, \psi(x, \alpha, \infty) \, \mathrm{F}\alpha; \tag{e}$$

the function F, and the limits λ and μ, being arbitrary, except so far as has been above defined. Consequently, unless the function of α, denoted here by $\psi(x, \alpha, \infty)$, be itself $= 0$, it must change sign at least once between the limits $\alpha = \lambda$, $\alpha = \mu$, however close these limits may be; and therefore must change sign indefinitely often, between the limits a and x, or x and b. A function which thus changes sign indefinitely often, within a finite range of a variable on which it depends, may be called a *fluctuating function*. We shall consider now a class of cases, in which such a function may present itself.

[3.] Let N_α be a real function of α, continuous or discontinuous in value, but always comprised between some finite limits, so as never to be numerically greater than $\pm\mathrm{c}$, in which c is a finite constant; let

$$\mathrm{M}_\alpha = \int_0^\alpha d\alpha \, \mathrm{N}_\alpha; \tag{f}$$

and let the equation

$$\mathrm{M}_\alpha = \mathrm{a}, \tag{g}$$

in which a is some finite constant, have infinitely many real roots, extending from $-\infty$ to $+\infty$, and such that the interval $\alpha_{n+1} - \alpha_n$, between any one root α_n and the next succeeding α_{n+1}, is never greater than some finite constant, b. Then,

$$0 = \mathrm{M}_{\alpha_{n+1}} - \mathrm{M}_{\alpha_n} = \int_{\alpha_n}^{\alpha_{n+1}} d\alpha \, \mathrm{N}_\alpha; \tag{h}$$

and consequently the function N_α must change sign at least once between the limits $\alpha = \alpha_n$ and $\alpha = \alpha_{n+1}$; and therefore at least m times between the limits $\alpha = \alpha_n$ and $\alpha = \alpha_{n+m}$, this latter limit being supposed, according to the analogy of this notation, to be the m^{th} root of the equation (g), after the root α_n. Hence the function $\mathrm{N}_{\beta\alpha}$, formed from N_α by multiplying α by β, changes sign at least m times between the limits $\alpha = \lambda$, $\alpha = \mu$, if*

$$\lambda \ngtr \beta^{-1}\alpha_n, \quad \mu \nless \beta^{-1}\alpha_{n+m};$$

the interval $\mu - \lambda$ between these limits being less than $\beta^{-1}(m+2)\mathrm{b}$, if

$$\lambda > \beta^{-1}\alpha_{n-1}, \quad \mu < \beta^{-1}\alpha_{n+m+1};$$

so that, under these conditions, (β being > 0,) we have

$$m > -2 + \beta\mathrm{b}^{-1}(\mu - \lambda).$$

However small, therefore, the interval $\mu - \lambda$ may be, provided that it be greater than 0, the number of changes of sign of the function $\mathrm{N}_{\beta\alpha}$, within this range of the variable α, will increase indefinitely with β. Passing then to the extreme or limiting supposition, $\beta = \infty$, we may say that the function $\mathrm{N}_{\infty\alpha}$ *changes sign infinitely often* within a finite range of the variable α on which it depends; and consequently that it is, in the sense of the last article, a FLUCTUATING FUNCTION. We shall next consider the integral of the product formed by multiplying together two functions of α, of which one is $\mathrm{N}_{\infty\alpha}$, and the other is arbitrary, but finite, and shall see that this integral vanishes.

[4.] It has been seen that the function N_α changes sign at least once between the limits $\alpha = \alpha_n$, $\alpha = \alpha_{n+1}$. Let it then change sign k times between those limits, and let the k corresponding values of α be denoted by $\alpha_{n,1}, \alpha_{n,2}, \dots \alpha_{n,k}$. Since the function N_α may be discontinuous in value, it will not necessarily vanish for these k values of α; but at least it will have one constant sign, being throughout not < 0, or else throughout not > 0, in the interval from $\alpha = \alpha_n$ to $\alpha = \alpha_{n,1}$; it will be, on the contrary, throughout not > 0, or throughout not < 0, from $\alpha_{n,1}$ to $\alpha_{n,2}$; again, not < 0, or not > 0, from $\alpha_{n,2}$ to $\alpha_{n,3}$; and so on. Let then N_α be never < 0 throughout the whole of the interval from $\alpha_{n,i}$ to $\alpha_{n,i+1}$; and let it be > 0 for at least some finite part of that interval; i being some integer number between the limits 0 and k, or even one of those limits themselves, provided that the symbols $\alpha_{n,0}$, $\alpha_{n,k+1}$ are understood to denote the same quantities as α_n, α_{n+1}. Let F_α be a finite function of α, which receives no sudden change of value, at least for that extent of the variable α, for which this function is to be employed; and let us consider the integral

$$\int_{\alpha_{n,i}}^{\alpha_{n,i+1}} d\alpha\, \mathrm{N}_\alpha \mathrm{F}_\alpha. \tag{i}$$

Let $F^{\backprime}$ be the algebraically least, and $F^{\backprime\backprime}$ the algebraically greatest value of the function F_α, between the limits of integration; so that, for every value of α between these limits, we shall have

$$\mathrm{F}_\alpha - \mathrm{F}^{\backprime} \nless 0, \quad \mathrm{F}^{\backprime\backprime} - \mathrm{F}_\alpha \nless 0;$$

* These notations $\ngtr$ and $\nless$ are designed to signify the contradictories of $>$ and $<$; so that "$a \ngtr b$" is equivalent to "a not $> b$," and "$a \nless b$" is equivalent to "a not $< b$."

these values $\mathrm{F}^{\backprime}$ and $\mathrm{F}^{\backprime\backprime}$, of the function F_α, corresponding to some values $\alpha^{\backprime}_{n,i}$ and $\alpha^{\backprime\backprime}_{n,i}$ of the variable α, which are not outside the limits $\alpha_{n,i}$ and $\alpha_{n,i+1}$. Then, since, between these latter limits, we have also

$$\mathrm{N}_\alpha \not< 0,$$

we shall have

$$\left.\begin{aligned}\int_{\alpha_{n,i}}^{\alpha_{n,i+1}} d\alpha\, \mathrm{N}_\alpha(\mathrm{F}_\alpha - \mathrm{F}^{\backprime}) \not< 0;\\ \int_{\alpha_{n,i}}^{\alpha_{n,i+1}} d\alpha\, \mathrm{N}_\alpha(\mathrm{F}^{\backprime\backprime} - \mathrm{F}_\alpha) \not< 0;\end{aligned}\right\} \tag{k}$$

the integral (i) will therefore be not $< s_{n,i}\mathrm{F}^{\backprime}$, and not $> s_{n,i}\mathrm{F}^{\backprime\backprime}$, if we put, for abridgment,

$$s_{n,i} = \int_{\alpha_{n,i}}^{\alpha_{n,i+1}} d\alpha\, \mathrm{N}_\alpha; \tag{l}$$

and consequently this integral (i) may be represented by $s_{n,i}\mathrm{F}'$, in which

$$\mathrm{F}' \not< \mathrm{F}^{\backprime}, \quad \mathrm{F}' \not> \mathrm{F}^{\backprime\backprime},$$

because, with the suppositions already made, $s_{n,i} > 0$. We may even write

$$\mathrm{F}' > \mathrm{F}^{\backprime}, \quad \mathrm{F}' < \mathrm{F}^{\backprime\backprime},$$

unless it happen that the function F_α has a constant value through the whole extent of the integration; or else that it is equal to one of its extreme values, $\mathrm{F}^{\backprime}$ or $\mathrm{F}^{\backprime\backprime}$, throughout a finite part of that extent, while, for the remaining part of the same extent, that is, for all other values of α between the same limits, the factor N_α vanishes. In all these cases, F' may be considered as a value of the function F_α, corresponding to a value $\alpha'_{n,i}$ of the variable α which is included between the limits of integration; so that we may express the integral (i) as follows:

$$\int_{\alpha_{n,i}}^{\alpha_{n,i+1}} d\alpha\, \mathrm{N}_\alpha \mathrm{F}_\alpha = s_{n,i}\mathrm{F}_{\alpha'_{n,i}}; \tag{m}$$

in which

$$\alpha'_{n,i} > \alpha_{n,i}, \quad < \alpha_{n,i+1}. \tag{n}$$

In like manner, the expression (m), with the inequalities (n), may be proved to hold good, if N_α be never > 0, and sometimes < 0, within the extent of the integration, the integral $s_{n,i}$ being in this case < 0; we have, therefore, rigorously,

$$\int_{\alpha_n}^{\alpha_{n+1}} d\alpha\, \mathrm{N}_\alpha \mathrm{F}_\alpha = s_{n,0}\mathrm{F}_{\alpha'_{n,0}} + s_{n,1}\mathrm{F}_{\alpha'_{n,1}} + \cdots + +s_{n,k}\mathrm{F}_{\alpha'_{n,k}}. \tag{o}$$

But also, we have, by (h)

$$0 = s_{n,0} + s_{n,1} + \cdots + s_{n,k}; \tag{p}$$

the integral in (o) may therefore be thus expressed, without any loss of rigour:

$$\int_{\alpha_n}^{\alpha_{n+1}} d\alpha\, \mathrm{N}_\alpha \mathrm{F}_\alpha = s_{n,0}\Delta_{n,0} + \cdots + s_{n,k}\Delta_{n,k}, \tag{q}$$

in which

$$\Delta_{n,i} = \mathrm{F}_{\alpha'_{n,i}} - \mathrm{F}_{\alpha_n}; \tag{r}$$

so that $\Delta_{n,i}$ is a finite difference of the function F_α, corresponding to the finite difference $\alpha'_{n,i} - \alpha_n$ of the variable α, which latter difference is less than $\alpha_{n+1} - \alpha_n$, and therefore less than the finite constant b of the last article. The theorem (q) conducts immediately to the following,

$$\int_{\beta^{-1}\alpha_n}^{\beta^{-1}\alpha_{n+1}} d\alpha\, \mathrm{N}_{\beta\alpha} \mathrm{F}_\alpha = \beta^{-1}(s_{n,0}\delta_{n,0} + \cdots + s_{n,k}\delta_{n,k}), \tag{s}$$

in which

$$\delta_{n,i} = \mathrm{F}_{\beta^{-1}\alpha'_{n,i}} - \mathrm{F}_{\beta^{-1}\alpha_n}; \tag{t}$$

so that, if β be large, $\delta_{n,i}$ is small, being the difference of the function F_α corresponding to a difference of the variable α, which latter difference is less than β^{-1}b. Let $\pm\delta_n$ be the greatest of the $k+1$ differences $\delta_{n,0}, \ldots \delta_{n,k}$, or let it be equal to one of those differences and not exceeded by any other, abstraction being made of sign; then, since the $k+1$ factors $s_{n,0}, \ldots s_{n,k}$ are alternately positive and negative, or negative and positive, the numerical value of the integral (s) cannot exceed that of the expression

$$\pm\beta^{-1}(s_{n,0} - s_{n,1} + s_{n,2} - \cdots + (-1)^k s_{n,k})\delta_n. \tag{u}$$

But, by the definition (l) of $s_{n,i}$, and by the limits $\pm$c of value of the finite function N_α, we have

$$\pm s_{n,i} \ \ngtr (\alpha_{n,i+1} - \alpha_{n,i})\mathrm{c}; \tag{v}$$

therefore

$$\pm(s_{n,0} - s_{n,1} + \cdots + (-1)^k s_{n,k}) \ \ngtr (\alpha_{n+1} - \alpha_n)\mathrm{c}; \tag{w}$$

and the following rigorous expression for the integral (s) results:

$$\int_{\beta^{-1}\alpha_n}^{\beta^{-1}\alpha_{n+1}} d\alpha\, \mathrm{N}_{\beta\alpha} \mathrm{F}_\alpha = \theta_n \beta^{-1}(\alpha_{n+1} - \alpha_n)\mathrm{c}\delta_n; \tag{x}$$

θ_n being a factor which cannot exceed the limits ± 1. Hence, if we change successively n to $n+1$, $n+2, \ldots n+m-1$, and add together all the results, we obtain this other rigorous expression, for the integral of the product $\mathrm{N}_{\beta\alpha}\mathrm{F}_\alpha$, extended from $\alpha = \beta^{-1}\alpha_n$ to $\alpha = \beta^{-1}\alpha_{n+m}$:

$$\int_{\beta^{-1}\alpha_n}^{\beta^{-1}\alpha_{n+m}} d\alpha\, \mathrm{N}_{\beta\alpha} \mathrm{F}_\alpha = \theta \beta^{-1}(\alpha_{n+m} - \alpha_n)\mathrm{c}\delta; \tag{y}$$

in which δ is the greatest of the m quantities $\delta_n, \delta_{n+1}, \ldots$, or is equal to one of those quantities, and is not exceeded by any other; and θ cannot exceed ± 1. By taking β sufficiently large, and suitably choosing the indices n and $n+m$, we may make the limits of integration in the formula (y) approach as nearly as we please to any given finite values, a and b; while, in the second member of that formula, the factor $\beta^{-1}(\alpha_{n+m}-\alpha_n)$ will tend to become the finite quantity $b-a$, and $\theta\mathrm{c}$ cannot exceed the finite limits $\pm\mathrm{c}$; but the remaining factor δ will tend indefinitely to 0, as β increases without limit, because it is the difference between two values of the function F_α, corresponding to two values of the variable α of which the difference diminishes indefinitely. Passing then to the limit $\beta=\infty$, we have, with the same rigour as before:

$$\int_a^b d\alpha\,\mathrm{N}_{\infty\alpha}\mathrm{F}_\alpha = 0; \tag{z}$$

which is the theorem that was announced at the end of the preceding article. And although it has been here supposed that the function F_α receives no sudden change of value, between the limits of integration; yet we see that if this function receive any finite number of such sudden changes between those limits, but vary gradually in value between any two such changes, the foregoing demonstration may be applied to each interval of gradual variation of value separately; and the theorem (z) will still hold good.

[5.] This theorem (z) may be thus written:

$$\lim_{\beta=\infty}\int_a^b d\alpha\,\mathrm{N}_{\beta\alpha}\mathrm{F}_\alpha = 0; \tag{a$'$}$$

and we may easily deduce from it the following:

$$\lim_{\beta=\infty}\int_a^b d\alpha\,\mathrm{N}_{\beta(\alpha-x)}\mathrm{F}_\alpha = 0; \tag{b$'$}$$

the function F_α being here also finite, within the extent of the integration, and x being independent of α and β. For the reasonings of the last article may easily be adapted to this case; or we may see, from the definitions in article [3.], that if the function N_α have the properties there supposed, then $\mathrm{N}_{\alpha-x}$ will also have those properties. In fact, if N_α be always comprised between given finite limits, then $\mathrm{N}_{\alpha-x}$ will be so too; and we shall have, by (f),

$$\int_0^\alpha d\alpha\,\mathrm{N}_{\alpha-x} = \int_{-x}^{\alpha-x} d\alpha\,\mathrm{N}_\alpha = \mathrm{M}_{\alpha-x}-\mathrm{M}_{-x}; \tag{c$'$}$$

in which M_{-x} is finite, because the suppositions of the third article oblige M_α to be always comprised between the limits $\mathrm{a}\pm\mathrm{bc}$; so that the equation

$$\int_0^\alpha d\alpha\,\mathrm{N}_{\alpha-x} = \mathrm{a}-\mathrm{M}_{-x}, \tag{d$'$}$$

which is of the form (g), has infinitely many real roots, of the form

$$\alpha = x+\alpha_n, \tag{e$'$}$$

and therefore of the kind assumed in the two last articles. Let us now examine what happens, when, in the first member of the formula (b′), we substitute, instead of the finite factor F_α, an expression such as $(\alpha - x)^{-1} f_\alpha$, which becomes infinite between the limits of integration, the value of x being supposed to be comprised between those limits, and the function f_α being finite between them. That is, let us inquire whether the integral

$$\int_a^b d\alpha\, \mathrm{N}_{\beta(\alpha-x)} (\alpha - x)^{-1} f_\alpha, \tag{f′}$$

(in which $x > a$, $< b$), tends to any and to what finite and determined limit, as β tends to become infinite.

In this inquiry, the theorem (b′) shows that we need only attend to those values of α which are extremely near to x, and are for example comprised between the limits $x \mp \epsilon$, the quantity ϵ being small. To simplify the question, we shall suppose that for such values of α, the function f_α varies gradually in value; we shall also suppose that $\mathrm{N}_0 = 0$, and that $\mathrm{N}_\alpha \alpha^{-1}$ tends to a finite limit as α tends to 0, whether this be by decreasing or by increasing; although the limit thus obtained, for the case of infinitely small and positive values of α, may possibly differ from that which corresponds to the case of infinitely small and negative values of that variable, on account of the discontinuity which the function N_α may have. We are then to investigate, with the help of these suppositions, the value of the double limit:

$$\lim_{\epsilon=0} . \lim_{\beta=\infty} . \int_{x-\epsilon}^{x+\epsilon} d\alpha\, \mathrm{N}_{\beta(\alpha-x)} (\alpha - x)^{-1} f_\alpha; \tag{g′}$$

this notation being designed to suggest, that we are first to assume a small but not evanescent value of ϵ, and a large but not infinite value of β, and to effect the integration, or conceive it effected, with these assumptions; then, retaining the same value of ϵ, make β larger and larger without limit; and then at last suppose ϵ to tend to 0, unless the result corresponding to an infinite value of β shall be found to be independent of ϵ. Or, introducing two new quantites y and η, determined by the definitions

$$y = \beta(\alpha - x), \quad \eta = \beta\epsilon, \tag{h′}$$

and eliminating α and β by means of these, we are led to seek the value of the double limit following:

$$\lim_{\epsilon=0} . \lim_{\eta=\infty} . \int_{-\eta}^{\eta} dy\, \mathrm{N}_y y^{-1} f_{x+\epsilon\eta^{-1}y}; \tag{i′}$$

in which η tends to ∞, before ϵ tends to 0. It is natural to conclude that since the sought limit (g′) can be expressed under the form (i′), it must be equivalent to the product

$$f_x \times \int_{-\infty}^{\infty} dy\, \mathrm{N}_y y^{-1}; \tag{k′}$$

and in fact it will be found that this equivalence holds good; but before finally adopting this conclusion, it is proper to consider in detail some difficulties which may present themselves.

[6.] Decomposing the function $f_{x+\epsilon\eta^{-1}y}$ into two parts, of which one is independent of y, and is $= f_x$, while the other part varies with y, although slowly, and vanishes with that variable; it is clear that the formula (i′) will be decomposed into two corresponding parts, of which the first conducts immediately to the expression (k′); and we are now to inquire whether the integral in this expression has a finite and determinate value. Admitting the suppositions made in the last article, the integral

$$\int_{-\zeta}^{\zeta} dy\,\mathrm{N}_y y^{-1}$$

will have a finite and determinate value, if ζ be finite and determinate; we are therefore conducted to inquire whether the integrals

$$\int_{-\infty}^{-\zeta} dy\,\mathrm{N}_y y^{-1}, \quad \int_{\zeta}^{\infty} dy\,\mathrm{N}_y y^{-1},$$

are also finite and determinate. The reasonings which we shall employ for the second of these integrals, will also apply to the first; and to generalize a little the question to which we are thus conducted, we shall consider the integral

$$\int_{a}^{\infty} d\alpha\,\mathrm{N}_\alpha \mathrm{F}_\alpha; \tag{l′}$$

F_α being here supposed to denote any function of α which remains always positive and finite, but decreases continually and gradually in value, and tends indefinitely towards 0, while α increases indefinitely from some given finite value which is not greater than a. Applying to this integral (l′) the principles of the fourth article, and observing that we have now $\mathrm{F}_{\alpha'_{n,i}} < \mathrm{F}_{\alpha_n}$, $\alpha'_{n,i}$ being $> \alpha_n$, and α_n being assumed $\not< a$; and also that

$$\pm(s_{n,0} + s_{n,2} + \cdots) = \mp(s_{n,1} + s_{n,3} + \cdots) \not> \tfrac{1}{2}\mathrm{bc}; \tag{m′}$$

we find

$$\pm \int_{\alpha_n}^{\alpha_{n+1}} d\alpha\,\mathrm{N}_\alpha \mathrm{F}_\alpha < \tfrac{1}{2}\mathrm{bc}(\mathrm{F}_{\alpha_n} - \mathrm{F}_{\alpha_{n+1}}); \tag{n′}$$

and consequently

$$\pm \int_{\alpha_n}^{\alpha_{n+m}} d\alpha\,\mathrm{N}_\alpha \mathrm{F}_\alpha < \tfrac{1}{2}\mathrm{bc}(\mathrm{F}_{\alpha_n} - \mathrm{F}_{\alpha_{n+m}}). \tag{o′}$$

This latter integral is therefore finite and numerically less than $\frac{1}{2}\mathrm{bc}\,\mathrm{F}_{\alpha_n}$, however great the upper limit α_{n+m} may be; it tends also to a determined value as m increases indefinitely, because the part which corresponds to values of α between any given value of the form α_{n+m} and any other of the form α_{n+m+p} is included between the limits $\pm\frac{1}{2}\mathrm{bc}\,\mathrm{F}_{\alpha_{n+m}}$, which limits approach indefinitely to each other and to 0, as m increases indefinitely. And in the integral (l′), if we suppose the lower limit of a to lie between α_{n-1} and α_n, while the upper limit, instead of being infinite, is at first assumed to be a large but finite quantity b, lying between

α_{n+m} and α_{n+m+1}, we shall only thereby add to the integral (o′) two parts, an initial and a final, of which the first is evidently finite and determinate, while the second is easily proved to tend indefinitely to 0 as m increases without limit. The integral (l′) is therefore itself finite and determined, under the conditions above supposed, which are satisfied, for example, by the function $\mathrm{F}_\alpha = \alpha^{-1}$, if a be > 0. And since the suppositions of the last article render also the integral

$$\int_0^a d\alpha\, \mathrm{N}_\alpha \alpha^{-1}$$

determined and finite, if the value of a be such, we see that with these suppositions we may write

$$\varpi' = \int_0^\infty d\alpha\, \mathrm{N}_\alpha \alpha^{-1}, \tag{p′}$$

ϖ' being itself a finite and determined quantity. By reasonings almost the same we are led to the analogous formula

$$\varpi'' = \int_{-\infty}^0 d\alpha\, \mathrm{N}_\alpha \alpha^{-1}; \tag{q′}$$

and finally to the result

$$\varpi = \varpi' + \varpi'' = \int_{-\infty}^\infty d\alpha\, \mathrm{N}_\alpha \alpha^{-1}; \tag{r′}$$

in which ϖ'' and ϖ are also finite and determined. The product (k′) is therefore itself determinate and finite, and may be represented by ϖf_x.

[7.] We are next to introduce, in (i′), the variable part of the function f, namely

$$f_{x+\epsilon\eta^{-1}y} - f_x,$$

which varies from $f_{x-\epsilon}$ to $f_{x+\epsilon}$, while y varies from $-\eta$ to $+\eta$, and in which ϵ may be any quantity > 0. And since it is clear, that under the conditions assumed in the fifth article,

$$\lim_{\epsilon=0} . \lim_{\eta=\infty} . \int_{-\zeta}^{\zeta} dy\, \mathrm{N}_y y^{-1} (f_{x+\epsilon\eta^{-1}y} - f_x) = 0, \tag{s′}$$

if ζ be any finite and determined quantity, however large, we are conducted to examine whether this double limit vanishes when the integration is made to extend from $y = \zeta$ to $y = \eta$. It is permitted to suppose that f_α continually increases, or continually decreases, from $\alpha = x$ to $\alpha = x + \epsilon$; let us therefore consider the integral

$$\int_\zeta^\eta d\alpha\, \mathrm{N}_\alpha \mathrm{F}_\alpha \mathrm{G}_\alpha, \tag{t′}$$

in which the function F_α decreases, while G_α increases, but both are positive and finite, within the extent of the integration.

By reasonings similar to those of the fourth article, we find under these conditions,

$$\pm \int_{\alpha_n}^{\alpha_{n+1}} d\alpha \, \mathrm{N}_\alpha \mathrm{F}_\alpha \mathrm{G}_\alpha < \mathrm{bc}(\mathrm{F}_{\alpha_n} \mathrm{G}_{\alpha_{n+1}} - \mathrm{F}_{\alpha_{n+1}} \mathrm{G}_{\alpha_n}); \tag{u$'$}$$

and therefore

$$\left. \begin{aligned} \pm\frac{1}{\mathrm{bc}} \int_{\alpha_n}^{\alpha_{n+m}} d\alpha \, \mathrm{N}_\alpha \mathrm{F}_\alpha \mathrm{G}_\alpha < \mathrm{F}_{\alpha_{n+m-1}} \mathrm{G}_{\alpha_{n+m}} - \mathrm{F}_{\alpha_{n+1}} \mathrm{G}_{\alpha_n} & \\ +(\mathrm{F}_{\alpha_n} - \mathrm{F}_{\alpha_{n+2}}) \mathrm{G}_{\alpha_{n+1}} + (\mathrm{F}_{\alpha_{n+2}} - \mathrm{F}_{\alpha_{n+4}}) \mathrm{G}_{\alpha_{n+3}} + \&\mathrm{c}. & \\ +(\mathrm{F}_{\alpha_{n+1}} - \mathrm{F}_{\alpha_{n+3}}) \mathrm{G}_{\alpha_{n+2}} + (\mathrm{F}_{\alpha_{n+3}} - \mathrm{F}_{\alpha_{n+5}}) \mathrm{G}_{\alpha_{n+4}} + \&\mathrm{c}. & \end{aligned} \right\} \tag{v$'$}$$

This inequality will still subsist, if we increase the second member by changing, in the positive products on the second and third lines, the factors G to their greatest value $\mathrm{G}_{\alpha_{n+m}}$; and, after adding the results, suppress the three negative terms which remain in the three lines of these expression, and change the functions F, in the first and third lines, to their greatest value F_{α_n}. Hence,

$$\pm \int_{\alpha_n}^{\alpha_{n+m}} d\alpha \, \mathrm{N}_\alpha \mathrm{F}_\alpha \mathrm{G}_\alpha < 3\mathrm{bc} \, \mathrm{F}_{\alpha_n} \mathrm{G}_{\alpha_{n+m}}; \tag{w$'$}$$

this integral will therefore ultimately vanish, if the product of the greatest values of the functions F and G tend to the limit 0. Thus, if we make

$$\mathrm{F}_\alpha = \alpha^{-1}, \quad \mathrm{G}_\alpha = \pm(f_{x+\epsilon\eta^{-1}\alpha} - f_x),$$

the upper sign being taken when f_α increases from $\alpha = x$ to $\alpha = x + \epsilon$; and if we suppose that ζ and η are of the forms α_n and α_{n+m}; we see that the integral (t$'$) is numerically less than $3\mathrm{bc}\,\alpha_n^{-1}(f_{x+\epsilon} - f_x)$, and therefore that it vanishes at the limit $\epsilon = 0$. It is easy to see that the same conclusion holds good, when we suppose that η does not coincide with any quantity of the form α_{n+m}, and where the limits of integration are changed to $-\eta$ and $-\zeta$. We have therefore, rigorously,

$$\lim_{\epsilon=0} . \lim_{\eta=\infty} . \int_{-\eta}^{\eta} dy \, \mathrm{N}_y y^{-1}(f_{x+\epsilon\eta^{-1}y} - f_x) = 0, \tag{x$'$}$$

nowithstanding the great and ultimately infinite extent over which the integration is conducted. The variable part of the function f may therefore be suppressed in the double limit (i$'$), without any loss of accuracy; and that limit is found to be exactly equal to the expression (k$'$); that is, by the last article, to the determined product ϖf_x. Such, therefore, is the value of the limit (g$'$), from which (i$'$) was derived by the transformation (h$'$); and such finally is the limit of the integral (f$'$), proposed for investigation in the fifth article. We have, then, proved that under the conditions of that article,

$$\lim_{\beta=\infty} . \int_a^b d\alpha \, \mathrm{N}_{\beta(\alpha-x)}(\alpha - x)^{-1} f_\alpha = \varpi f_x; \tag{y$'$}$$

and consequently that the arbitrary but finite and gradually varying function f_x, between the limits $x = a$, $x = b$, may be transformed as follows:

$$f_x = \varpi^{-1} \int_a^b d\alpha \, \mathrm{N}_{\infty(\alpha - x)} (\alpha - x)^{-1} f_\alpha; \tag{z$'$}$$

which is a result of the kind denoted by (d) in the second article, and includes the theorem (a) of FOURIER. For all the suppositions made in the foregoing articles, respecting the form of the function N, are satisfied by assuming this function to be the sine of the variable on which it depends; and then the constant ϖ, determined by the formula (r′), becomes coincident with π, that is, with the ratio of the circumference to the diameter of a circle, or with the least positive root of the equation

$$\frac{\sin x}{x} = 0.$$

[8.] The known theorem just alluded to, namely, that the definite integral (r′) becomes $= \pi$, when $\mathrm{N}_\alpha = \sin \alpha$, may be demonstrated in the following manner. Let

$$\mathrm{A} = \int_0^\infty d\alpha \, \frac{\sin \beta\alpha}{\alpha};$$

$$\mathrm{B} = \int_0^\infty d\alpha \, \frac{\cos \beta\alpha}{1 + \alpha^2};$$

then these two definite integrals are connected with each other by the relation

$$\mathrm{A} = \left(\int_0^\beta d\beta - \frac{d}{d\beta} \right) \mathrm{B},$$

because

$$\int_0^\beta d\beta \, \mathrm{B} = \int_0^\infty d\alpha \, \frac{\sin \beta\alpha}{\alpha(1 + \alpha^2)},$$

$$-\frac{d}{d\beta} \mathrm{B} = \int_0^\infty d\alpha \, \frac{\alpha \sin \beta\alpha}{1 + \alpha^2};$$

and all these integrals, by the principles of the foregoing articles, receive determined and finite (that is, not infinite) values, whatever finite or infinite value may be assigned to β. But for all values of $\beta > 0$, the value of A is constant; therefore, for all such values of β, the relation between A and B gives, by integration,

$$e^{-\beta} \left\{ \left(\int_0^\beta d\beta + 1 \right) \mathrm{B} - \mathrm{A} \right\} = \text{const.};$$

and this constant must be $= 0$, because the factor of $e^{-\beta}$ does not tend to become infinite with β. That factor is therefore itself $= 0$, so that we have

$$\mathrm{A} = \left(\int_0^\beta d\beta + 1 \right) \mathrm{B}, \text{ if } \beta > 0.$$

Comparing the two expressions for A, we find

$$\mathrm{B} + \frac{d}{d\beta}\mathrm{B} = 0,\ \text{if } \beta > 0;$$

and therefore, for all such values of β,

$$\mathrm{B}e^{\beta} = \text{const.}$$

The constant in this last result is easily proved to be equal to the quantity A, by either of the two expressions already established for that quantity; we have therefore

$$\mathrm{B} = \mathrm{A}e^{-\beta},$$

however little the value of β may exceed 0; and because B tends to the limit $\frac{\pi}{2}$ as β tends to 0, we find finally, for all values of β greater than 0,

$$\mathrm{A} = \frac{\pi}{2}, \quad \mathrm{B} = \frac{\pi}{2}e^{-\beta}.$$

These values, and the result

$$\int_{-\infty}^{\infty} d\alpha\, \frac{\sin\alpha}{\alpha} = \pi,$$

to which they immediately conduct, have long been known; and the first relation, above mentioned, between the integrals A and B, has been employed by LEGENDRE to deduce the former integral from the latter; but it seemed worth while to indicate a process by which that relation may be made to conduct to the values of both those integrals, without the necessity of expressly considering the second differential coefficient of B relative to β, which coefficient presents itself at first under an indeterminate form.

[9.] The connexion of the formula (z′) with FOURIER'S theorem (a), will be more distinctly seen, if we introduce a new function P_α defined by the condition

$$\mathrm{N}_\alpha = \int_0^\alpha d\alpha\, \mathrm{P}_\alpha, \tag{a″}$$

which is consistent with the suppositions already made respecting the function N_α. According to those suppositions the new function P_α is not necessarily continuous, nor even always finite, since its integral N_α may be discontinuous; but P_α is supposed to be finite for small values of α, in order that N_α may vary gradually for such values, and may bear a finite ratio to α. The value of the first integral of P_α is supposed to be always comprised between given finite limits, so as never to be numerically greater than ±c; and the second integral,

$$\mathrm{M}_\alpha = \left(\int_0^\alpha d\alpha\right)^2 \mathrm{P}_\alpha, \tag{b″}$$

becomes infinitely often equal to a given constant, a, for values of α which extend from negative to positive infinity, and are such that the interval between any one and the next following is never greater than a given finite constant, b. With these suppositions respecting the otherwise arbitrary function P_α, the theorems (z) and (z′) may be expressed as follows:

$$\lim_{\beta=\infty} . \int_a^b d\alpha \left(\int_0^{\beta\alpha} d\gamma\, \mathrm{P}_\gamma \right) f_\alpha = 0; \tag{A}$$

and

$$f_x = \varpi^{-1} \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-x)} f_\alpha; \quad (x > a, \ < b) \tag{B}$$

ϖ being determined by the equation

$$\varpi = \int_{-\infty}^\infty d\alpha \int_0^1 d\beta\, \mathrm{P}_{\beta\alpha}. \tag{c$''$}$$

Now, by making

$$\mathrm{P}_\alpha = \cos\alpha,$$

(a supposition which satisfies all the conditions above assumed), we find, as before

$$\varpi = \pi,$$

and the theorem (B) reduces itself to the less general formula (a), so that it includes the theorem of FOURIER.

[10.] If we suppose that x coincides with one of the limits, a or b, instead of being included between them, we find easily, by the foregoing analysis,

$$f_a = \varpi^{\backprime -1} \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-a)} f_\alpha; \tag{d$''$}$$

$$f_b = \varpi^{\backprime\backprime -1} \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-b)} f_\alpha; \tag{e$''$}$$

in which

$$\varpi^\backprime = \int_0^\infty d\alpha \int_0^1 d\beta\, \mathrm{P}_{\beta\alpha}; \tag{f$''$}$$

$$\varpi^{\backprime\backprime} = \int_{-\infty}^0 d\alpha \int_0^1 d\beta\, \mathrm{P}_{\beta\alpha}; \tag{g$''$}$$

so that, as before,

$$\varpi = \varpi^\backprime + \varpi^{\backprime\backprime}.$$

Finally, when x is outside the limits a and b, the double integral in (B) vanishes; so that

$$0 = \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-x)} f_\alpha, \text{ if } x < a \text{ or } > b. \tag{h$''$}$$

And the foregoing theorems will still hold good, if the function f_α receive any number of sudden changes of value, between the limits of integration, provided that it remain finite between them; except that for those very values $\alpha^{\backprime}$ of the variable α, for which the finite function f_α receives any such sudden variation, so as to become $= f^{\backprime}$ for values of α infinitely little greater than $\alpha^{\backprime}$, after having been $= f^{\backprime\backprime}$ for values infinitely little less than $\alpha^{\backprime}$, we shall have, instead of (B), the formula

$$\omega^{\backprime} f^{\backprime} + \omega^{\backprime\backprime} f^{\backprime\backprime} = \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-\alpha^{\backprime})} f_\alpha. \tag{i$''$}$$

[11.] If P_α be not only finite for small values of α, but also vary gradually for such values, then, whether α be positive or negative, we shall have

$$\lim_{\alpha=0} .\mathrm{N}_\alpha \alpha^{-1} = \mathrm{P}_0; \tag{k$''$}$$

and if the equation

$$\mathrm{N}_{\alpha-x} = 0 \tag{l$''$}$$

have no real root α, except the root $\alpha = x$, between the limits a and b, nor any which coincides with either of those limits, then we may change f_α to $\dfrac{(\alpha-x)\mathrm{P}_0}{\mathrm{N}_{\alpha-x}} f_\alpha$, in the formula (z$'$), and we shall have the expression:

$$f_x = \varpi^{-1} \mathrm{P}_0 \int_a^b d\alpha\, \mathrm{N}_{\infty(\alpha-x)} \mathrm{N}_{\alpha-x}^{-1} f_\alpha. \tag{m$''$}$$

Instead of the infinite factor in the index, we may substitute any large number, for example, an uneven integer, and take the limit with respect to it; we may, therefore, write

$$f_x = \varpi^{-1} \mathrm{P}_0 \lim_{n=\infty} \int_a^b d\alpha \frac{\displaystyle\int_0^{(2n+1)(\alpha-x)} d\alpha\, \mathrm{P}_\alpha}{\displaystyle\int_0^{\alpha-x} d\alpha\, \mathrm{P}_\alpha} f_\alpha. \tag{n$''$}$$

Let

$$\int_{(2n-1)\alpha}^{(2n+1)\alpha} d\alpha\, \mathrm{P}_\alpha = \mathrm{Q}_{\alpha,n} \int_0^\alpha d\alpha\, \mathrm{P}_\alpha; \tag{o$''$}$$

then

$$1 + \mathrm{Q}_{\alpha,1} + \mathrm{Q}_{\alpha,2} + \cdots + \mathrm{Q}_{\alpha,n} = \frac{\displaystyle\int_0^{(2n+1)\alpha} d\alpha\, \mathrm{P}_\alpha}{\displaystyle\int_0^{\alpha} d\alpha\, \mathrm{P}_\alpha}, \tag{p$''$}$$

and the formula (n″) becomes

$$f_x = \varpi^{-1}\mathrm{P}_0 \left(\int_a^b d\alpha\, f_\alpha + \sum\nolimits_{(n)1}^{\infty} \int_a^b d\alpha\, \mathrm{Q}_{\alpha-x,n} f_\alpha \right); \tag{C}$$

in which development, the terms corresponding to large values of n are small. For example, when $\mathrm{P}_\alpha = \cos\alpha$, then

$$\varpi = \pi, \quad \mathrm{P}_0 = 1, \quad \mathrm{Q}_{\alpha,n} = 2\cos 2n\alpha,$$

and the theorem (C) reduces itself to the following known result:

$$f_x = \pi^{-1} \left(\int_a^b d\alpha\, f_\alpha + 2\sum\nolimits_{(n)1}^{\infty} \int_a^b d\alpha\, \cos(2n\alpha - 2nx) f_\alpha \right); \tag{q″}$$

in which it is supposed that $x > a$, $x < b$, and that $b - a \ngtr \pi$, in order that $\alpha - x$ may be comprised between the limits $\pm\pi$, for the whole extent of the integration; and the function f_α is supposed to remain finite within the same extent, and to vary gradually in value, at least for values of the variable α which are extremely near to x. The result (q″) may also be thus written:

$$f_x = \pi^{-1} \sum\nolimits_{(n)-\infty}^{\infty} \int_a^b d\alpha\, \cos(2n\alpha - 2nx) f_\alpha; \tag{r″}$$

and if we write

$$\alpha = \frac{\beta}{2}, \quad x = \frac{y}{2}, \quad f_{\frac{y}{2}} = \phi_y,$$

it becomes

$$\phi_y = \frac{1}{2\pi} \sum\nolimits_{(n)-\infty}^{\infty} \int_{2a}^{2b} d\beta\, \cos(n\beta - ny)\phi_\beta, \tag{s″}$$

the interval between the limits of integration relatively to β being now not greater than 2π, and the value of y being included between those limits. For example, we may assume

$$2a = -\pi, \quad 2b = \pi,$$

and then we shall have, by writing α, x, and f, instead of β, y, and ϕ,

$$f_x = \frac{1}{2\pi} \sum\nolimits_{(n)-\infty}^{\infty} \int_{-\pi}^{\pi} d\alpha\, \cos(n\alpha - nx) f_\alpha, \tag{t″}$$

in which $x > -\pi$, $x < \pi$. It is permitted to assume the function f_α such as to vanish when $\alpha < 0$, $> -\pi$; and then the formula (t″) resolves itself into the two following, which (with a slightly different notation) occur often in the writings of Poisson, as does also the formula (t″):

$$\tfrac{1}{2} \int_0^\pi d\alpha\, f_\alpha + \sum\nolimits_{(n)1}^{\infty} \int_0^\pi d\alpha\, \cos(n\alpha - nx) f_\alpha = \pi f_x; \tag{u″}$$

$$\tfrac{1}{2} \int_0^\pi d\alpha\, f_\alpha + \sum\nolimits_{(n)1}^{\infty} \int_0^\pi d\alpha\, \cos(n\alpha + nx) f_\alpha = 0; \tag{v″}$$

x being here supposed > 0, but $< \pi$; and the function f_α being arbitrary, but finite, and varying gradually, from $\alpha = 0$ to $\alpha = \pi$, or at least not receiving any sudden change of value for any value x of the variable α, to which the formula (u″) is to be applied. It is evident that the limits of integration in (t″) may be made to become $\mp l$, l being any finite quantity, by merely multiplying $n\alpha - nx$ under the sign cos., by $\dfrac{\pi}{l}$, and changing the external factor $\dfrac{1}{2\pi}$ to $\dfrac{1}{2l}$; and it is under this latter form that the theorem (t″) is usually presented by POISSON: who has also remarked, that the difference of the two series (u″) and (v″) conducts to the expression first assigned by LAGRANGE, for developing an arbitrary function between finite limits, in a series of sines of multiples of the variable on which it depends.

[12.] In general, in the formula (m″), from which the theorem (C) was derived, in order that x may be susceptible of receiving all values $> a$ and $< b$ (or at least all for which the function f_x receives no sudden change of value), it is necessary, by the remark made at the beginning of the last article, that the equation

$$\int_0^\alpha d\alpha\, \mathrm{P}_\alpha = 0, \tag{w″}$$

should have no real root α different from 0, between the limits $\mp(b-a)$. But it is permitted to suppose, consistently with this restriction, that a is < 0, and that b is > 0, while both are finite and determined; and then the formula (m″), or (C) which is a consequence of it, may be transformed so as to receive new limits of integration, which shall approach as nearly as may be desired to negative and positive infinity. In fact, by changing α to $\lambda\alpha$, x to λx, and $f_{\lambda x}$ to f_x, the formula (C) becomes

$$f_x = \lambda\varpi^{-1}\mathrm{P}_0 \left(\int_{\lambda^{-1}a}^{\lambda^{-1}b} d\alpha\, f_\alpha + \sum\nolimits_{(n)1}^{\infty} \int_{\lambda^{-1}a}^{\lambda^{-1}b} d\alpha\, \mathrm{Q}_{\lambda\alpha-\lambda x,n} f_\alpha \right); \tag{x″}$$

in which $\lambda^{-1}a$ will be large and negative, while $\lambda^{-1}b$ will be large and positive, if λ be small and positive, because we have supposed that a is negative, and b positive; and the new variable x is only obliged to be $> \lambda^{-1}a$ and $< \lambda^{-1}b$, if the new function f_x be finite and vary gradually between these new and enlarged limits. At the same time, the definition (o″) shows that $\mathrm{P}_0\mathrm{Q}_{\lambda\alpha-\lambda x,n}$ will tend indefinitely to become equal to $2\mathrm{P}_{2n\lambda(\alpha-x)}$; in such a manner that

$$\lim_{\lambda=0} . \frac{\mathrm{P}_0\mathrm{Q}_{\lambda\alpha-\lambda x,n}}{2\mathrm{P}_{2n\lambda(\alpha-x)}} = 1, \tag{y″}$$

at least if the function P be finite and vary gradually. Admitting then that we may adopt the following ultimate transformation of a sum into an integral, at least under the sign $\displaystyle\int_{-\infty}^{\infty} d\alpha$,

$$\lim_{\lambda=0} .2\lambda \left(\tfrac{1}{2}\mathrm{P}_0 + \sum\nolimits_{(n)1}^{\infty} \mathrm{P}_{2n\lambda(\alpha-x)} \right) = \int_0^\infty d\beta\, \mathrm{P}_{\beta(\alpha-x)}, \tag{z″}$$

we shall have, as the limit of (x″), this formula:

$$f_x = \varpi^{-1} \int_{-\infty}^{\infty} d\alpha \int_0^{\infty} d\beta \, \mathrm{P}_{\beta(\alpha - x)} f_\alpha; \tag{D}$$

which holds good for all real values of the variable x, at least under the conditions lately supposed, and may be regarded as an extension of the theorem (B), from finite to infinite limits. For example, by making P a cosine, the theorem (D) becomes

$$f_x = \pi^{-1} \int_{-\infty}^{\infty} d\alpha \int_0^{\infty} d\beta \, \cos(\beta\alpha - \beta x) f_\alpha, \tag{a‴}$$

which is a more usual form than (a) for the theorem of FOURIER. In general, the deduction in the present article, of the theorem (D) from (C), may be regarded as a verification of the analysis employed in this paper, because (D) may also be obtained from (B), by making the limits of integration infinite; but the demonstration of the theorem (B) itself, in former articles, was perhaps more completely satisfactory, besides that it involved fewer suppositions; and it seems proper to regard the formula (D) as only a limiting form of (B).

[13.] This formula (D) may also be considered as a limit in another way, by introducing, under the sign of integration relatively to β, a factor $\mathrm{F}_{k\beta}$ such that

$$\mathrm{F}_0 = 1, \quad \mathrm{F}_\infty = 0, \tag{b‴}$$

in which k is supposed positive but small, and the limit taken with respect to it, as follows:

$$f_x = \lim_{k=0} . \varpi^{-1} \int_{-\infty}^{\infty} d\alpha \left(\int_0^{\infty} d\beta \, \mathrm{P}_{\beta(\alpha - x)} \mathrm{F}_{k\beta} \right) f_\alpha. \tag{E}$$

It is permitted to suppose that the function F decreases continually and gradually, at a finite and decreasing rate, from 1 to 0, while the variable on which it depends increases from 0 to ∞; the first differential coefficient F' being thus constantly finite and negative, but constantly tending to 0, while the variable is positive and tends to ∞. Then, by the suppositions already made respecting the function P, if $\alpha - x$ and k be each different from 0, we shall have

$$\int_0^{\beta} d\beta \, \mathrm{P}_{\beta(\alpha - x)} \mathrm{F}_{k\beta} = \mathrm{F}_{k\beta} \mathrm{N}_{\beta(\alpha - x)} (\alpha - x)^{-1} - k(\alpha - x)^{-1} \int_0^{\beta} d\beta \, \mathrm{N}_{\beta(\alpha - x)} \mathrm{F}'_{k\beta}; \tag{c‴}$$

and therefore, because $\mathrm{F}_\infty = 0$, while N is always finite, the integral relative to β in the formula (E) may be thus expressed:

$$\int_0^{\beta} d\beta \, \mathrm{P}_{\beta(\alpha - x)} \mathrm{F}_{k\beta} = (\alpha - x)^{-1} \psi_{k^{-1}(\alpha - x)}, \tag{d‴}$$

the function ψ being assigned by the equation

$$\psi_\lambda = - \int_0^{\infty} d\gamma \, \mathrm{N}_{\lambda\gamma} \mathrm{F}'_\gamma. \tag{e‴}$$

For any given value of λ, the value of this function ψ is finite and determinate, by the principles of the sixth article; and as λ tends to ∞, the function ψ tends to 0, on account of the fluctuation of N, and because F' tends to 0, while γ tends to ∞; the integral (d‴) therefore tends to vanish with k, if α be different from x; so that

$$\lim_{k=0} . \int_0^\infty d\beta \, \mathrm{P}_{\beta(\alpha-x)} \mathrm{F}_{k\beta} = 0, \text{ if } \alpha \gtrless x. \tag{f‴}$$

On the other hand, if $\alpha = x$, that integral tends to become infinite, because we have, by (b‴),

$$\lim_{k=0} . \mathrm{P}_0 \int_0^\infty d\beta \, \mathrm{F}_{k\beta} = \infty. \tag{g‴}$$

Thus, while the formula (d‴) shows that the integral relative to β in (E) is a homogeneous function of $\alpha - x$ and k, of which the dimension is negative unity, we see also, by (f‴) and (g‴), that this function is such as to vanish or become infinite at the limit $k = 0$, according as $\alpha - x$ is different from or equal to zero. When the difference between α and x, whether positive or negative, is very small and of the same order as k, the value of the last mentioned integral (relative to β) varies very rapidly with α; and in this way of considering the subject, the proof of the formula (E) is made to depend on the verification of the equation

$$\varpi^{-1} \int_{-\infty}^\infty d\lambda \, \psi_\lambda \lambda^{-1} = 1. \tag{h‴}$$

But this last verification is easily effected; for when we substitute the expression (e‴) for ψ_λ, and integrate first relatively to λ, we find, by (r′),

$$\int_{-\infty}^\infty d\lambda \, \mathrm{N}_{\lambda\gamma} \lambda^{-1} = \varpi; \tag{i‴}$$

it remains then to show that

$$-\int_0^\infty d\gamma \, \mathrm{F}'_\gamma = 1; \tag{k‴}$$

and this follows immediately from the conditions (b‴). For example, when P is a cosine, and F a negative neperian exponential, so that

$$\mathrm{P}_\alpha = \cos\alpha, \quad \mathrm{F}_\alpha = e^{-\alpha},$$

then, making $\lambda = k^{-1}(\alpha - x)$, we have

$$\int_0^\infty d\beta \, e^{-k\beta} \cos(\beta\alpha - \beta x) = (\alpha - x)^{-1} \psi_\lambda;$$

$$\psi_\lambda = \int_0^\infty d\gamma \, e^{-\gamma} \sin\lambda\gamma = \frac{\lambda}{1+\lambda^2};$$

and

$$\varpi^{-1}\int_{-\infty}^{\infty} d\lambda\,\psi_\lambda\lambda^{-1} = \pi^{-1}\int_{-\infty}^{\infty}\frac{d\lambda}{1+\lambda^2} = 1.$$

It is nearly thus that POISSON has, in some of his writings, demonstrated the theorem of FOURIER, after putting it under a form which differs only slightly from the following:

$$f_x = \pi^{-1}\lim_{k=0}\int_{-\infty}^{\infty} d\alpha \int_0^{\infty} d\beta\, e^{-k\beta}\cos(\beta\alpha - \beta x) f_\alpha; \tag{l'''}$$

namely, by substituting for the integral relative to β its value

$$\frac{k}{k^2 + (\alpha - x)^2};$$

and then observing that, if k be very small, this value is itself very small, unless α be extremely near to x, so that f_α may be changed to f_x; while, making $\alpha = x + k\lambda$, and integrating relatively to λ between limits indefinitely great, the factor by which this function f_x is multiplied in the second member of (l'''), is found to reduce itself to unity.

[14.] Again, the function F_α retaining the same properties as in the last article for positive values of α, and being further supposed to satisfy the condition

$$\mathrm{F}_{-\alpha} = \mathrm{F}_\alpha, \tag{m'''}$$

while k is still supposed to be positive and small, the formula (D) may be presented in this other way, as the limit of the result of two integrations, of which the first is to be effected with respect to the variable α:

$$f_x = \lim_{k=0}.\varpi^{-1}\int_0^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha\, \mathrm{F}_{k\alpha}\mathrm{P}_{\beta(\alpha-x)} f_\alpha. \tag{F}$$

Now it often happens that if the function f_α be obliged to satisfy conditions which determine all its values by means of the arbitrary values which it may have for a given finite range, from $\alpha = a$ to $\alpha = b$, the integral relative to α in the formula (F) can be shown to vanish at the limit $k = 0$, for all real and positive values of β, except those which are roots of a certain equation

$$\Omega_\rho = 0; \tag{G}$$

while the same integral is, on the contrary, infinite, for these particular values of β; and then the integration relatively to β will in general change itself into a summation relatively to the real and positive roots ρ of the equation (G), which is to be combined with an integration relatively to α between the given limits a and b; the resulting expression being of the form

$$f_x = \sum\nolimits_\rho \int_a^b d\alpha\,\phi_{x,\alpha,\rho} f_\alpha. \tag{H}$$

For example, in the case where P is a cosine, and F a negative exponential, if the conditions relative to the function f be supposed such as to conduct to expressions of the forms

$$\int_0^\infty d\alpha\, e^{-h\alpha} f_\alpha = \frac{\psi(h)}{\phi(h)}, \tag{n'''}$$

$$\int_0^{-\infty} d\alpha\, e^{h\alpha} f_\alpha = \frac{\psi(-h)}{\phi(-h)}, \tag{o'''}$$

in which h is any real or imaginary quantity, independent of α, and having its real part positive; it will follow that

$$\int_{-\infty}^\infty d\alpha\, e^{-k\sqrt{\alpha^2}}(\cos\beta\alpha - \sqrt{-1}\sin\beta\alpha) f_\alpha = \frac{\psi(\beta\sqrt{-1}+k)}{\phi(\beta\sqrt{-1}+k)} - \frac{\psi(\beta\sqrt{-1}-k)}{\phi(\beta\sqrt{-1}-k)}, \tag{p'''}$$

in which $\sqrt{\alpha^2}$ is $= \alpha$ or $-\alpha$, according as α is $>$ or < 0, and the quantities β and k are real, and k is positive. The integral in (p'''), and consequently also that relative to α in (F), in which, now

$$\mathrm{P}_\alpha = \cos\alpha, \quad \mathrm{F}_\alpha = e^{-k\sqrt{\alpha^2}},$$

will therefore, under these conditions, tend to vanish with k, unless β be a root ρ of the equation

$$\phi(\rho\sqrt{-1}) = 0, \tag{q'''}$$

which here corresponds to (G); but the same integral will on the contrary tend to become infinite, as k tends to 0, if β be a root of the equation (q'''). Making therefore $\beta = \rho + k\lambda$, and supposing $k\lambda$ to be small, while ρ is a real and positive root of (q'''), the integral (p''') becomes

$$\frac{k^{-1}}{1+\lambda^2}(\mathrm{A}_\rho - \sqrt{-1}\mathrm{B}_\rho), \tag{r'''}$$

in which A_ρ and B_ρ are real, namely,

$$\left.\begin{aligned} \mathrm{A}_\rho &= \frac{\psi(\rho\sqrt{-1})}{\phi'(\rho\sqrt{-1})} + \frac{\psi(-\rho\sqrt{-1})}{\phi'(-\rho\sqrt{-1})}, \\ \mathrm{B}_\rho &= \sqrt{-1}\left(\frac{\psi(\rho\sqrt{-1})}{\phi'(\rho\sqrt{-1})} - \frac{\psi(-\rho\sqrt{-1})}{\phi'(-\rho\sqrt{-1})}\right); \end{aligned}\right\} \tag{s'''}$$

ϕ' being the differential coefficient of the function ϕ. Multiplying the expression (r''') by $\pi^{-1}\,d\beta\,(\cos\beta x + \sqrt{-1}\sin\beta x)$, which may be changed to $\pi^{-1}k\,d\lambda\,(\cos\rho x + \sqrt{-1}\sin\rho x)$; integrating relatively to λ between indefinitely great limits, negative and positive; taking the real part of the result, and summing it relatively to ρ; there results,

$$f_x = \sum\nolimits_\rho (\mathrm{A}_\rho \cos\rho x + \mathrm{B}_\rho \sin\rho x); \tag{t'''}$$

a development which has been deduced nearly as above, by POISSON and LIOUVILLE, from the suppositions (n′′′), (o′′′), and from the theorem of FOURIER presented under a form equivalent to the following:

$$f_x = \lim_{k=0} .\pi^{-1} \int_0^\infty d\beta \int_{-\infty}^\infty d\alpha\, e^{-k\sqrt{\alpha^2}} \cos(\beta\alpha - \beta x) f_\alpha; \qquad \text{(u}'''\text{)}$$

and in which it is to be remembered that if 0 be a root of the equation (q′′′), the corresponding terms in the development of f_x must in general be modified by the circumstance, that in calulating these terms, the integration relatively to λ extends only from 0 to ∞.

For example, when the function f is obliged to satisfy the conditions

$$f_{-\alpha} = f_\alpha, \quad f_{l-\alpha} = -f_{l+\alpha}, \qquad \text{(v}'''\text{)}$$

the suppositions (n′′′) (o′′′) are satisfied; the functions ϕ and ψ being here such that

$$\phi(h) = e^{hl} + e^{-hl},$$
$$\psi(h) = \int_0^l d\alpha\, (e^{h(l-\alpha)} - e^{h(\alpha-l)}) f_\alpha;$$

therefore the equation (q′′′) becomes in this case

$$\cos \rho l = 0, \qquad \text{(w}'''\text{)}$$

and the expressions (s′′′) for the coefficients of the development (t′′′) reduce themselves to the following:

$$\mathrm{A}_\rho = \frac{2}{l} \int_0^l d\alpha\, \cos \rho\alpha\, f_\alpha; \quad \mathrm{B}_\rho = 0; \qquad \text{(x}'''\text{)}$$

so that the method conducts to the following expression for the function f, which satisfies the conditions (v′′′),

$$f_x = \frac{2}{l} \sum\nolimits_{(n)1}^{\infty} \cos \frac{(2n-1)\pi x}{2l} \int_0^l d\alpha\, \cos \frac{(2n-1)\pi\alpha}{2l} f_\alpha; \qquad \text{(y}'''\text{)}$$

in which f_α is arbitrary from $\alpha = 0$ to $\alpha = l$, except that f_l must vanish. The same method has been applied, by the authors already cited, to other and more difficult questions; but it will harmonize better with the principles of the present paper to treat the subject in another way, to which we shall now proceed.

[15.] Instead of introducing, as in (E) and (F), a factor which has unity for its limit, we may often remove the apparent indeterminateness of the formula (D) in another way, by the principles of fluctuating functions. For if we integrate first relatively to α between indefinitely great limits, negative and positive, then, under the conditions which conduct to developments of the form (H), we shall find that the resulting function of β is usually a fluctuating one, of which the integral vanishes, except in the immediate neighbourhood of certain particular values determined by an equation such as (G); and then, by integrating only in such immediate neighbourhood, and afterwards summing the results, the development (H) is obtained. For example, when P is a cosine, and when the conditions (v''') are satisfied by the function f, it is not difficult to prove that

$$\int_{-2ml-l}^{2ml+l} d\alpha\, \cos(\beta\alpha - \beta x) f_\alpha = \frac{2\cos(2m\beta l + \beta l + m\pi)}{\cos\beta l} \cos\beta x \int_0^l d\alpha\, \cos\beta\alpha f_\alpha; \qquad \text{(z''')}$$

m being here an integer number, which is to be supposed large, and ultimately infinite. The equation (G) becomes therefore, in the present question and by the present method, as well as by that of the last article,

$$\cos\rho l = 0;$$

and if we make $\beta = \rho + \gamma$, ρ being a root of this equation, we may neglect γ in the second member of (z'''), except in the denominator

$$\cos\beta l = -\sin\rho l\, \sin\gamma l,$$

and in the fluctuating factor of the numerator

$$\cos(2m\beta l + \beta l + m\pi) = -\sin\rho l\, \sin(2m\gamma l + \gamma l);$$

consequently, multiplying by $\pi^{-1}\, d\gamma$, integrating relatively to γ between any two small limits of the forms $\mp\epsilon$, and observing that

$$\lim_{m=\infty} . \frac{2}{\pi} \int_{-\epsilon}^{\epsilon} d\gamma\, \frac{\sin(2ml\gamma + l\gamma)}{\sin l\gamma} = \frac{2}{l},$$

the development

$$f_x = \frac{2}{l} \sum\nolimits_\rho \cos\rho x \int_0^l d\alpha\, \cos\rho\alpha\, f_\alpha,$$

which coincides with (y'''), and is of the form (H), is obtained.

[16.] A more important application of the method of the last article is suggested by the expression which FOURIER has given for the arbitrary initial temperature of a solid sphere, on the supposition that this temperature is the same for all points at the same distance from the centre. Denoting the radius of the sphere by l, and that of any layer or shell of it by α, while the initial temperature of the same layer is denoted by $\alpha^{-1} f_\alpha$, we have the equations

$$f_0 = 0, \quad f'_l + \nu f_l = 0, \qquad (\text{a}^{IV})$$

which permit us to suppose

$$f_\alpha + f_{-\alpha} = 0, \quad f'_{l+\alpha} + f'_{l-\alpha} + \nu(f_{l+\alpha} + f_{l-\alpha}) = 0; \qquad (\text{b}^{IV})$$

ν being here a constant quantity not less than $-l^{-1}$, and f' being the first differential coefficient of the function f, which function remains arbitrary for all values of α greater than 0, but not greater than l. The equations (b^{IV}) give

$$\begin{aligned}(\beta\cos\beta l + \nu\sin\beta l)\int_{l-\alpha}^{l+\alpha} d\alpha\, \sin\beta\alpha\, f_\alpha \\ = (\beta\sin\beta l - \nu\cos\beta l)\int_{\alpha-l}^{\alpha+l} d\alpha\, \cos\beta\alpha\, f_\alpha - \cos\beta\alpha\,(f_{\alpha+l} + f_{\alpha-l});\end{aligned} \qquad (\mathrm{c}^{IV})$$

so that

$$(\rho\sin\rho l - \nu\cos\rho l)\int_{\alpha-l}^{\alpha+l} d\alpha\, \cos\rho\alpha\, f_\alpha = \cos\rho\alpha\,(f_{\alpha+l} + f_{\alpha-l}), \qquad (\mathrm{d}^{IV})$$

if ρ be a root of the equation

$$\rho\cos\rho l + \nu\sin\rho l = 0. \qquad (\mathrm{e}^{IV})$$

This latter equation is that which here corresponds to (G); and when we change β to $\rho + \gamma$, γ being very small, we may write, in the first member of (c^{IV}),

$$\beta\cos\beta l + \nu\sin\beta l = \gamma\{(1 + \nu l)\cos\rho l + \rho l\sin\rho l\}, \qquad (\mathrm{f}^{IV})$$

and change β to ρ in all the terms of the second member, except in the fluctuating factor $\cos\beta\alpha$, in which α is to be made extremely large. Also, after making $\cos\beta\alpha = \cos\rho\alpha\,\cos\gamma\alpha - \sin\rho\alpha\,\sin\gamma\alpha$, we may suppress $\cos\gamma\alpha$ in the second member of (c^{IV}), before integrating with respect to γ, because by (d^{IV}) the terms involving $\cos\gamma\alpha$ tend to vanish with γ, and because $\gamma^{-1}\cos\gamma\alpha$ changes sign with γ. On the other hand, the integral of $\dfrac{d\gamma\,\sin\gamma\alpha}{\gamma}$ is to be replaced by π, though it be taken only for very small values, negative and positive, of γ, because α is here indefinitely large and positive. Thus in the present question, the formula

$$f_x = \frac{1}{\pi}\,.\lim_{\alpha=\infty}.\int_0^\infty d\beta\int_{l-\alpha}^{l+\alpha} d\alpha\,\sin\beta\alpha\, f_\alpha, \qquad (\mathrm{g}^{IV})$$

(which is obtained from (a''') by suppressing the terms which involve $\cos\beta x$, on account of the first condition (b^{IV}),) may be replaced by a sum relative to the real and positive roots of the equation (e^{IV}); the term corresponding to any one such root being

$$\frac{\mathrm{R}_\rho\sin\rho x}{(1 + \nu l)\cos\rho l - \rho l\sin\rho l}, \qquad (\mathrm{h}^{IV})$$

if we suppose $\rho > 0$, and make for abridgment

$$\mathrm{R}_\rho = (\nu\cos\rho l - \rho\sin\rho l)\int_{\alpha-l}^{\alpha+l} d\alpha\,\sin\rho\alpha\, f_\alpha + \sin\rho\alpha\,(f_{\alpha+l} + f_{\alpha-l}). \qquad (\mathrm{i}^{IV})$$

The equations (b^{IV}) show that the quantity R_ρ does not vary with α, and therefore that it may be rigorously thus expressed:

$$\mathrm{R}_\rho = 2(\nu \cos \rho l - \rho \sin \rho l) \int_0^l d\alpha \, \sin \rho\alpha \, f_\alpha; \tag{k^{IV}}$$

we have also, by (e^{IV}), ρ being > 0,

$$\frac{2(\nu \cos \rho l - \rho \sin \rho l)}{\cos \rho l + l(\nu \cos \rho l - \rho \sin \rho l)} = \frac{2\rho}{\rho l - \sin \rho l \, \cos \rho l}. \tag{l^{IV}}$$

And if we set aside the particular case where

$$\nu l + 1 = 0, \tag{m^{IV}}$$

the term corresponding to the root

$$\rho = 0, \tag{n^{IV}}$$

of the equation (e^{IV}), vanishes in the development of f_x; because this term is, by (g^{IV}),

$$\frac{x}{\pi} \int_0^\beta d\beta \left(\beta \int_{l-\alpha}^{l+\alpha} d\alpha \, \sin \beta\alpha \, f_\alpha \right), \tag{o^{IV}}$$

α being very large, and β small, but both being positive; and unless the condition (m^{IV}) be satisfied, the equation (c^{IV}) shows that the quantity to be integrated in (o^{IV}), with respect to β, is a finite and fluctuating function of that variable, so that its integral vanishes, at the limit $\alpha = \infty$. Setting aside then the case (m^{IV}) which corresponds physically to the absence of exterior radiation, we see that the function f_x, which represents the initial temperature of any layer of the sphere multiplied by the distance x of that layer from the centre, and which is arbitrary between the limits $x = 0$, $x = l$, that is, between the centre and the surface, (though it is obliged to satisfy at those limits the conditions (a^{IV})), may be developed in the following series, which was discovered by FOURIER, and is of the form (H):

$$f_x = \sum\nolimits_\rho \frac{2\rho \sin \rho x \displaystyle\int_0^l d\alpha \, \sin \rho\alpha \, f_\alpha}{\rho l - \sin \rho l \, \cos \rho l}; \tag{p^{IV}}$$

the sum extending only to those roots of the equation (e^{IV}) which are greater than 0. In the particular case (m^{IV}), in which the root (n^{IV}) of the equation (e^{IV}) must be employed, the term (o^{IV}) becomes, by (c^{IV}) and (d^{IV}),

$$\frac{3x}{\pi l^3} \left\{ \int_{\alpha-l}^{\alpha+l} d\alpha \, f_\alpha \alpha \mathrm{C} - l(f_{\alpha+l} + f_{\alpha-l})\alpha \mathrm{C} \right\}, \tag{q^{IV}}$$

in which, at the limit here considered,

$$\text{C} = \int_0^\infty d\theta \, \frac{\operatorname{vers} \theta}{\theta^2} = \frac{\pi}{2}; \qquad (\text{r}^{IV})$$

but also, by the equations (b^{IV}), (m^{IV}),

$$\int_{\alpha-l}^{\alpha+l} d\alpha \, f_\alpha \alpha - l(f_{\alpha+l} + f_{\alpha-l})\alpha = 2\int_0^l d\alpha \, f_\alpha \alpha; \qquad (\text{s}^{IV})$$

the sought term of f_x becomes, therefore, in the present case,

$$\frac{3x}{l^3} \int_0^l d\alpha \, f_\alpha \alpha, \qquad (\text{t}^{IV})$$

and the corresponding term in the expression of the temperature $x^{-1} f_x$ is equal to the mean initial temperature of the sphere; a result which has been otherwise obtained by Poisson, for the case of no exterior radiation, and which might have been anticipated from physical considerations. The supposition

$$\nu l + 1 < 0, \qquad (\text{u}^{IV})$$

which is inconsistent with the physical conditions of the question, and in which Fourier's development (p^{IV}) may fail, is excluded in the foregoing analysis.

[17.] When a converging series of the form (H) is arrived at, in which the coefficients ϕ of the arbitrary function f, under the sign of integration, do not tend to vanish as they correspond to larger and larger roots ρ of the equation (G); then those coefficients $\phi_{x,\alpha,\rho}$ must in general tend to become fluctuating functions of α, as ρ becomes larger and larger. And the sum of those coefficients, which may be thus denoted,

$$\sum\nolimits_\rho \phi_{x,\alpha,\rho} = \psi_{x,\alpha,\rho}, \qquad (\text{I})$$

and which is here supposed to be extended to all real and positive roots of the equation (G), as far as some given root ρ, must tend to become a fluctuating function of α, and to have its mean value equal to zero, as ρ tends to become infinite, for all values of α and x which are different from each other, and are both comprised between the limits of the integration relative to α; in such a manner as to satisfy the equation

$$\int_\lambda^\mu d\alpha \, \psi_{x,\alpha,\infty} f_\alpha = 0, \qquad (\text{K})$$

which is of the form (e), referred to in the second article; provided that the arbitrary function f is finite, and that the quantities λ, μ, x, α are all comprised between the limits a and b, which enter into the formula (H); while α is, but x is not, comprised also between the new limits λ and μ. But when $\alpha = x$, the sum (i) tends to become infinite with ρ, so that we have

$$\psi_{x,x,\infty} = \infty, \qquad (\text{L})$$

and

$$\int_{x-\epsilon}^{x+\epsilon} d\alpha\, \psi_{x,\alpha,\infty} f_\alpha = f_x, \tag{M}$$

ϵ being here a quantity indefinitely small. For example, in the particular question which conducts to the development (y‴), we have

$$\phi_{x,\alpha,\rho} = \frac{2}{l} \cos \rho x \, \cos \rho\alpha, \tag{v^{IV}}$$

and

$$\rho = \frac{(2n-1)\pi}{2l}; \tag{w^{IV}}$$

therefore, summing relatively to ρ, or to n, from $n = 1$ to any given positive value of the integer number n, we have, by (I),

$$\psi_{x,\alpha,\rho} = \frac{\sin \dfrac{n\pi(\alpha - x)}{l}}{2l \sin \dfrac{\pi(\alpha - x)}{2l}} + \frac{\sin \dfrac{n\pi(\alpha + x)}{l}}{2l \sin \dfrac{\pi(\alpha + x)}{2l}}; \tag{x^{IV}}$$

and it is evident that this sum tends to become a fluctuating function of α, and to satisfy the equation (K), as ρ, or n, tends to become infinite, while α and x are different from each other, and are both comprised between the limits 0 and l. On the other hand, when α becomes equal to x, the first part of the expression (x^{IV}) becomes $= \dfrac{n}{l}$, and therefore tends to become infinite with n, so that the equation (L) is true. And the equation (M) is verified by observing, that if $x > 0$, $< l$, we may omit the second part of the sum (x^{IV}), as disappearing in the integral through fluctuation, while the first part gives, at the limit,

$$\lim_{n=\infty} \int_{x-\epsilon}^{x+\epsilon} d\alpha \, \frac{\sin \dfrac{n\pi(\alpha - x)}{l}}{2l \sin \dfrac{\pi(\alpha - x)}{2l}} f_\alpha = f_x. \tag{y^{IV}}$$

If x be equal to 0, the integral is to be taken only from 0 to ϵ, and the result is only half as great, namely,

$$\lim_{n=\infty} . \int_0^{\epsilon} d\alpha \, \frac{\sin \dfrac{n\pi\alpha}{l}}{2l \sin \dfrac{\pi\alpha}{2l}} f_\alpha = \tfrac{1}{2} f_0; \tag{z^{IV}}$$

but, in this case, the other part of the sum (x^{IV}) contributes an equal term, and the whole result is f_0. If $x = l$, the integral is to be taken from $l - \epsilon$ to l, and the two parts of the expression (x^{IV}) contribute the two terms $\frac{1}{2} f_l$ and $-\frac{1}{2} f_l$, which neutralize each other. We may therefore in this way prove, *à posteriori*, by consideration of fluctuating functions, the truth of the development (y‴) for any arbitrary but finite function f_x, and for all values of the real variable x from $x = 0$ to $x = l$, the function being supposed to vanish at the latter limit; observing only that if this function f_x undergo any sudden change of value, for any value $x^\backprime$ of the variable between the limits 0 and l, and if x be made equal to $x^\backprime$ in the development (y‴), the process shows that this development then represents the semisum of the two values which the function f receives, immediately before and after it undergoes this sudden change.

[18.] The same mode of *à posteriori* proof, through the consideration of fluctuating functions, may be applied to a great variety of other analogous developments, as has indeed been indicated by FOURIER, in a passage of his Theory of Heat. The spirit of POISSON'S method, when applied to the establishment, *à posteriori*, of developments of the form (H) would lead us to multiply, before the summation, each coefficient $\phi_{x,\alpha,\rho}$ by a factor $\mathrm{F}_{k,\rho}$ which tends to unity as k tends to 0, but tends to vanish as ρ tends to ∞; and then instead of a *generally fluctuating sum* (I), there results a *generally evanescent sum* (k being evanescent), namely,

$$\sum_{\rho} \mathrm{F}_{k,\rho}\phi_{x,\alpha,\rho} = \chi_{x,\alpha,k,\rho}, \tag{N}$$

which conducts to equations analogous to (K) (L) (M), namely,

$$\lim_{k=0} \int_{\lambda}^{\mu} d\alpha\, \chi_{x,\alpha,k,\infty} f_{\alpha} = 0; \tag{O}$$

$$\lim_{k=0} \chi_{x,x,k,\infty} = \infty; \tag{P}$$

$$\lim_{k=0} \int_{x-\epsilon}^{x+\epsilon} d\alpha\, \chi_{x,\alpha,k,\infty} f_{\alpha} = f_x. \tag{Q}$$

It would be interesting to inquire what form the generally evanescent function χ would take immediately before its vanishing when

$$\mathrm{F}_{k,\rho} = e^{-k\rho},$$

and

$$\phi_{x,\alpha,\rho} = \frac{2\rho \sin\rho x\, \sin\rho\alpha}{\rho l - \sin\rho l\, \cos\rho l},$$

ρ being a root of the equation

$$\rho l \operatorname{cotan} \rho l = \text{const.},$$

and the constant in the second member being supposed not greater than unity.

[19.] The development (C), which, like (H), expresses an arbitrary function, at least between given limits, by a combination of summation and integration, was deduced from the expression (m″) of the eleventh article, which conducts also to many other analogous developments, according to the various ways in which the factor with the infinite index, $\mathrm{N}_{\infty(\alpha-x)}$, may be replaced by an infinite sum, or other equivalent form. Thus, if, instead of (o″), we establish the following equation,

$$\int_{(2n-2)\alpha}^{2n\alpha} d\alpha\, \mathrm{P}_{\alpha} = \mathrm{R}_{\alpha,n} \int_{0}^{\alpha} d\alpha\, \mathrm{P}_{\alpha}, \tag{a^V}$$

we shall have, instead of (C), the development:

$$f_x = \varpi^{-1}\mathrm{P}_0 \sum\nolimits_{(n)1}^{\infty} \int_{a}^{b} d\alpha\, \mathrm{R}_{\alpha-x,n} f_{\alpha}; \tag{R}$$

which, when P is a cosine, reduces itself to the form,

$$f_x = \frac{2}{\pi} \sum\nolimits_{(n)1}^{\infty} \int_a^b d\alpha \, \cos(\overline{2n-1} \, . \, \overline{\alpha - x}) f_\alpha, \tag{b^V}$$

x being $> a$, $< b$, and $b - a$ being not $> \pi$; and easily conducts to the known expression

$$f_x = \frac{1}{l} \sum\nolimits_{(n)1}^{\infty} \int_{-l}^{l} d\alpha \, \cos \frac{(2n-1)\pi(\alpha - x)}{2l} f_\alpha, \tag{c^V}$$

which holds good for all values of x between $-l$ and $+l$. By supposing $f_\alpha = f_{-\alpha}$, we are conducted to the expression (y$'''$); and by supposing $f_\alpha = -f_{-\alpha}$ we are conducted to this other known expression,

$$f_x = \frac{2}{l} \sum\nolimits_{(n)1}^{\infty} \sin \frac{(2n-1)\pi x}{2l} \int_0^l d\alpha \, \sin \frac{(2n-1)\pi\alpha}{2l} f_\alpha; \tag{d^V}$$

which holds good even at the limit $x = l$, by the principles of the seventeenth article, and therefore offers the following transformation for the arbitrary function f_l:

$$f_l = -\frac{2}{l} \sum\nolimits_{(n)1}^{\infty} (-1)^n \int_0^l d\alpha \, \sin \frac{(2n-1)\pi\alpha}{2l} f_\alpha. \tag{e^V}$$

For example, by making $f_\alpha = \alpha^i$, and supposing i to be an uneven integer number; effecting the integration indicated in (e^V), and dividing both members by l^i, we find the following relation between the sums of the reciprocals of even powers of odd whole numbers:

$$1 = [i]^1 \omega_2 - [i]^3 \omega_4 + [i]^5 \omega_6 - \cdots; \tag{f^V}$$

in which

$$[i]^k = i(i-1)(i-2) \cdots (i-k+1); \tag{g^V}$$

and

$$\omega_{2k} = 2 \left(\frac{2}{\pi} \right)^{2k} \sum\nolimits_{(n)1}^{\infty} (2n-1)^{-2k}; \tag{h^V}$$

thus

$$1 = \omega_2 = 3\omega_2 - 3 \, . \, 2 \, . \, 1 \, . \, \omega_4 = 5\omega_2 - 5 \, . \, 4 \, . \, 3 \, \omega_4 + 5 \, . \, 4 \, . \, 3 \, . \, 2 \, . \, 1 \, \omega_6, \tag{i^V}$$

so that

$$\omega_2 = 1, \quad \omega_4 = \tfrac{1}{3}, \quad \omega_6 = \tfrac{2}{15}. \tag{k^V}$$

Again, by making $f_\alpha = \alpha^i$, but supposing $i =$ an uneven number $2k$, we get the following additional term in the second member of the equation (f^V),

$$(-1)^k [2k]^{2k} \omega_{2k+1}, \tag{l^V}$$

in which

$$\omega_{2k+1} = -2\left(\frac{2}{\pi}\right)^{2k+1} \sum\nolimits_{(n)1}^{\infty} (-1)^n (2n-1)^{-2k-1}; \tag{m^V}$$

thus

$$1 = \omega_1 = 2\omega_2 - 2\,.\,1\,\omega_3 = 4\omega_2 - 4\,.\,3\,.\,2\,\omega_4 + 4\,.\,3\,.\,2\,.\,1\,\omega_5, \tag{n^V}$$

so that

$$\omega_1 = 1, \quad \omega_3 = \tfrac{1}{2}, \quad \omega_5 = \tfrac{5}{24}. \tag{o^V}$$

Accordingly, if we multiply the values (k^V) by $\dfrac{\pi^2}{8}$, $\dfrac{\pi^4}{32}$, $\dfrac{\pi^6}{128}$, we get the known values for the sums of the reciprocals of the squares, fourth powers, and sixth powers of the odd whole numbers; and if we multiply the values (o^V) by $\dfrac{\pi}{4}$, $\dfrac{\pi^3}{16}$, $\dfrac{\pi^5}{64}$, we get the known values for the sums of the reciprocals of the first, third, and fifth powers of the same odd numbers, taken however with alternately positive and negative signs. Again, if we make $f_\alpha = \sin\alpha$, in (e^V), and divide both members of the resulting equation by $\cos l$, we get this known expression for a tangent,

$$\tan l = \sum\nolimits_{(n)-\infty}^{\infty} \frac{2}{(2n-1)\pi - 2l}; \tag{p^V}$$

which shows that, with the notation (h^V),

$$\tan l = \omega_2 l^1 + \omega_4 l^3 + \omega_6 l^5 + \cdots; \tag{q^V}$$

so that the coefficients of the ascending powers of the arc in the development of its tangent are connected with each other by the relations (f^5), which may be briefly represented thus:

$$\sqrt{-1} = (1 + \sqrt{-1}\mathrm{D}_0)^{2k-1} \tan 0; \tag{r^V}$$

the second member of this symbolic equation being supposed to be developed, and $\mathrm{D}_0^i \tan 0$ being understood to denote the value which the i^{th} differential coefficient of the tangent of α, taken with respect to α, acquires when $\alpha = 0$; thus

$$1 = \mathrm{D}_0 \tan 0 = 3\mathrm{D}_0 \tan 0 - \mathrm{D}_0^3 \tan 0 = 5\mathrm{D}_0 \tan 0 - 10\mathrm{D}_0^3 \tan 0 + \mathrm{D}_0^5 \tan 0. \tag{s^V}$$

Finally, if we make $f_\alpha = \cos\alpha$, and attend to the expression (p^V), we obtain, for the secant of an arc l, the known expression:

$$\sec l = \sum\nolimits_{(n)-\infty}^{\infty} \frac{2(-1)^{n+1}}{(2n-1)\pi - 2l}; \tag{t^V}$$

which shows that, with the notation (m^V),

$$\sec l = \omega_1 l^0 + \omega_3 l^2 + \omega_5 l^4 + \cdots, \tag{u^V}$$

and therefore, by the relations of the form (n^V),

$$\sqrt{-1}(1 - (\sqrt{-1}\mathrm{D}_0)^{2k} \sec 0) = (1 + \sqrt{-1}\mathrm{D}_0)^{2k} \tan 0; \tag{v^V}$$

thus

$$1 = \sec 0 = 2\mathrm{D}_0 \tan 0 - \mathrm{D}_0^2 \sec 0 = 4\mathrm{D}_0 \tan 0 - 4\mathrm{D}_0^3 \tan 0 + \mathrm{D}_0^4 \sec 0. \tag{w^V}$$

Though several of the results above deduced are known, the writer does not remember to have elsewhere seen the symbolic equations (r^V), (v^V), as expressions for the laws of the coefficients of the developments of the tangent and secant, according to ascending powers of the arc.

[20.] In the last article, the symbol R was such, that

$$\sum\nolimits_{(n)1}^{n} \mathrm{R}_{\alpha,n} = \mathrm{N}_{2n\alpha}\mathrm{N}_{\alpha}^{-1}; \tag{x^V}$$

and in article [11.], we had

$$1 + \sum\nolimits_{(n)1}^{n} \mathrm{Q}_{\alpha,n} = \mathrm{N}_{2n\alpha+\alpha}\mathrm{N}_{\alpha}^{-1}. \tag{y^V}$$

Assume, now, more generally,

$$\nabla_\beta \mathrm{S}_{\alpha,\beta} = \mathrm{N}_{\beta\alpha}\mathrm{N}_{\alpha}^{-1}; \tag{z^V}$$

and let the operation ∇_β admit of being effected after, instead of before, the integration relatively to α; the expression (m$''$) will then acquire this very general form:

$$f_x = \varpi^{-1}\mathrm{P}_0 \nabla_\infty \int_a^b d\alpha\, \mathrm{S}_{\alpha-x,\beta} f_\alpha; \tag{S}$$

which includes the transformations (C) and (R), and in which the notation ∇_∞ is designed to indicate that after performing the operation ∇_β we are to make the variable β infinite, according to some given law of increase, connected with the form of the operation denoted by ∇.

[21.] In order to deduce the theorems (C), (R), (S), we have hitherto supposed (as was stated in the twelfth article), that the equation $\mathrm{N}_\alpha = 0$ has no real root different from 0 between the limits $\mp(b-a)$, in which a and b are the limits of the integration relative to a, between which latter limits it is also supposed that the variable x is comprised. If these conditions be not satisfied, the factor $\mathrm{N}_{\alpha-x}^{-1}$, in the formula (m$''$), may become infinite within the proposed extent of integration, for values of α and x which are not equal to each other; and it will then be necessary to change the first member of each of the equations (m$''$), (C), (R), (S), to a function different from f_x, but to be determined by similar principles. To simplify the question, let it be supposed that the function N_α receives no sudden change of value, and that the equation

$$\mathrm{N}_\alpha = 0, \tag{a^{VI}}$$

which coincides with (w$''$) has all its real roots unequal. These roots must here coincide with the quantities $\alpha_{n,i}$ of the fourth and other articles, for which the function N_α changes sign; but as the double index is now unnecessary, while the notation α_n has been appropriated to the roots of the equation (g), we shall denote the roots of the equation (a^{VI}), in their order, by the symbols

$$\nu_{-\infty}, \ldots \nu_{-1}, \nu_0, \nu_1, \ldots \nu_\infty; \tag{b^{VI}}$$

and choosing ν_0 for that root of (a^{VI}) which has already been supposed to vanish, we shall have

$$\nu_0 = 0, \tag{c^{VI}}$$

while the other roots will be $>$ or $<$ 0, according as their indices are positive or negative. If the differential coefficient P_α be also supposed to remain always finite, and to receive no sudden change of value in the immediate neighbourhood of any root ν of (a^{VI}), we shall have, for values of α in that neighbourhood, the limiting equation:

$$\lim_{\alpha=\nu} .\mathrm{N}_\alpha(\alpha-\nu)^{-1} = \mathrm{P}_\nu; \tag{d^{VI}}$$

and P_ν will be different from 0, because the real roots of the equation (a^{VI}) have been supposed unequal. Conceive also that the integral

$$\int_{-\infty}^{\infty} d\alpha\, \mathrm{N}_{\alpha+\beta\nu}\alpha^{-1} = \varpi_{\nu,\beta} \tag{e^{VI}}$$

tends to some finite and determined limit, which may perhaps be different for different roots ν, and therefore may be thus denoted,

$$\varpi_{\nu,\infty} = \varpi_\nu, \tag{f^{VI}}$$

as β tends to ∞, after the given law referred to at the end of the last article. Then, by writing

$$\alpha = x+\nu+\beta^{-1}y, \tag{g^{VI}}$$

and supposing β to be very large, we easily see, by reasoning as in former articles, that the part of the integral

$$\int_a^b d\alpha\, \mathrm{N}_{\beta(\alpha-x)}\mathrm{N}^{-1}_{\alpha-x} f_\alpha, \tag{h^{VI}}$$

which corresponds to values of $\alpha - x$ in the neighbourhood of the root ν, is very nearly expressed by

$$\varpi_\nu \mathrm{P}_\nu^{-1} f_{x+\nu}; \tag{i^{VI}}$$

and that this expression is accurate at the limit. Instead of the equation (S), we have therefore now this other equation:

$$\sum\nolimits_\nu \varpi_\nu \mathrm{P}_\nu^{-1} f_{x+\nu} = \nabla_\infty \int_a^b d\alpha\, \mathrm{S}_{\alpha-x,\beta} f_\alpha; \tag{T}$$

the sum in the first member being extended to all those roots ν of the equation (a^{VI}), which satisfy the conditions

$$x+\nu > a, \quad < b. \tag{k^{VI}}$$

If one of the roots ν should happen to satisfy the condition

$$x+\nu = a, \tag{l^{VI}}$$

the corresponding term in the first member of (T) would be, by the same principles,

$$\varpi'_\nu \mathrm{P}_\nu^{-1} f_a, \tag{m^{VI}}$$

in which

$$\varpi'_\nu = \lim_{\beta=\infty} \int_0^\infty d\alpha\, \mathrm{N}_{\alpha+\beta\nu}\alpha^{-1}. \tag{n^{VI}}$$

And if a root ν of (a^{VI}) should satisfy the condition

$$x + \nu = b, \tag{o^{VI}}$$

the corresponding term in the first member of (T) would then be

$$\varpi''_\nu \mathrm{P}_\nu^{-1} f_b, \tag{p^{VI}}$$

in which

$$\varpi''_\nu = \lim_{\beta=\infty} \int_{-\infty}^0 d\alpha\, \mathrm{N}_{\alpha+\beta\nu}\alpha^{-1}. \tag{q^{VI}}$$

Finally, if a value of $x+\nu$ satisfy the conditions (k^{VI}), and if the function f undergo a sudden change of value for this particular value of the variable on which that function depends, so that $f = f''$ immediately before, and $f = f'$ immediately after the change, then the corresponding part of the first member of the formula (T) is

$$\mathrm{P}_\nu^{-1}(\varpi'_\nu f' + \varpi''_\nu f''). \tag{r^{VI}}$$

And in the formulæ for ϖ_ν, ϖ'_ν, ϖ''_ν, it is permitted to write

$$\mathrm{N}_{\alpha+\beta\nu}\alpha^{-1} = \int_0^1 dt\, \mathrm{P}_{t\alpha+\beta\nu}. \tag{s^{VI}}$$

[22.] One of the simplest ways of rendering the integral (e^{VI}) determinate at its limit, is to suppose that the function P_α is of the periodical form which satisfies the two following equations,

$$\mathrm{P}_{-\alpha} = \mathrm{P}_\alpha, \quad \mathrm{P}_{\alpha+p} = -\mathrm{P}_\alpha; \tag{t^{VI}}$$

p being some given positive constant. Multiplying these equations by $d\alpha$, and integrating from $\alpha = 0$, we find, by (a$''$),

$$\mathrm{N}_{-\alpha} + \mathrm{N}_\alpha = 0, \quad \mathrm{N}_{\alpha+p} + \mathrm{N}_\alpha = \mathrm{N}_p; \tag{u^{VI}}$$

therefore

$$\mathrm{N}_p = \mathrm{N}_{\frac{p}{2}} + \mathrm{N}_{-\frac{p}{2}} = 0, \tag{v^{VI}}$$

and

$$\mathrm{N}_{\alpha+p} = -\mathrm{N}_\alpha, \quad \mathrm{N}_{\alpha+2p} = \mathrm{N}_\alpha, \quad \&\text{c.} \tag{w^{VI}}$$

Consequently, if the equations (t^{VI}) be satisfied, the multiples (by whole numbers) of p will all be roots of the equation (a^{VI}); and reciprocally that equation will have no other real roots, if we suppose that the function P_α, which vanishes when α is any odd multiple of $\dfrac{p}{2}$, preserves one constant sign between any one such multiple and the next following, or simply between $\alpha = 0$ and $\alpha = \dfrac{p}{2}$. We may then, under these conditions, write

$$\nu_i = ip, \tag{x^{VI}}$$

i being any integer number, positive or negative, and ν_i denoting generally, as in (b^{VI}), any root of the equation (a^{VI}). And we shall have

$$\int_{-\infty}^{\infty} d\alpha\, \mathrm{N}_{\alpha+kp}\alpha^{-1} = (-1)^k \varpi, \tag{y^{VI}}$$

k being any integer number, and ϖ still retaining the same meaning as in the former articles. Also, for any integer value of k,

$$\mathrm{P}_{kp} = (-1)^k \mathrm{P}_0. \tag{z^{VI}}$$

These things being laid down, let us resume the integral (e^{VI}), and let us suppose that the law by which β increases to ∞ is that of coinciding successively with the several uneven integer numbers 1, 3, 5, &c., as was supposed in deducing the formula (C). Then $\beta\nu$ in (e^{VI}) will be an odd or even multiple of p, according as ν is the one or the other, so that we shall have by (x^{VI}), (y^{VI}), the following determined expression for the sought limit (f^{VI}):

$$\varpi_{\nu_i} = (-1)^i \varpi; \tag{a^{VII}}$$

but also, by (x^{VI}), (z^{VI}),

$$\mathrm{P}_{\nu_i} = (-1)^i \mathrm{P}_0; \tag{b^{VII}}$$

therefore

$$\varpi_\nu \mathrm{P}_\nu^{-1} = \varpi \mathrm{P}_0^{-1}, \tag{c^{VII}}$$

the value of this expression being thus the same for all the roots of (a^{VI}). At the same time, in (i^{VI}),

$$f_{x+\nu} = f_{x+ip}; \tag{d^{VII}}$$

the equation (T) becomes therefore now

$$\sum\nolimits_i f_{x+ip} = \varpi^{-1}\mathrm{P}_0 \nabla_\infty \int_a^b d\alpha\, \mathrm{S}_{\alpha-x,\beta} f_\alpha, \tag{U}$$

β tending to infinity by passing through the successive positive odd numbers, and i receiving all integer values which allow $x+ip$ to be comprised between the limits a and b. If any integer value of i render $x+ip$ equal to either of these limits, the corresponding term of the sum in the first member of (U) is to be $\frac{1}{2}f_a$, or $\frac{1}{2}f_b$; and if the function f receive any sudden change of value between the same limits of integration, corresponding to a value of the variable which is of the form $x+ip$, the term introduced thereby will be of the form $\frac{1}{2}f^{\backprime} + \frac{1}{2}f^{\backprime\backprime}$.

For example, when

$$\mathrm{P}_\alpha = \cos\alpha, \quad \varpi = \pi, \quad p = \pi, \tag{e^{VII}}$$

we obtain the following known formula, instead of (r″),

$$\sum\nolimits_i f_{x+i\pi} = \pi^{-1} \sum\nolimits_{(n)-\infty}^{\infty} \int_a^b d\alpha \, \cos(2n\alpha - 2nx)\, f_\alpha; \tag{f^{VII}}$$

which may be transformed in various ways, by changing the limits of integration, and in which halves of functions are to be introduced in extreme cases, as above.

On the other hand, if the law of increase of β be, as in (R), that of coinciding successively with large and larger even numbers, then

$$\varpi_\nu = \varpi, \quad \mathrm{P}_\nu = \mp \mathrm{P}_0, \tag{g^{VII}}$$

and the equation (T) becomes

$$\sum\nolimits_i (-1)^i f_{x+i\pi} = \varpi^{-1} \mathrm{P}_0 \nabla_\infty \int_a^b d\alpha \, \mathrm{S}_{\alpha-x,\beta} f_\alpha. \tag{V}$$

For example, in the case (e^{VII}), we obtain this extension of formula (b^{V}),

$$\sum\nolimits_i (-1)^i f_{x+i\pi} = \pi^{-1} \sum\nolimits_{(n)-\infty}^{\infty} \int_a^b d\alpha \, \cos(\overline{2n-1} \,.\, \overline{\alpha - x})\, f_\alpha. \tag{h^{VII}}$$

We may verify the equations (f^{VII}) (h^{VII}) by remarking that both members of the former equation remain unchanged, and that both members of the latter are changed in sign, when x is increased by π. A similar verification of the equations (U) and (V) requires that in general the expression

$$\nabla_\infty \int_a^b d\alpha \, \mathrm{S}_{\alpha-x,\beta} f_\alpha \tag{i^{VII}}$$

should either receive no change, or simply change its sign, when x is increased by p, according as β tends to ∞ by coinciding with large and odd or with large and even numbers.

[23.] In all the examples hitherto given to illustrate the general formulæ of this paper, it has been supposed for the sake of simplicity, that the function P is a cosine; and this supposition has been sufficient to deduce, as we have seen, a great variety of known results. But it is evident that this function P may receive many other forms, consistently with the suppositions made in deducing those general formulæ; and many new results may thus be obtained by the method of the foregoing articles.

For instance, it is permitted to suppose

$$\mathrm{P}_\alpha = 1, \text{ if } \alpha^2 < 1; \tag{k^{VII}}$$

$$\mathrm{P}_1 = 0; \tag{l^{VII}}$$

$$\mathrm{P}_{\alpha+2} = -\mathrm{P}_\alpha; \tag{m^{VII}}$$

and then the equations (t^{VI}) of the last article, with all that were deduced from them, will still hold good. We shall now have

$$p = 2; \tag{n^{VII}}$$

and the definite integral denoted by ϖ, and defined by the equation (r′), may now be computed as follows. Because the function N_α changes sign with α, we have

$$\varpi = 2 \int_0^\infty d\alpha \, \mathrm{N}_\alpha \alpha^{-1}; \tag{o^{VII}}$$

but

$$\left.\begin{array}{lll} \mathrm{N}_\alpha = \alpha, \text{ from } \alpha = 0 & \text{to } \alpha = 1; \\ \ldots 2 - \alpha, \qquad\qquad \ldots 1 & \ldots 3; \\ \ldots \alpha - 4, \qquad\qquad \ldots 3 & \ldots 4; \end{array}\right\} \tag{p^{VII}}$$

and

$$\mathrm{N}_{\alpha+4} = \mathrm{N}_\alpha. \tag{q^{VII}}$$

Hence

$$\int_0^4 d\alpha \, \mathrm{N}_\alpha \alpha^{-1} = 6 \log 3 - 4 \log 4, \tag{r^{VII}}$$

the logarithms being Napierian; and generally, if m be any positive integer number, or zero,

$$\begin{aligned} \int_{4m}^{4m+4} d\alpha \, \mathrm{N}_\alpha \alpha^{-1} &= \int_0^4 \mathrm{N}_\alpha (\alpha + 4m)^{-1} \\ &= 4m \log(4m) - (8m+2) \log(4m+1) \\ &\quad + (8m+6) \log(4m+3) - (4m+4) \log(4m+4) \\ &= \sum\nolimits_{(k)1}^{\infty} \frac{1 - 2^{-2k}}{k(k+\frac{1}{2})} (2m+1)^{-2k}. \end{aligned} \tag{s^{VII}}$$

But, by (h^V),

$$\sum\nolimits_{(m)0}^{\infty} (2m+1)^{-2k} = \tfrac{1}{2} \left(\frac{\pi}{2}\right)^{2k} \omega_{2k}, \tag{t^{VII}}$$

if k be any integer number > 0; therefore

$$\varpi = \sum\nolimits_{(k)1}^{\infty} \frac{1 - 2^{-2k}}{k(k+\frac{1}{2})} \left(\frac{\pi}{2}\right)^{2k} \omega_{2k}; \tag{u^{VII}}$$

ω_{2k} being by (q^V) the coefficient of x^{2k-1} in the development of $\tan x$. From this last property, we have

$$\sum\nolimits_{(k)1}^{\infty} \frac{\omega_{2k} x^{2k}}{k(k+\frac{1}{2})} = \frac{4}{x} \left(\int_0^x dx\right)^2 \tan x = \frac{4}{x} \int_0^x dx \, \log \sec x; \tag{v^{VII}}$$

therefore, substituting successively the values $x = \dfrac{\pi}{2}$ and $x = \dfrac{\pi}{4}$, and subtracting the result of the latter substitution from that of the former, we find, by (u^{VII}),

$$\begin{aligned}\varpi &= \frac{8}{\pi}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{4}}\right) dx\,\log\sec x\\ &= \frac{8}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx\,\log\tan x\\ &= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} dx\,\log\operatorname{cotan} x. \end{aligned} \qquad (\mathrm{w}^{VII})$$

Such, in the present question, is an expression for the constant ϖ; its numerical value may be approximately calculated by multiplying the Napierian logarithm of ten by the double of the average of the ordinary logarithms of the cotangents of the middles of any large number of equal parts into which the first octant may be divided; thus, if we take the ninetieth part of the sum of the logarithms of the cotangents of the ninety angles $\dfrac{1^\circ}{4}, \dfrac{3^\circ}{4}, \dfrac{5^\circ}{4}, \ldots \dfrac{177^\circ}{4}, \dfrac{179^\circ}{4}$, as given by the ordinary tables, we obtain nearly, as the average of these ninety logarithms, the number $0,5048$; of which the double, being multiplied by the Napierian logarithm of ten, gives, nearly, the number $2,325$, as an approximate value of the constant ϖ. But a much more accurate value may be obtained with little more trouble, by computing separately the doubles of the part (r^{VII}), and of the sum of (s^{VII}) taken from $m = 1$ to $m = \infty$; for thus we obtain the expression

$$\varpi = 12\log 3 - 8\log 4 + 2\sum\nolimits_{(k)1}^{\infty} \frac{1 - 2^{-2k}}{k(k+\frac{1}{2})} \sum\nolimits_{(m)1}^{\infty} (2m+1)^{-2k}, \qquad (\mathrm{x}^{VII})$$

in which each sum relative to m can be obtained from known results, and the sum relative to k converges tolerably fast; so that the second line of the expression (x^{VII}) is thus found to be nearly $= 0,239495$, while the first line is nearly $= 2,092992$; and the whole value of the expression (x^{VIII}) is nearly

$$\varpi = 2,332487. \qquad (\mathrm{y}^{VII})$$

There is even an advantage in summing the double of the expression (s^{VII}) only from $m = 2$ to $m = \infty$, because the series relative to k converges then more rapidly; and having thus found $2\displaystyle\int_8^\infty d\alpha\,\mathrm{N}_\alpha \alpha^{-1}$, it is only necessary to add thereto the expression

$$2\int_0^8 d\alpha\,\mathrm{N}_\alpha \alpha^{-1} = 12\log 3 - 20\log 5 + 28\log 7 - 16\log 8. \qquad (\mathrm{z}^{VII})$$

The form of the function P and the value of the constant ϖ being determined as in the present article, it is permitted to substitute them in the general equations of this paper; and thus to deduce new transformations for portions of arbitrary functions, which might have been employed instead of those given by FOURIER and POISSON, if the discontinuous function P, which receives alternately the values 1, 0, and -1, had been considered simpler in its properties than the trigonometrical function cosine.

[24.] Indeed, when the conditions (t^{VI}) are satisfied, the function P_x can be developed according to cosines of the odd multiples of $\dfrac{\pi x}{p}$, by means of the formula (y'''), which here becomes, by changing l to $\dfrac{p}{2}$, and f to P,

$$\mathrm{P}_x = \sum\nolimits_{(n)1}^{\infty} \mathrm{A}_{2n-1} \cos \frac{(2n-1)\pi x}{p}, \tag{a^{VIII}}$$

in which

$$\mathrm{A}_{2n-1} = \frac{4}{p} \int_0^{\frac{p}{2}} d\alpha \, \cos \frac{(2n-1)\pi\alpha}{p} \mathrm{P}_\alpha; \tag{b^{VIII}}$$

the function N_x at the same time admitting a development according to sines of the same odd multiples, namely

$$\mathrm{N}_x = \frac{p}{\pi} \sum\nolimits_{(n)1}^{\infty} \frac{\mathrm{A}_{2n-1}}{2n-1} \sin \frac{(2n-1)\pi x}{p}; \tag{c^{VIII}}$$

and the constant ϖ being equal to the following series,

$$\varpi = p \sum\nolimits_{(n)1}^{\infty} \frac{\mathrm{A}_{2n-1}}{2n-1}. \tag{d^{VIII}}$$

Thus, in the case of the last article, where $p = 2$, and $\mathrm{P}_\alpha = 1$ from $\alpha = 0$ to $\alpha = 1$, we have

$$\mathrm{A}_{2n-1} = \frac{4}{\pi} \frac{(-1)^{n+1}}{2n-1}; \tag{e^{VIII}}$$

$$\mathrm{P}_x = \frac{4}{\pi} \left(\cos \frac{\pi x}{2} - 3^{-1} \cos \frac{3\pi x}{2} + 5^{-1} \cos \frac{5\pi x}{2} - \cdots \right); \tag{f^{VIII}}$$

$$\mathrm{N}_x = \frac{8}{\pi^2} \left(\sin \frac{\pi x}{2} - 3^{-2} \sin \frac{3\pi x}{2} + 5^{-2} \sin \frac{5\pi x}{2} - \cdots \right); \tag{g^{VIII}}$$

$$\varpi = \frac{8}{\pi} (1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \cdots); \tag{h^{VIII}}$$

so that, from the comparision of (w^{VII}) and (h^{VIII}), the folowing relation results:

$$\int_0^{\frac{\pi}{4}} dx \, \log \cot x = \sum\nolimits_{(n)0}^{\infty} (-1)^n (2n+1)^{-2}. \tag{i^{VIII}}$$

But most of the suppositions made in former articles may be satisfied, without assuming for the function P the periodical form assigned by the conditions (t^{VI}). For example, we might assume

$$\mathrm{P}_\alpha = \frac{4}{\pi} \int_0^{\pi} d\theta \, \sin \theta^2 \cos(2\alpha \sin \theta); \tag{k^{VIII}}$$

which would give, by (a″) and (b″),

$$\text{N}_\alpha = \frac{2}{\pi}\int_0^\pi d\theta \,\sin\theta \sin(2\alpha\sin\theta); \tag{l^{VIII}}$$

$$\text{M}_\alpha = \frac{1}{\pi}\int_0^\pi d\theta \,\text{vers}(2\alpha\sin\theta); \tag{m^{VIII}}$$

and finally, by (r′)

$$\varpi = 2\int_0^\pi d\theta \,\sin\theta = 4. \tag{n^{VIII}}$$

This expression (k^{VIII}) for P_α satisfies all the conditions of the ninth article; for it is clear that it gives a value to N_α, which is numerically less than $\frac{4}{\pi}$; and the equation

$$\text{M}_\alpha = 1, \tag{o^{VIII}}$$

which is of the form (g), is satisfied by all the infinitely many real and unequal roots of the equation

$$\int_0^\pi d\theta \,(2\alpha\sin\theta) = 0, \tag{p^{VIII}}$$

which extend from $\alpha = -\infty$ to $\alpha = \infty$, and of which the interval between any one and the next following is never greater than π, nor even so great; because (as it is not difficult to prove) these several roots are contained in alternate or even octants, in such a manner that we may write

$$\alpha_n > \frac{n\pi}{2} - \frac{\pi}{4}, \quad < \frac{n\pi}{2}. \tag{q^{VIII}}$$

We may, therefore, substitute the expression (k^{VIII}) for P, in the formulæ (A), (B), (C), &c.; and we find, by (B), if $x > a$, $< b$,

$$f_x = \pi^{-1}\int_a^b d\alpha \int_0^\infty d\beta \int_0^\pi d\theta \,\sin\theta^2 \,\cos\{2\beta(\alpha - x)\sin\theta\} f_\alpha; \tag{r^{VIII}}$$

that is,

$$f_x = \frac{1}{2\pi}\lim_{\beta=\infty}\int_0^\pi d\theta \,\sin\theta \int_a^b d\alpha \,\sin\{2\beta(\alpha - x)\sin\theta\}(\alpha - x)^{-1} f_\alpha; \tag{s^{VIII}}$$

a theorem which may be easily proved *à posteriori*, by the principles of fluctuating functions, because those principles show, that (if x be comprised between the limits of integration) the limit relative to β of the integral relative to α, in (s^{VIII}), is equal to πf_x. In like manner, the theorem (C), when applied to the present form of the function P, gives the following other expression for the arbitrary function f_x:

$$f_x = \tfrac{1}{2}\int_a^b d\alpha \, f_\alpha + \sum\nolimits_{(n)1}^{\infty}\int_a^b d\alpha \, f_\alpha \frac{\int_0^\pi d\theta \,\sin\theta \,\sin(2(\alpha - x)\sin\theta)\cos(4n(\alpha - x)\sin\theta)}{\int_0^\pi d\theta \,\sin\theta \,\sin(2(\alpha - x)\sin\theta)}; \tag{t^{VIII}}$$

x being between a and b, and $b - a$ being not greater than the least positive root ν of the equation

$$\frac{1}{\nu}\int_0^\pi d\theta\,\sin\theta\sin(2\nu\sin\theta) = 0. \qquad (\mathrm{u}^{VIII})$$

And if we wish to prove, *à posteriori*, this theorem of transformation (t^{VIII}), by the same principles of fluctuating functions, we have only to observe that

$$1 + 2\sum\nolimits_{(n)1}^{\infty}\cos 2ny = \frac{\sin(2ny + y)}{\sin y}, \qquad (\mathrm{v}^{VIII})$$

and therefore that the second member of (t^{VIII}) may be put under the form

$$\lim_{n=\infty}\int_a^b d\alpha\, f_\alpha \frac{\displaystyle\int_0^\pi d\theta\,\sin\theta\,\sin((4n+2)(\alpha - x)\sin\theta)}{\displaystyle 2\int_0^\pi d\theta\,\sin\theta\,\sin(2(\alpha - x)\sin\theta)}; \qquad (\mathrm{w}^{VIII})$$

in which the presence of the fluctuating factor

$$\sin((4n+2)(\alpha - x)\sin\theta),$$

combined with the condition that $\alpha - x$ is numerically less than the least root of the equation (u^{VIII}), shows that we need only attend to values of α indefinitely near to x, and may therefore write in the denominator,

$$\int_0^\pi d\theta\,\sin\theta\,\sin(2(\alpha - x)\sin\theta) = \pi(\alpha - x); \qquad (\mathrm{x}^{VIII})$$

for thus, by inverting the order of the two remaining integrations, that is by writing

$$\int_a^b d\alpha\int_0^\pi d\theta\,\ldots = \int_0^\pi d\theta\int_a^b d\alpha\,\ldots, \qquad (\mathrm{y}^{VIII})$$

we find first

$$\lim_{n=\infty}\int_a^b d\alpha\, f_\alpha\frac{\sin((4n+2)(\alpha - x)\sin\theta)}{2\pi(\alpha - x)} = \tfrac{1}{2}f_x, \qquad (\mathrm{z}^{VIII})$$

for every value of θ between 0 and π, and of x between a and b; and finally,

$$\tfrac{1}{2}f_x\int_0^\pi d\theta\,\sin\theta = f_x.$$

[25.] The results of the foregoing articles may be extended by introducing, under the functional signs N, P, a product such as $\beta\gamma$, instead of $\beta\alpha$, γ being an arbitrary function of α; and by considering the integral

$$\int_a^b d\alpha\,\mathrm{N}_{\beta\gamma}\mathrm{F}_\alpha, \qquad (\mathrm{a}^{IX})$$

in which F is any function which remains finite between the limits of integration. Since γ is a function of α, it may be denoted by γ_α, and α will be reciprocally a function of γ, which may be denoted thus:

$$\alpha = \phi_{\gamma_\alpha}. \tag{b^{IX}}$$

While α increases from a to b, we shall suppose, at first, that the function γ_α increases constantly and continuously from γ_a to γ_b, in such a manner as to give always, within this extent of variation, a finite and determined and positive value to the differential coefficient of the function ϕ, namely,

$$\frac{d\alpha}{d\gamma} = \phi'_\gamma. \tag{c^{IX}}$$

We shall also express, for abridgment, the product of this coefficient and of the function F by another function of γ, as follows,

$$\phi'_\gamma \mathrm{F}_\alpha = \psi. \tag{d^{IX}}$$

Then the integral (a^{IX}) becomes

$$\int_{\gamma_a}^{\gamma_b} d\gamma\, \mathrm{N}_{\beta\gamma}\psi_\gamma; \tag{e^{IX}}$$

and a rigorous expression for it may be obtained by the process of the fourth article, namely

$$\left(\int_{\gamma_a}^{\beta^{-1}\alpha_n} + \int_{\beta^{-1}\alpha_{n+m}}^{\gamma_b}\right) d\gamma\, \mathrm{N}_{\beta\gamma}\psi_\gamma + \theta\beta^{-1}(\alpha_{n+m} - \alpha_n)\mathrm{c}\delta; \tag{f^{IX}}$$

in which, as before, α_n, α_{n+m} are suitably chosen roots of the equation (g); c is a finite constant; θ is included between the limits ± 1; and δ is the difference between two values of the function ψ_γ, corresponding to two values of the variable γ of which the difference is less than β^{-1}b, b being another finite constant. The integral (a^{IX}) therefore diminishes indefinitely when β increases indefinitely; and thus, or simply by the theorem (Z) combined with the expression (e^{IX}), we have, rigorously, at the limit, without supposing here that N_0 vanishes,

$$\int_a^b d\alpha\, \mathrm{N}_{\infty\gamma}\mathrm{F}_\alpha = 0. \tag{W}$$

The same conclusion is easily obtained, by reasonings almost the same, for the case where γ continually decreases from γ_a to γ_b, in such a manner as to give, within this extent of variation, a finite and determined and negative value to the differential coefficient (c^{IX}). And with respect to the case where the function γ is for a moment stationary in value, so that its differential coefficient vanishes between the limits of integration, it is sufficient to observe that although ψ in (e^{IX}) becomes then infinite, yet F in (a^{IX}) remains finite, and the integral of the finite product $d\alpha\, \mathrm{N}_{\beta\gamma}\mathrm{F}_\alpha$, taken between infinitely near limits, is zero. Thus, generally, the theorem (W), which is an extension of the theorem (Z), holds good between any finite limits a and b, if the function F be finite between those limits, and if, between the same limits of integration, the function γ never remain unchanged throughout the whole extent of any finite change of α.

[26.] It may be noticed here, that if β be only very large, instead of being infinite, an approximate expression for the integral (a^{IX}) may be obtained, on the same principles, by attending only to values of α which differ very little from those which render the coefficient (c^{IX}) infinite. For example, if we wish to find an approximate expression for a large root of the equation (p^{VIII}), or to express approximately the function

$$f_\beta = \frac{1}{\pi} \int_0^\pi d\alpha \, \cos(2\beta \sin\alpha), \tag{g^{IX}}$$

when β is a large positive quantity, we need only attend to values of α which differ little from $\frac{\pi}{2}$; making then

$$\sin\alpha = 1 - y^2, \quad d\alpha = \frac{2\,dy}{\sqrt{2-y^2}}, \tag{h^{IX}}$$

and neglecting y^2 in the denominator of this last expression, the integral (g^{IX}) becomes

$$f_\beta = \mathrm{A}_\beta \cos 2\beta + \mathrm{B}_\beta \sin 2\beta, \tag{i^{IX}}$$

in which, nearly,

$$\left.\begin{aligned} \mathrm{A}_\beta &= \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} dy \, \cos(2\beta y^2) = \frac{1}{\sqrt{2\pi\beta}}; \\ \mathrm{B}_\beta &= \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} dy \, \sin(2\beta y^2) = \frac{1}{\sqrt{2\pi\beta}}; \end{aligned}\right\} \tag{k^{IX}}$$

so that the large values of β which make the function (g^{IX}) vanish are nearly of the form

$$\frac{n\pi}{2} - \frac{\pi}{8}, \tag{l^{IX}}$$

n being an integer number; and such is therefore the approximate form of the large roots α_n of the equation (p^{VIII}): results which agree with the relations (q^{VIII}), and to which Poisson has been conducted, in connexion with another subject, and by an entirely different analysis.

The theory of fluctuating functions may also be employed to obtain a more close approximation; for instance, it may be shown, by reasonings of the kind lately employed, that the definite integral (g^{IX}) admits of being expressed (more accurately as β is greater) by the following semiconvergent series, of which the first terms have been assigned by Poisson:

$$f_\beta = \frac{1}{\sqrt{\pi\beta}} \sum\nolimits_{(i)1}^{\infty} [0]^{-i} ([-\tfrac{1}{2}]^i)^2 (4\beta)^{-i} \cos\left(2\beta - \frac{\pi}{4} - \frac{i\pi}{2}\right); \tag{m^{IX}}$$

and in which, according to a known notation of factorials,

$$\left.\begin{aligned} [0]^{-i} &= 1^{-1} \,.\, 2^{-1} \,.\, 3^{-1} \ldots . \, i^{-1}; \\ [-\tfrac{1}{2}]^i &= \frac{-1}{2} \,.\, \frac{-3}{2} \,.\, \frac{-5}{2} \cdots \frac{1-2i}{2}. \end{aligned}\right\} \tag{n^{IX}}$$

For the value $\beta = 20$, the 3 first terms of the series (m^{IX}) give

$$\left.\begin{aligned} f_{20} &= \left(1 - \frac{9}{204800}\right) \frac{\cos 86^\circ\, 49'\, 52''}{\sqrt{20\pi}} + \frac{1}{320} \frac{\sin 86^\circ\, 49'\, 52''}{\sqrt{20\pi}} \\ &= 0,0069736 + 0,0003936 = +0,0073672. \end{aligned}\right\} \tag{o^{IX}}$$

For the same value of β, the sum of the first sixty terms of the ultimately convergent series

$$f_\beta = \sum\nolimits_{(i)0}^{\infty} ([0]^{-i})^2 (-\beta^2)^i \tag{p^{IX}}$$

gives

$$\left.\begin{aligned} f_{20} = {} & +7\,447\,387\,396\,709\,949,9657957 \\ & -7\,447\,387\,396\,709\,949,9584289 \\ & = +0,0073668. \end{aligned}\right\} \tag{q^{IX}}$$

The two expressions (m^{IX}) (p^{IX}) therefore agree, and we may conclude that the following numerical value is very nearly correct:

$$\frac{1}{\pi} \int_0^\pi d\alpha\, \cos(40 \sin \alpha) = +0,007367. \tag{r^{IX}}$$

[27.] Resuming the rigorous equation (W), and observing that

$$\int_0^\infty d\beta\, \mathrm{P}_{\beta\gamma} = \lim_{\beta=\infty} .\mathrm{N}_{\beta\gamma} \gamma_\alpha^{-1}, \tag{s^{IX}}$$

we easily see that in calculating the definite integral

$$\int_a^b d\alpha \int_0^\beta d\beta\, \mathrm{P}_{\beta\gamma} f_\alpha, \tag{t^{IX}}$$

in which the function f is finite, it is sufficient to attend to those values of α which are not only between the limits a and b, but are also very nearly equal to real roots x of the equation

$$\gamma_x = 0. \tag{u^{IX}}$$

The part of the integral (t^{IX}), corresponding to values of α in the neighbourhood of any one such root x, between the above-mentioned limits, is equal to the product

$$\frac{f_x}{\gamma'_x} \times \int_{-\infty}^{\infty} d\alpha\, \frac{\mathrm{N}_{\beta\gamma'_x(\alpha - x)}}{\alpha - x}, \tag{v^{IX}}$$

in which β is indefinitely large and positive, and the differential coefficient γ'_x of the function γ is supposed to be finite, and different from 0. A little consideration shows that the integral in this last expression is $= \pm\varpi$, ϖ being the same constant as in former articles, and the upper or lower sign being taken according as γ'_x is positive or negative. Denoting then by $\sqrt{\gamma'^2_x}$ the positive quantity, which is $= +\gamma'_x$ or $= -\gamma'_x$, according as γ'_x is > 0 or < 0, the part (v^{IX}) of the integral (t^{IX}) is

$$\frac{\varpi f_x}{\sqrt{\gamma'^2_x}}; \tag{w^{IX}}$$

and we have the expression

$$\int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta\gamma} f_\alpha = \varpi \sum_x \frac{f_x}{\sqrt{\gamma'^2_x}}, \tag{x^{IX}}$$

the sum being extended to all those roots x of the equation (u^{IX}) which are $> a$ but $< b$. If any root of that equation should coincide with either of these limits a or b, the value of α in its neighbourhood would introduce, into the second member of the expression (x^{IX}), one or other of the terms

$$\frac{\varpi' f_a}{\gamma'_a}, \quad \frac{-\varpi'' f_a}{\gamma'_a}, \quad \frac{\varpi'' f_b}{\gamma'_b}, \quad \frac{-\varpi' f_b}{\gamma'_b}; \tag{y^{IX}}$$

the first to be taken when $\gamma_a = 0$, $\gamma'_a > 0$; the second when $\gamma_a = 0$, $\gamma'_a < 0$; the third when $\gamma_b = 0$, $\gamma'_b > 0$; and the fourth when $\gamma_b = 0$, $\gamma'_b < 0$. If, then, we suppose for simplicity, that neither γ_a nor γ_b vanishes, the expression (x^{IX}) conducts to the theorem

$$\sum\nolimits_x f_x = \varpi^{-1} \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta\gamma} f_\alpha \sqrt{\gamma'^2_\alpha}; \tag{X}$$

and the sign of summation may be omitted, if the equation $\gamma_x = 0$ have only one real root between the limits a and b. For example, that one root itself may then be expressed as follows:

$$x = \varpi^{-1} \int_a^b d\alpha \int_0^\infty d\beta\, \mathrm{P}_{\beta\gamma} \alpha \sqrt{\gamma'^2_\alpha}; \tag{z^{IX}}$$

The theorem (X) includes some analogous results which have been obtained by CAUCHY, for the case when P is a cosine.

[28.] It is also possible to extend the foregoing theorem in other ways; and especially be applying similar reasonings to functions of several variables. Thus, if $\gamma, \gamma^{(1)} \ldots$ be each a function of several real variables $\alpha, \alpha^{(1)} \ldots$; if P and N be still respectively functions of the kinds supposed in former articles, while $\mathrm{P}^{(1)}$, $\mathrm{N}^{(1)}, \ldots$ are other functions of the same kinds; then the theorem (W) may be extended as follows:

$$\int_a^b d\alpha \int_{a^{(1)}}^{b^{(1)}} d\alpha^{(1)} \ldots \mathrm{N}_{\infty\gamma} \mathrm{N}^{(1)}_{\infty\gamma^{(1)}} \ldots \mathrm{F}_{\alpha,\alpha^{(1)},\ldots} = 0, \tag{Y}$$

the function F being finite for all values of the variables $\alpha, \alpha^{(1)}, \ldots$, within the extent of the integrations; and the theorem (X) may be thus extended:

$$\left.\begin{aligned} \sum f_{x,x^{(1)},\ldots} = \varpi^{-1}\varpi^{(1)-1} \ldots \int_a^b d\alpha \int_{a^{(1)}}^{b^{(1)}} d\alpha^{(1)} \ldots \int_0^\infty d\beta \int_0^\infty d\beta^{(1)} \ldots \mathrm{P}_{\beta\gamma} \mathrm{P}^{(1)}_{\beta^{(1)}\gamma^{(1)}} \ldots \\ \ldots f_{\alpha,\alpha^{(1)},\ldots} \sqrt{\mathrm{L}^2}; \end{aligned}\right\} \tag{Z}$$

in which, according to the analogy of the foregoing notation,

$$\varpi^{(i)} = \int_{-\infty}^{\infty} d\alpha \int_{0}^{1} d\beta \, \mathrm{P}_{\beta\alpha}^{(i)}; \tag{a^X}$$

and L is the coefficient which enters into the expression, supplied by the principles of the transformation of multiple integrals,

$$\mathrm{L}\, d\alpha\, d\alpha^{(1)} \, \ldots = d\gamma\, d\gamma^{(1)} \, \ldots; \tag{b^X}$$

while the summation in the first member is to be extended to all those values of $x, x^{(1)}, \ldots$ which, being respectively between the respective limits of integration relatively to the variables $\alpha, \alpha^{(1)}, \ldots$ are values of those variables satisfying the system of equations

$$\gamma_{x,x^{(1)},\ldots} = 0, \quad \gamma^{(1)}_{x,x^{(1)},\ldots} = 0, \ldots. \tag{c^X}$$

And thus may other remarkable results of CAUCHY be presented under a generalized form. But the theory of such extensions appears likely to suggest itself easily enough to any one who may have considered with attention the remarks already made; and it is time to conclude the present paper by submitting a few general observations on the nature and history of this important branch of analysis.

LAGRANGE appears to have been the first who was led (in connexion with the celebrated problem of vibrating cords) to assign, as the result of a species of interpolation, an expression for an arbitrary function, continuous or discontinuous in form, between any finite limits, by a series of sines of multiples, in which the coefficients are definite integrals. Analogous expressions, for a particular class of rational and integral functions, were derived by DANIEL BERNOUILLI, through successive integrations, from the results of certain trigonometric summations, which he had characterized in a former memoir as being *incongruously true.* No farther step of importance towards the improvement of this theory seems to have been made, till FOURIER, in his researches on Heat, was led to the discovery of his well-known theorem, by which any arbitrary function of any real variable is expressed, between finite or infinite limits, by a double definite integral. POISSON and CAUCHY have treated the same subject since, and enriched it with new views and applications; and through the labours of these, and, perhaps, of other writers, the theory of the development or transformation of arbitrary functions, through functions of determined forms, has become one of the most important and interesting departments of modern algebra.

It must, however, be owned that some obscurity seems still to hang over the subject, and that a farther examination of its principles may not be useless or unnecessary. The very existence of such transformations as in this theory are sought for and obtained, appears at first sight paradoxical; it is difficult at first to conceive the possibility of expressing a perfectly arbitrary function through any series of sines or cosines; the variable being thus made the subject of known and determined operations, whereas it had offered itself originally as the subject of operations unknown and undetermined. And even after this first feeling of paradox is removed, or relieved, by the consideration that the number of the operations of known form is infinite, and that the operation of arbitrary form reappears in another part of the expression, as performed on an auxiliary variable; it still requires attentive consideration to see clearly how it is possible that none of the values of this new variable should have any influence on the final result, except those which are extremely nearly equal to the variable originally proposed. This latter difficulty has not, perhaps, been removed to the complete satisfaction of those who desire to examine the question with all the diligence its importance deserves, by any of the published works upon the subject. A conviction, doubtless, may be attained, that the results are true, but something is, perhaps, felt to be still wanting for the full rigour of mathematical demonstration. Such has, at least, been the impression left on the mind of the present writer, after an attentive study of the reasonings usually employed, respecting the transformations of arbitrary functions.

Poisson, for example, in treating this subject, sets out, most commonly, with a series of cosines of multiple arcs; and because the sum is generally indeterminate, when continued to infinity, he alters the series by multiplying each term by the corresponding power of an auxiliary quantity which he assumes to be less than unity, in order that its powers may diminish, and at last vanish; but in order that the new series may tend indefinitely to coincide with the old one, he conceives, after effecting its summation, that the auxiliary quantity tends to become unity. The limit thus obtained is generally zero, but becomes on the contrary infinite when the arc and its multiples vanish; from which it is inferred by Poisson, that if this arc be the difference of two variables, an original and an auxiliary, and if the series be multiplied by any arbitrary function of the latter variable, and integrated with respect thereto, the effect of all the values of that variable will disappear from the result, except the effect on those which are extremely nearly equal to the variable originally proposed.

Poisson has made, with consummate skill, a great number of applications of this method; yet it appears to present, on close consideration, some difficulties of the kind above alluded to. In fact, the introduction of the system of factors, which tend to vanish before the integration, as their indices increase, but tend to unity, after the integration, for all finite values of those indices, seems somewhat to change the nature of the question, by the introduction of a foreign element. Nor is it perhaps manifest that the original series, of which the sum is indeterminate, may be replaced by the convergent series with determined sum, which results from multiplying its terms by the powers of a factor infinitely little less than unity; while it is held that to multiply by the powers of a factor infinitely greater than unity would give an useless or even false result. Besides there is something unsatisfactory in employing an apparently arbitrary contrivance for annulling the effect of those terms of the proposed series which are situated at a great distance from the origin, but which do not themselves originally tend to vanish as they become more distant therefrom. Nor is this difficulty entirely removed, when integration by parts is had recourse to, in order to show that the effect of these distant terms is insensible in the ultimate result; because it then becomes necessary to differentiate the arbitrary function; but to treat its differential coefficient as always finite, is to diminish the generality of the inquiry.

Many other processes and proofs are subject to similar or different difficulties; but there is one method of demonstration employed by FOURIER, in his separate Treatise on Heat, which has, in the opinion of the present writer, received less notice than it deserves, and of which it is proper here to speak. The principle of the method here alluded to may be called the *Principle of Fluctuation*, and is the same which was enunciated under that title in the remarks prefixed to this paper. In virtue of this principle (which may thus be considered as having been indicated by FOURIER, although not expressly stated by him), if any function, such as the sine or cosine of an infinite multiple of an arc, changes sign infinitely often within a finite extent of the variable on which it depends, and has for its mean value zero; and if this, which may be called a *fluctuating function*, be multiplied by any arbitrary but finite function of the same variable, and afterwards integrated between any finite limits; the integral of the product will be zero, on account of the mutual destruction or neutralization of all its elements.

It follows immediately from this principle, that if the factor by which the fluctuating function is multiplied, instead of remaining always finite, becomes infinite between the limits of integration, for one or more particular values of the variable on which it depends; it is then only necessary to attend to values in the immediate neighbourhood of these, in order to obtain the value of the integral. And in this way FOURIER has given what seems to be the most satisfactory published proof, and (so to speak) the most natural explanation of the theorem called by his name; since it exhibits the actual process, one might almost say the interior mechanism, which, in the expression assigned by him, destroys the effect of all those values of the auxiliary variable which are not required for the result. So clear, indeed, is this conception, that it admits of being easily translated into geometrical constructions, which have accordingly been used by FOURIER for that purpose.

There are, however, some remaining difficulties connected with this mode of demonstration, which may perhaps account for the circumstance that it seems never to be mentioned, nor alluded to, in any of the historical notices which POISSON has given on the subject of these transformations. For example, although FOURIER, in the proof just referred to, of the theorem called by his name, shows clearly that in integrating the product of an arbitrary but finite function, and the sine or cosine of an infinite multiple, each successive positive portion of the integral is destroyed by the negative portion which follows it, if infinitely small quantities be neglected, yet he omits to show that the infinitely small outstanding difference of values of the positive and negative portions, corresponding to the single period of the trigonometrical function introduced, is of the second order; and, therefore, a doubt may arise whether the infinite number of such infinitely small periods, contained in any finite interval, may not produce, by their accumulation, a finite result. It is also desirable to be able to state the argument in the language of limits, rather than that of infinitesimals, and to exhibit, by appropriate definitions and notations, what was evidently foreseen by FOURIER, that the result depends rather on the *fluctuating* than on the *trigonometric* character of the auxiliary function employed.

The same view of the question had occurred to the present writer, before he was aware that indications of it were to be found among the published works of FOURIER; and he still conceives that the details of the demonstration to which he was thus led may be not devoid of interest and utility, as tending to give greater rigour and clearness to the proof and the conception of a widely applicable and highly remarkable theorem.

Yet, if he did not suppose that the present paper contains something more than a mere expansion or improvement of a known proof of a known result, the Author would scarcely have ventured to offer it to the Transactions* of the Royal Irish Academy. It aims not merely to give a more perfectly satisfactory demonstration of FOURIER'S celebrated theorem than any which the writer has elsewhere seen, but also to present that theorem, and many others analogous thereto, under a greatly generalized form, deduced from the principle of fluctuation. Functions more general than sines or cosines, yet having some correspondent properties, are introduced throughout; and constants, distinct from the ratio of the circumference to the diameter of a circle, present themselves in connexion therewith. And thus, if the intention of the writer have been in any degree accomplished, it will have been shown, according to the opinion expressed in the remarks prefixed to this paper, that the development of the important principle above referred to gives not only a new clearness, but also (in some respects) a new extension, to this department of science.

* The Author is desirous to acknowledge, that since the time of his first communicating the present paper to the Royal Irish Academy, in June, 1840, he has had an opportunity of entirely rewriting it, and that the last sheet is only now passing through the press, in June, 1842. Yet it may be proper to mention that the theorems (A) (B) (C), which sufficiently express the character of the communication, were printed (with some slight differences of notation) in the year 1840, as part of the *Proceedings* of the Academy for the date prefixed to this paper.