ON SOME EXTENSIONS OF QUATERNIONS

By

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NOTE ON THE TEXT

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The headings '*Section* I.', '*Section* II.' and '*Section* III.' were not included for the first three sections of the original text, but analogous headings were included for the final two sections.

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On some Extensions of Quaternions. By* Sir William Rowan Hamilton, *LL.D., M.R.I.A., F.R.A.S., Corresponding Member of the French Institute, Hon. or Corr. Member of several other Scientific Societies in British and Foreign Countries, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland†.*

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Section I.

[1.] Conceive that in the polynomial expressions,

$$\left.\begin{aligned} \mathrm{P} &= \iota_0 x_0 + \iota_1 x_1 + \ldots + \iota_n x_n = \Sigma\iota x, \\ \mathrm{P}' &= \iota_0 x_0' + \iota_1 x_1' + \ldots + \iota_n x_n' = \Sigma\iota x', \\ \mathrm{P}'' &= \iota_0 x_0'' + \iota_1 x_1'' + \ldots + \iota_n x_n'' = \Sigma\iota x'', \end{aligned}\right\} \tag{1}$$

the symbols $x_0 \ldots x_n$, which we shall call the *constituents* of the polynome P, and in like manner that the constituents $x_0' \ldots x_n'$ of P′, and $x_0'' \ldots x_n''$ of P″, are subject to all the usual rules of algebra, and to no others; but that the other symbols $\iota_0 \ldots \iota_n$, by which those constituents of each polynome are here symbolically multiplied, are not all subject to all those usual rules: and that, on the contrary, these latter symbols are subject, *as a system*, to some *peculiar laws*, of comparison and combination. More especially, let us conceive, in the first place, that these $n+1$ symbols, of the form ι_f, are and must remain unconnected with each other by any *linear* relation, with ordinary algebraical coefficients; whence it will follow that an *equality* between any *two* polynomial expressions of the present class requires that *all* their *corresponding constituents* should be *separately* equal, or that

$$\text{if} \quad \mathrm{P}' = \mathrm{P}, \quad \text{then} \quad x_0' = x_0,\ x_1' = x_1, \ldots\ x_n' = x_n : \tag{2}$$

and therefore, in particular, that the *evanescence* of any *one* such polynome P requires the vanishing of *each* constituent separately; so that

$$\text{if} \quad \mathrm{P} = 0, \quad \text{then} \quad x_0 = 0,\ x_1 = 0, \ldots\ x_n = 0. \tag{3}$$

* See the work entitled, "Lectures on Quaternions," by the present writer. (Hodges and Smith, Dublin, 1853.)

† Communicated by the Author.

In the second place, we shall suppose that all the usual rules of *addition* and *subtraction* extend to these new polynomes, and to their terms; and that the symbols ι, like the symbols x, are *distributive* in their operation; whence it will follow that

$$\mathrm{P}' \pm \mathrm{P} = \iota_0(x'_0 \pm x_0) + \ldots + \iota_n(x'_n \pm x_n), \tag{4}$$

or that

$$\Sigma\iota x' \pm \Sigma\iota x = \Sigma\iota(x' \pm x): \tag{5}$$

and as a further connexion with common algebra, we shall conceive that each separate symbol of the form ι may combine commutatively as a factor with each of the form x, and with every other algebraic quantity, so that $\iota x = x\iota$, and that therefore the polynome P may be thus written,

$$\mathrm{P} = x_0\iota_0 + x_1\iota_1 + \ldots + x_n\iota_n = \Sigma x\iota. \tag{6}$$

But, third, *instead* of supposing that the symbols ι combine thus in general *commutatively, among themselves*, as factors or as operators, we shall *distinguish generally* between the two *inverted* (*or opposite*) *products*, $\iota\iota'$ and $\iota'\iota$, or $\iota_f\iota_g$ and $\iota_g\iota_f$; and shall conceive that all the $(n+1)^2$ *binary* products $(\iota\iota')$, *including squares* $(\iota^2 = \iota\iota)$, of the $n+1$ symbols ι, are *defined* as being each equal to a certain *given* or originally *assumed polynome*, of the general form (1), by $(n+1)^2$ equations of the following type,

$$\iota_f\iota_g = (fg0)\iota_0 + (fg1)\iota_1 + \ldots + (fgh)\iota_h + \ldots + (fgn)\iota_n; \tag{7}$$

the $(n+1)^3$ coefficients, or constituents, of the form (fgh), which we shall call the "constants of multiplication," being so many given, or assumed, algebraic constants, of which some may vanish, and which we do not here suppose to satisfy *generally* the relation, $(fgh) = (gfh)$. And thus the *product of any two given polynomes*, P and P$'$, of the form (1), combined in a *given order* as factors, becomes equal to a *third given polynome*, P$''$, of the same general form,

$$\mathrm{P}'' = \mathrm{PP}' = \Sigma x_f\iota_f \,.\, \Sigma x'_g\iota_g = \Sigma x''_h\iota_h; \tag{8}$$

the summations extending still from 0 to n, and the constituent x''_h of the product admitting of being thus expressed:

$$x''_h = \Sigma(fgh)x_fx'_g. \tag{9}$$

As regards the subjection of the symbols ι to the *associative* law of multiplication, expressed by the formula,

$$\iota \,.\, \iota'\iota'' = \iota\iota' \,.\, \iota'',$$

we shall make no supposition at present.

[2.] As a first simplification of the foregoing very general* conception, let it be now supposed that

$$\iota_0 = 1; \tag{10}$$

* Some account of a connected conception respecting *Sets*, considered as including Quaternions, may be found in the Preface to the Lectures already cited.

the n other symbols, $\iota_1, \iota_2, \ldots \iota_n$, being thus the only ones which are not subject to all the ordinary rules of algebra. Then because

$$\iota_0 \iota_g = \iota_g, \quad \iota_f \iota_0 = \iota_f, \tag{11}$$

it will follow that if either of the two indices f and g be $= 0$, the constant of multiplication (fgh) is either $= 1$ or $= 0$, according as h is equal or unequal to the other of those two indices; and we may write,

$$(0fh) = (f0h) = 0, \quad \text{if} \quad h \gtrless f; \tag{12}$$

$$(0ff) = (f0f) = 1. \tag{13}$$

With this simplification, the *number* of the arbitrary or disposable constants of the form (fgh), which are not thus known already to have the value 0 or 1, is reduced from $(n+1)^3$ to $(n+1)n^2$; because we may now suppose that f and g are each > 0, or that they vary only from 1 to n. For we may write,

$$\mathrm{P} = p + \varpi, \quad \mathrm{P}' = p' + \varpi', \tag{14}$$

where

$$\left.\begin{array}{ll} p = \iota_0 x_0 = x_0, & \varpi = \iota_1 x_1 + \ldots + \iota_f x_f + \ldots + \iota_n x_n, \\ p' = \iota_0 x'_0 = x'_0, & \varpi' = \iota_1 x'_1 + \ldots + \iota_g x'_g + \ldots + \iota_n x'_n; \end{array}\right\} \tag{15}$$

and then, by observing that p and p' are symbols of the usual and algebraical kind, shall have this expression for the product of two polynomes:

$$\mathrm{P}'' = \mathrm{PP}' = (p + \varpi)(p' + \varpi') = pp' + p\varpi' + p'\varpi + \varpi\varpi'; \tag{16}$$

where the last term, or partial product, $\varpi\varpi'$, is now the only one for which any peculiar rules are required.

[3.] When the polynome P has thus been decomposed into two parts, p and ϖ, of which the one (p) is subject to all the usual rules of algebraical calculation, but the other (ϖ) to peculiar rules; and when these two parts are thus in such a sense *heterogeneous*, that an *equation* between two such polynomes resolves itself immediately into *two* separate equations, one between parts of the one kind, and the other between parts of the other kind; so that

$$\text{if} \quad \mathrm{P} = \mathrm{P}', \quad \text{or} \quad p + \varpi = p' + \varpi', \quad \text{then} \quad p = p, \quad \text{and} \quad \varpi = \varpi'; \tag{17}$$

we shall call the former part (p) the *scalar part*, or simply THE SCALAR, of the polynome P, and shall denote it, as such, by the symbol S . P, or SP; and we shall call the latter part (ϖ) the *vector part*, or simply THE VECTOR, of the same polynome, and shall denote this other part by the symbol V . P, or VP: these *names* (scalar and vector), and these *characteristics* (S and V), being here adopted as an extension of the phraseology and notation of the Calculus of Quaternions*, in which such scalars and vectors receive useful geometrical interpretations.

* See Lectures, *passim.*

From the same calculus we shall here borrow also the conception and the sign of *conjugation*; and shall say that any two polynomes (such as those represented by $p + \varpi$ and $p - \varpi$) are CONJUGATE, if they have *equal scalars* (p), but *opposite vectors* ($\pm\varpi$): and if either of these two polynomes be denoted by P, then the symbol K.P, or KP, shall be employed to represent the other; K being thus used (as in quaternions) as the *characteristic of conjunction.* With these notations, and with the recent significations of p and ϖ,

$$p = \mathrm{S}(p + \varpi), \quad \varpi = \mathrm{V}(p + \varpi), \quad p - \varpi = \mathrm{K}(p + \varpi); \tag{18}$$

or, writing P and P′ for $p + \varpi$ and $p - \varpi$,

$$\mathrm{P}' = \mathrm{KP}, \quad \text{if} \quad \mathrm{SP}' = \mathrm{SP}, \quad \text{and} \quad \mathrm{VP}' = -\mathrm{VP}; \tag{19}$$

and generally, for any polynome P, of the kind here considered,

$$\mathrm{P} = \mathrm{SP} + \mathrm{VP}, \quad \mathrm{KP} = \mathrm{SP} - \mathrm{VP}. \tag{20}$$

We may also propose to call the n symbols $\iota_1 \ldots \iota_n$ by the general name of VECTOR-UNITS, as the symbol ι_0 has been equated in (10) to the SCALAR-UNIT, or to 1; and may call that equation (10) the UNIT-LAW, or more fully, the *law of the primary unit.*

[4.] Already, from these few definitions and notations, a variety of symbolical consequences can be deduced, which have indeed already occurred in the Calculus of Quaternions, but which are here taken with enlarged significations, and without reference to interpretation in geometry. For example, in the general equations (20), we may *abstract from the operand*, that is, from the polynome P, and may write more briefly (as in quaternions),

$$1 = \mathrm{S} + \mathrm{V}, \quad \mathrm{K} = \mathrm{S} - \mathrm{V}; \tag{21}$$

whence

$$\mathrm{S} = \tfrac{1}{2}(1 + \mathrm{K}), \quad \mathrm{V} = \tfrac{1}{2}(1 - \mathrm{K}); \tag{22}$$

or more fully,

$$\mathrm{SP} = \tfrac{1}{2}(\mathrm{P} + \mathrm{P}'), \quad \mathrm{VP} = \tfrac{1}{2}(\mathrm{P} - \mathrm{P}'), \quad \text{if} \quad \mathrm{P}' = \mathrm{KP}. \tag{23}$$

Again, since (with the recent meanings of p and ϖ),

$$\left.\begin{array}{l} \mathrm{S}p = p, \quad \mathrm{V}p = 0, \quad \mathrm{K}p = p, \quad \mathrm{S}\varpi = 0, \quad \mathrm{V}\varpi = \varpi, \quad \mathrm{K}\varpi = -\varpi, \\ \mathrm{S}(p - \varpi) = p, \quad \mathrm{V}(p - \varpi) = -\varpi, \quad \mathrm{K}(p - \varpi) = p + \varpi, \end{array}\right\} \tag{24}$$

we may write

$$\left.\begin{array}{l} \mathrm{SSP} = \mathrm{SP}, \quad \mathrm{VSP} = 0 = \mathrm{SVP}, \quad \mathrm{VVP} = \mathrm{VP}, \\ \mathrm{SKP} = \mathrm{SP} = \mathrm{KSP}, \quad \mathrm{VKP} = -\mathrm{VP} = \mathrm{KVP}, \quad \mathrm{KKP} = \mathrm{P}; \end{array}\right\} \tag{25}$$

or more concisely,

$$\left.\begin{array}{l} \mathrm{S}^2 = \mathrm{S}, \quad \mathrm{VS} = \mathrm{SV} = 0, \quad \mathrm{V}^2 = \mathrm{V}, \\ \mathrm{SK} = \mathrm{KS} = \mathrm{S}, \quad \mathrm{VK} = \mathrm{KV} = -\mathrm{V}, \quad \mathrm{K}^2 = 1. \end{array}\right\} \tag{26}$$

The operations, S, V, K are evidently *distributive*,

$$\mathrm{S}\Sigma = \Sigma\mathrm{S}, \quad \mathrm{V}\Sigma = \Sigma\mathrm{V}, \quad \mathrm{K}\Sigma = \Sigma\mathrm{K}; \tag{27}$$

and hence it is permitted to multiply together any two of the equations (21), (22), or to square any one of them, as if S, V, K were ordinary algebraical symbols, and the results must be found to be consistent with those equations themselves, and with the relations (26). Thus, squaring and multiplying the equations (21), we obtain

$$\left.\begin{array}{r} 1^2 = (\mathrm{S}+\mathrm{V})^2 = \mathrm{S}^2 + \mathrm{V}^2 + 2\mathrm{SV} = \mathrm{S}+\mathrm{V} = 1, \\ \mathrm{K}^2 = (\mathrm{S}-\mathrm{V})^2 = \mathrm{S}^2 + \mathrm{V}^2 - 2\mathrm{SV} = \mathrm{S}+\mathrm{V} = 1, \\ 1\mathrm{K} = (\mathrm{S}+\mathrm{V})(\mathrm{S}-\mathrm{V}) = \mathrm{S}^2 - \mathrm{V}^2 = \mathrm{S}-\mathrm{V} = \mathrm{K}; \end{array}\right\} \tag{28}$$

and the equations (22) give similarly,

$$\left.\begin{array}{r} \mathrm{S}^2 = \tfrac{1}{4}(1+\mathrm{K})^2 = \tfrac{1}{4}(1+\mathrm{K}^2+2\mathrm{K}) = \tfrac{1}{2}(1+\mathrm{K}) = \mathrm{S}; \\ \mathrm{V}^2 = \tfrac{1}{4}(1-\mathrm{K})^2 = \tfrac{1}{4}(1+\mathrm{K}^2-2\mathrm{K}) = \tfrac{1}{2}(1-\mathrm{K}) = \mathrm{V}; \\ \mathrm{SV} = \mathrm{VS} = \tfrac{1}{4}(1+\mathrm{K})(1-\mathrm{K}) = \tfrac{1}{4}(1-\mathrm{K}^2) = \tfrac{1}{4}(1-1) = 0. \end{array}\right\} \tag{29}$$

Again, if we multiply (22) by K, we get

$$\left.\begin{array}{r} \mathrm{KS} = \tfrac{1}{2}\mathrm{K}(1+\mathrm{K}) = \tfrac{1}{2}(\mathrm{K}+\mathrm{K}^2) = \tfrac{1}{2}(\mathrm{K}+1) = \mathrm{S}, \\ \mathrm{KV} = \tfrac{1}{2}\mathrm{K}(1-\mathrm{K}) = \tfrac{1}{2}(\mathrm{K}-\mathrm{K}^2) = \tfrac{1}{2}(\mathrm{K}-1) = -\mathrm{V}; \end{array}\right\} \tag{30}$$

all which results are seen to be symbolically true, and other verifications of this sort may easily be derived, among which the following may be not unworthy of notice:

$$(\mathrm{S}\pm\mathrm{V})^{2m} = 1, \quad (\mathrm{S}\pm\mathrm{V})^{2m+1} = \mathrm{S}\pm\mathrm{V}, \quad \left(\frac{1\pm\mathrm{K}}{2}\right)^m = \frac{1\pm\mathrm{K}}{2}, \tag{31}$$

where m is any positive whole number.

[5.] As a second simplification of the general conception of polynomes of the form (1), which will tend to render the laws of their operations on each other still more analogous to those of the quaternions, let it be now conceived that the choice of the "constants of multiplication," (fgh), is restricted by the following condition, which may be called the "Law of Conjugation:"

$$\mathrm{K}\,.\,\iota\iota' = \iota'\iota, \quad \text{or} \quad \mathrm{K}\,.\,\iota_f\iota_g = \iota_g\iota_f; \tag{32}$$

namely the condition that, "opposite (or inverted) products of any two of the n symbols $\iota_1, \ldots \iota_n$, shall always be conjugate polynomes." The indices f and g being still supposed to be each > 0, the constants of multiplication (fgh), which had remained arbitrary and disposable in [2.], after that first simplification which consisted in supposing $\iota_0 = 1$, come now to be still further reduced in number, from $(n+1)n^2$ to $\frac{1}{2}n(n^2+1)$. For we have now, by operating with S on the equation (32), the following formula of relation between those constants,

$$(fg0) = (gf0); \tag{33}$$

and by comparing coefficients of ι_h, this other formula is obtained,

$$-(fgh) = (gfh), \quad \text{if} \quad h > 0; \tag{34}$$

whence

$$(ffh) = 0, \quad \text{if} \quad h > 0. \tag{35}$$

Writing, for conciseness,

$$(fg0) = (fg), \quad (ff) = (f), \tag{36}$$

the *squares*, ι^2, of the n *vector-units* ι, will thus reduce themselves to so many constant *scalars*,

$$\iota_1^2 = (1), \quad \iota_2^2 = (2), \quad \ldots \quad \iota_f^2 = (f), \quad \ldots \quad \iota_n^2 = (n); \tag{37}$$

and besides these, we shall have $(n+1) \times \dfrac{n(n-1)}{2} = \frac{1}{2}(n^3 - n)$ *other* scalars, as constants of multiplication; namely the constituents (fgh) of the polynomial expansions of all the binary products $\iota\iota'$ or $\iota_f\iota_g$, of unequal vector-units, taken in any one selected *order*, for instance so that $g > f$; it being unnecessary now, on account of the formulæ of relation (33), (34), to attend *also* to the *opposite* order of the two factors, if the object be merely to determine the *number* of the independent constants, which number is thus found to be $n + \frac{1}{2}(n^3 - n) = \frac{1}{2}(n^3 + n)$, as above stated. Such then is the number of the constants of multiplication, including n of the form (f), and $\frac{1}{2}n(n-1)$ of the form (fg), besides others of the form (fgh), which remain still arbitrary, or disposable, after satisfying, first, the *Unit-Law*, $\iota_0 = 1$, and, second, the *Law of Conjugation*, $\mathrm{K} \,.\, \iota\iota' = \iota'\iota$.

[6.] From this law of conjugation, (32), several general consequences follow. For, *first*, we see from it that "the *square* of every *vector* is a *scalar*," which may be thus expanded:

$$\left.\begin{aligned} \varpi = (\iota_1 x_1 + \ldots + \iota_n x_n)^2 &= (1)x_1^2 + (2)x_2^2 + \ldots + (n)x_n^2 \\ &+ 2(12)x_1x_2 + 2(13)x_1x_3 + \ldots + 2(fg)x_fx_g + \ldots; \end{aligned}\right\} \tag{38}$$

that is, more briefly,

$$(\Sigma\iota x)^2 = \Sigma(e)x_e^2 + 2\Sigma(fg)x_fx_g, \tag{39}$$

the summations extending to values of the indices > 0 and g being $> f$. In the *second* place, and more generally, "*inverted products of any two vectors* are equal to *conjugate polynomes*;" or in symbols,

$$\varpi'\varpi = \mathrm{K} \,.\, \varpi\varpi', \tag{40}$$

whatever two vectors may be denoted by ϖ and ϖ'. In fact, these two products have (according to the definition [3.] of conjugates) one *common scalar part*, but *opposite vector parts*,

$$\left.\begin{aligned} \mathrm{S}\,.\,\varpi'\varpi = \mathrm{S}\,.\,\varpi\varpi' = \Sigma(e)x_e x'_e + \Sigma(fg)(x_f x'_g + x_g x'_f); \\ -\mathrm{V}\,.\,\varpi'\varpi = \mathrm{V}\,.\,\varpi\varpi' = \Sigma(fgh)(x_f x'_g - x_g x'_f)\iota_h : \end{aligned}\right\} \tag{41}$$

whence also we may write, as in quaternions,

$$\mathrm{S}\,.\,\varpi\varpi' = \tfrac{1}{2}(\varpi\varpi' + \varpi'\varpi), \quad \mathrm{V}\,.\,\varpi\varpi' = \tfrac{1}{2}(\varpi\varpi' - \varpi'\varpi). \tag{42}$$

And, *thirdly*, the result (40) may be still further generalized as follows: "The *conjugate of the product of any two polynomes* is equal to the *product of their conjugates*, taken in an *inverted order*;" or in symbols,

$$\mathrm{K}\,.\,\mathrm{PP}' = \mathrm{KP}'\,.\,\mathrm{KP}. \tag{43}$$

In fact, we have now, by (16), (24), (27) and (40),

$$\begin{aligned} \mathrm{KP}'' &= \mathrm{K}\,.\,\mathrm{PP}' = \mathrm{K}\,.\,(p+\varpi)(p'+\varpi') \\ &= \mathrm{K}(pp' + p\varpi' + p'\varpi + \varpi\varpi') \\ &= pp' - p\varpi' - p'\varpi + \varpi'\varpi \\ &= (p' - \varpi')(p - \varpi) = \mathrm{KP}'\,.\,\mathrm{KP}, \end{aligned} \tag{44}$$

as asserted in (43). It follows also, *fourthly*, that "the product of any two conjugate polynomes is a *scalar*, independent of their order, and equal to the difference of the squares of the scalar and vector parts of either of them;" for,

$$\text{if} \quad \mathrm{P}' = \mathrm{KP} \quad \text{then} \quad \mathrm{PP}' = (p+\varpi)(p-\varpi) = p^2 - \varpi^2; \tag{45}$$

where ϖ^2 is, by (38) or (39), a scalar. And if we agree to call the *square root* (taken with a suitable sign) of this scalar product of two conjugate polynomes, P and KP, the common TENSOR of each, and to denote it by the symbol TP; if also we give the name of VERSOR to the quotient of *a polynome divded by its own tensor*, and denote this quotient by the symbol UP: we shall then be able to establish several *general formulæ*, as extensions from the theory of quaternions. For we shall have

$$\mathrm{TP} = \mathrm{TKP} = \surd(\mathrm{PKP}) = \{(\mathrm{SP})^2 - (\mathrm{VP})^2\}^{\frac{1}{2}}; \tag{46}$$

$$\mathrm{T}(p \pm \varpi) = (p^2 - \varpi^2)^{\frac{1}{2}}; \quad \mathrm{T}p = (p^2)^{\frac{1}{2}}, \quad \mathrm{T}\varpi = (-\varpi^2)^{\frac{1}{2}}; \tag{47}$$

$$\mathrm{UP} = \frac{\mathrm{P}}{\surd(\mathrm{PKP})}, \quad \mathrm{U}(p \pm \varpi) = \frac{p \pm \varpi}{(p^2 - \varpi^2)^{\frac{1}{2}}}; \tag{48}$$

$$\mathrm{P} = \mathrm{TP}\,.\,\mathrm{UP} = \mathrm{UP}\,.\,\mathrm{TP}; \tag{49}$$

$$\mathrm{TUP} = \mathrm{UTP} = 1; \quad \mathrm{TTP} = \mathrm{TP}; \quad \mathrm{UUP} = \mathrm{UP} : \tag{50}$$

with some other connected equations. But, although the chief *terms* (such as scalar, vector, conjugate, tensor, versor), and the main *notations* answering thereto (namely S, V, K, T, U), of the calculus of quaternions, along with several general *formulæ* resulting, come thus to receive extended significations, as applying to certain *polynomial* expressions which involve n vector-units, and for which as many as $\frac{1}{2}(n^3+n)$ constants of multiplication are still left arbitrary and disposable; yet it must be observed, that we have not hitherto established any *modular property* of either of the two functions, which have been called above the *tensor* and *versor* of a polynome; nor any *associative law*, for the multiplication of three such polynomes together.

Observatory of Trinity College, Dublin,
June 6, 1854.

Section II.

[7.] Let us now consider generally the *associative* law of multiplication, which may be expressed by the formula already mentioned but reserved in [1.],

$$\iota \,.\, \iota'\iota'' = \iota\iota' \,.\, \iota''; \tag{51}$$

or by this other equation,

$$\iota_e \,.\, \iota_f\iota_g = \iota_e\iota_f \,.\, \iota_g : \tag{52}$$

and let us inquire into the conditions under which this law shall be fulfilled, for any 3 unequal or equal symbols of the form ι.

If the conception of the polynomial expression

$$\mathrm{P} = \Sigma\iota x = \iota_0 x_0 + \iota_1 x_1 + \ldots + \iota_n x_n, \tag{1}$$

be no further restricted than it was in [1.], then *each* of the three indices e, f, g, in the equation (52), may receive any one of the $n+1$ values from 0 to n; so that there are in this case $(n+1)^3$ associative conditions of this form (52), whereof *each*, by comparison of the coefficients of the $n+1$ symbols ι, breaks itself up into $n+1$ separate equations, of the ordinary algebraical kind, making in all no fewer than $(n+1)^4$ algebraical relations, to be satisfied, if possible, by the $(n+1)^3$ constants of multiplication, of the form (fgh): respecting which constants, it will be remembered that the general formula has been established,

$$\iota_f\iota_g = (fg0)\iota_0 + \ldots + (fgh)\iota_h + \ldots + (fgn)\iota_n. \tag{7}$$

We may therefore substitute, in (52), the expressions,

$$\left.\begin{array}{ll} \iota_f\iota_g = \Sigma_h(fgh)\iota_h, & \iota_e\iota_f = \Sigma_h(efh)\iota_h, \\ \iota_e\iota_h = \Sigma_k(ehk)\iota_k, & \iota_h\iota_g = \Sigma_k(hgk)\iota_k; \end{array}\right\} \tag{53}$$

and then, by comparing coefficients of ι_k, this associative formula (52) breaks itself up, as was just now remarked, into $(n+1)^4$ equations between the $(n+1)^3$ constants, which are all included in the following*:

$$\Sigma_h(fgh)(ehk) = \Sigma_h(efh)(hgk); \tag{54}$$

where the *four* indices $e\,f\,g\,k$ may each separately receive any one of the $n+1$ values from 0 to n, and the summations relatively to h are performed between the same limits.

* This formula (54) may be deduced from the equation (214) in p. 239 of the writer's "Researches respecting Quaternions", published in the Transactions of the Royal Irish Academy, vol. xxi, part 2, by changing there the letters $r\,s\,t\,r'\,s'$ to $f\,h\,g\,e\,k$, and substituting the symbol (fgh) for $n_{g,f,h}$. Or the same formula (54) may be derived from one given in page (30) of the Preface to the same author's Lectures on Quaternions, (Dublin, Hodges and Smith, 1853), by writing $g\,f\,e\,k$ instead of $f\,g\,g'\,h'$, and changing each of the two symbols $1_{g,f,h}$, $1'_{g,f,h}$, to (fgh). But the *general reductions* of the present paper have not been hitherto published.

[8.] Introducing next the simplification (10) of article [2.], or supposing $\iota_0 = 1$, which has been seen to reduce the number of the constants of multiplication from $(n+1)^3$ to $(n+1)n^2$, we find that the number of equations to be satisfied by them is reduced in a still greater ratio, namely from $(n+1)^4$ to $(n+1)n^3$. For, if we suppose the index g to become 0, and observe that each of the constants $(f0h)$ and $(0fh)$ is equal, by (12) and (13), to 0 or to 1, according as h is unequal or equal to f, we shall see that the sum in the left-hand member of the formula (54) reduces itself to the term (efk): but such is also in this case the value of the right-hand sum in the same formula, because in calculating that sum we need attend only to the value $h = k$, if g be still $= 0$. In like manner, if $f = 0$, each sum reduces itself to (egk); and if $e = 0$, the two sums become each $= (fgk)$. If then any one of these three indices, e, f, g, be $= 0$, the formula (54) is satisfied: which might indeed have been foreseen, by observing that, in each of these three cases, one factor of each member of the equation (52) becomes $= 1$. We may therefore henceforth suppose that each of the three indices, e, f, g, varies only from 1 to n, or that

$$e > 0, \quad f > 0, \quad g > 0; \tag{55}$$

while k may still receive any value from 0 to n, and h still varies in the summations between these latter limits: and thus the number of equations, supplied by the formula (54), between the constants (fgh), is reduced, as was lately stated, to $(n+1)n^3$; while the number of those constants themselves had been seen to be reduced to $(n+1)n^2$, by the same supposition $\iota_0 = 1$.

[9.] Additional reductions are obtained by introducing the law of conjugation (32), or by supposing $\mathrm{K} \,.\, \iota_f \iota_g = \iota_g \iota_f$, with the consequences already deduced from that law or equation in [5.]. Using Σ' to denote a summation relatively to h from 1 to n, and taking separately the two cases where $k = 0$ and where $k > 0$, we have, for the first case, by (54),

$$\Sigma'(efh)(gh) = \Sigma'(fgh)(eh); \tag{56}$$

and for the second case,

$$(ef)(g0k) - (fg)(e0k) = \Sigma'\{(efh)(ghk) + (fgh)(ehk)\}. \tag{57}$$

No new conditions would be obtained by interchanging e and g; but if we cyclically change efg to fge, each of the two sums (56) is seen to be equal to another of the same form; and two new equations are obtained from (57), by adding which thereto we find,

$$0 = \Sigma'\{(efh)(ghk) + (fgh)(ehk) + (geh)(fhk)\}; \tag{58}$$

and therefore,

$$(fg)(e0k) - (ef)(g0k) = \Sigma'(geh)(fhk). \tag{59}$$

When $e = f$, the equations (56) and (59) become, respectively,

$$0 = \Sigma'(fh)(fgh), \tag{60}$$

and

$$(fg)(f0k) - (f)(g0k) = \Sigma'(gfh)(fhk); \tag{61}$$

which are identically satisfied, if we suppose also $f = g$; the properties [5.] of the symbols (fgh) being throughout attended to: while, by the earlier properties [2.], the symbol $(e0k)$ or $(0ek)$ is equal to 0 or to 1, according as e and k are unequal or equal to each other. And no equations distinct from these are obtained by supposing $e = g$, or $f = g$, in (56) and (59). The associative conditions for which $k = 0$ are, therefore, in number, $n(n-1)$ of the form (60), and $\frac{1}{3}n(n-1)(n-2)$ of the form (56); or $\frac{1}{3}(n^3-n)$ in all. And the other associative conditions, for which $k > 0$, are, in number, $n^2(n-1)$ of the form (61), and $\frac{1}{2}n^2(n-1)(n-2)$ of the form (59), or $\frac{1}{2}(n^4-n^3)$ in all. It will, however, be found that this last number admits of being diminished by $\frac{1}{2}(n^2-n)$, namely by one for each of the symbols of the form (fg); and that if, before or after this reduction, the associative equations for which $k > 0$ be satisfied, then those other $\frac{1}{3}(n^3-n)$ conditions lately mentioned, for which $k = 0$, are satisfied also, as a necessary consequence. The *total number* of the *equations of association*, included in the formula (54), will thus come to be reduced to

$$\tfrac{1}{2}(n^4 - n^3) - \tfrac{1}{2}(n^2 - n), \quad \text{or to} \quad \tfrac{1}{2}n(n-1)(n^2-1);$$

but it may seem unlikely that even so large a number of conditions as this can be satisfied *generally*, by the $\frac{1}{2}n(n^2+1)$ constants of multiplication [5.]. Yet I have found, not only for the case $n = 2$, in which we have thus 5 constants and 3 equations, but also for the cases $n = 3$ and $n = 4$, for the former of which we have 15 constants and 24 equations, while for the latter we have 34 constants and 90 equations, that all these associative conditions can be satisfied: even in such a manner as to leave some degree of indetermination in the results, or some constants of multiplication disposable.

[10.] Without expressly introducing the symbols (fgh), results essentially equivalent to the foregoing may be deduced in the following way, with the help of the characteristics [3.] of operation, S, V, K. The formula of association (51) may first be written thus*:

$$\iota\,\mathrm{S}\iota'\iota'' + \iota\mathrm{V}\iota'\iota'' = \mathrm{S}\iota\iota' \,.\, \iota'' + \mathrm{V}\iota\iota' \,.\, \iota''; \tag{62}$$

in which the symbols $\mathrm{S}\iota\iota'$ and $\mathrm{V}\iota\iota'$ are used to denote concisely, without a point interposed, the scalar and vector parts of the product $\iota\iota'$, but a point is inserted, after those symbols, and before ι'', in the second member, as a mark of multiplication: so that, in this abridged notation, $\mathrm{S}\iota\iota' \,.\, \iota''$ and $\mathrm{V}\iota\iota' \,.\, \iota''$ denote the products which might be more fully expressed as $(\mathrm{S} \,.\, \iota\iota') \times \iota''$ and $(\mathrm{V} \,.\, \iota\iota') \times \iota''$; while it has been thought unnecessary to write any point in the first member, where the factor ι occurs at the left hand. Operating on (62) by S and V, we find the two following equations of association, which are respectively of the scalar and vector kinds:

$$\mathrm{S}(\iota\,\mathrm{V}\iota'\iota'' - \iota''\,\mathrm{V}\iota\iota') = 0; \tag{63}$$

* There is here a slight departure from the notation of the Lectures on Quaternions, by the suppression of certain points, which circumstance in the present connexion cannot produce ambiguity.

$$\mathrm{V}(\iota\,\mathrm{V}\iota'\iota'' + \iota''\,\mathrm{V}\iota\iota') = \iota''\,\mathrm{S}\iota\iota' - \iota\,\mathrm{S}\iota'\iota''; \tag{64}$$

because the law (32) of conjugation, $\iota'\iota = \mathrm{K}\iota\iota'$, gives, by (41),

$$\mathrm{S}\varpi'\varpi = +\mathrm{S}\varpi\varpi', \quad \mathrm{V}\varpi'\varpi = -\mathrm{V}\varpi\varpi'.$$

For the same reason, no essential change is made in either of the two equations, (63), (64), by interchanging ι and ι''; but if we cyclically permute the three vector-units, $\iota\,\iota'\,\iota''$, then (63) gives

$$\mathrm{S}(\iota\,\mathrm{V}\iota'\iota'') = \mathrm{S}(\iota'\,\mathrm{V}\iota''\iota) = \mathrm{S}(\iota''\,\mathrm{V}\iota\iota'); \tag{65}$$

and there arise three equations of the form (64), which give, by addition,

$$\mathrm{V}(\iota\,\mathrm{V}\iota'\iota'' + \iota'\,\mathrm{V}\iota''\iota + \iota''\,\mathrm{V}\iota\iota') = 0; \tag{66}$$

and therefore conduct to three other equations, of the form*

$$\mathrm{V}(\iota\,\mathrm{V}\iota'\iota'') = \iota''\,\mathrm{S}\iota\iota' - \iota'\,\mathrm{S}\iota''\iota. \tag{67}$$

Equating ι'' to ι, the two equations (65) reduce themselves to the single equation,

$$\mathrm{S}(\iota\,\mathrm{V}\iota\iota') = 0; \tag{68}$$

and the formula (67) becomes

$$\mathrm{V}(\iota\,\mathrm{V}\iota\iota') = \iota^2\iota' - \iota\mathrm{S}\iota\iota' : \tag{69}$$

both which results become identities, when we further equate ι' to ι. And no equations of condition, distinct from these, are obtained by supposing $\iota'' = \iota'$, or $\iota' = \iota$, in (65) and (67). The number of the symbols ι being still supposed $= n$, and therefore by [5.] the number of the constants which enter into the expressions of their n^2 binary products (including squares) being $= \frac{1}{2}(n^3 + n)$, these constants are thus (if possible) to be made to satisfy $\frac{1}{3}(n^3 - n)$ associative and scalar equations of condition, obtained through (63), from the comparison of the scalar parts of the two ternary products, $\iota \,.\, \iota'\iota''$ and $\iota\iota' \,.\, \iota''$; namely, $n(n-1)$ scalar equations of the form (68) and $\frac{1}{3}n(n-1)(n-2)$ such equations, of the forms (65). And the same constants of multiplication must also (if the associative law is to be fulfilled) be so chosen as to satisfy $\frac{1}{2}(n^3 - n^2)$ vector equations, equivalent each to n scalar equations, or in all to $\frac{1}{2}(n^4 - n^3)$ scalar conditions, obtained through (64) from the comparison of the vector parts of the same two ternary products (51); namely, $n(n-1)$ vector equations of the form (69), and $\frac{1}{2}n(n-1)(n-2)$ other vector equations, included in the formula (64). This new analysis therefore confirms completely the conclusion of the foregoing paragraph respecting the general existence of $\frac{1}{2}(n^4 - n^3) + \frac{1}{3}(n^3 - n)$ associative and scalar equations of condition, between the $\frac{1}{2}(n^3 + n)$ disposable constants of multiplication, when the general conception of the polynomial expression P of [1.] is modified by the suppositions, $\iota_0 = 1$ in [2.], and $\iota'\iota = K\iota\iota'$ in [5.]. At least the analysis of the present paragraph [10.] confirms what has been lately proved in [9.], that the number of conditions of association can be *reduced so far*; but the same analysis will also admit of being soon applied, so as to assist in proving the existence of those *additional* and *general* reductions which have been lately mentioned without proof, and which depress the number of conditions to be satisfied to $\frac{1}{2}(n^4 - n^3) - \frac{1}{2}(n^2 - n)$. Meanwhile it may be useful to exemplify briefly the foregoing general reasonings for the cases $n = 2$, $n = 3$, that is, for trinomial and quadrinomial polynomes.

* This formula is one continually required in calculating with quaternions (compare page li of the Contents, prefixed to the author's Lectures).

[11.] For the case $n = 2$, the two distinct symbols of the form ι may be denoted simply by ι and ι'; and the equations of association to be satisfied are all included in these two,

$$\iota \,.\, \iota\iota' = \iota^2\iota', \quad \iota' \,.\, \iota'\iota = \iota'^2\iota; \tag{70}$$

which give, when we operate on them by S and V, two scalar equations of the form (68), and two vector equations of the form (69), equivalent on the whole to six scalar equations of condition, between the five constants of multiplication, (1) (2) (12) (121) (122), if we write, on the plan of the preceding articles,

$$\iota^2 = (1), \quad \iota'^2 = (2), \quad \mathrm{S}\iota\iota' = (12), \quad \mathrm{V}\iota\iota' = -\mathrm{V}\iota'\iota = (121)\iota + (122)\iota'. \tag{71}$$

From (68), or from (60), or in so easy a case by more direct and less general considerations, we find that the comparison of the scalar parts of the products (70) conducts to the two equations,

$$0 = (121)(1) + (122)(12) = (122)(2) + (121)(12). \tag{72}$$

From (69), or (61), we find that the comparison of the vector parts of the same products (70) gives immediately four scalar equations, which however are seen to reduce themselves to the three following:

$$(121)(122) = -(12); \quad (122)^2 = (1); \quad (121)^2 = (2); \tag{73}$$

the first of these occurring twice. And it is clear that the equations (72) are satisfied, as soon as we assign to (1), (2) and (12) the values given by (73). If then we write, for conciseness,

$$(121) = a, \quad (122) = b, \tag{74}$$

we shall have, for the present case ($n = 2$), the values,

$$(1) = b^2, \quad (2) = a^2, \quad (12) = -ab. \tag{75}$$

And hence, (writing κ instead of ι',) we see that the *trinome**,

$$\mathrm{P} = z + \iota x + \kappa y, \tag{76}$$

where $x\,y\,z$ are ordinary variables, will possess all the properties of those *polynomial expressions* which have been hitherto considered in this paper, and especially the associative property, if we establish the formula of multiplication,

$$(\iota x + \kappa y)(\iota x' + \kappa y') = (bx - ay)(bx' - ay') + (a\iota + b\kappa)(xy' - yx'); \tag{77}$$

wherein a and b are any two constants of the ordinary and algebraical kind. In this trinomial system,

$$z'' + \iota x'' + \kappa y'' = (z + \iota x + \kappa y)(z' + \iota x' + \kappa y'), \tag{78}$$

* I am not aware that this trinomial expression (76), with the formula of multiplication (77), coincides with any of the triplet-forms of Professor De Morgan, or of Messrs. John and Charles Graves: but it is given here merely by way of illustration.

if

$$\left.\begin{aligned} x'' &= zx' + z'x + a(xy' - yx'), \\ y'' &= zy' + z'y + b(xy' - yx'), \\ z'' &= zz' + (bx - ay)(bx' - ay'); \end{aligned}\right\} \tag{79}$$

we have therefore the two *modular relations*,

$$\left.\begin{aligned} z'' + bx'' - ay'' &= (z + bx - ay)(z' + bx' - ay'), \\ z'' - bx'' + ay'' &= (z - bx + ay)(z' - bx' + ay'); \end{aligned}\right\} \tag{80}$$

that is to say, the functions $z \pm (bx - ay)$ are two *linear moduli* of the system. A general theory with which this result is connected will be mentioned a little further on. Geometrical interpretations (of no great interest) might easily be proposed, but they would not suit the plan of this communication.

[12.] For the case $n = 3$, or for the *quadrinome*

$$\mathrm{P} = x_0 + \iota_1 x_1 + \iota_2 x_2 + \iota_3 x_3, \tag{81}$$

we may assume

$$\left.\begin{aligned} \iota_1^2 &= a_1, & \iota_2^2 &= a_2, & \iota_3^2 &= a_3, \\ \mathrm{S}\iota_2\iota_3 &= b_1, & \mathrm{S}\iota_3\iota_1 &= b_2, & \mathrm{S}\iota_1\iota_2 &= b_3, \end{aligned}\right\} \tag{82}$$

and

$$\left.\begin{aligned} \mathrm{V}\iota_2\iota_3 = -\mathrm{V}\iota_3\iota_2 &= \iota_1 l_1 + \iota_2 m_3 + \iota_3 n_2, \\ \mathrm{V}\iota_3\iota_1 = -\mathrm{V}\iota_1\iota_3 &= \iota_2 l_2 + \iota_3 m_1 + \iota_1 n_3, \\ \mathrm{V}\iota_1\iota_2 = -\mathrm{V}\iota_2\iota_1 &= \iota_3 l_3 + \iota_1 m_2 + \iota_2 n_1; \end{aligned}\right\} \tag{83}$$

and then the $\frac{1}{2}(n^4 - n^3) = 27$ scalar equations of condition, included in the vector form,

$$\mathrm{V}(\iota \,.\, \iota'\iota'') = \mathrm{V}(\iota\iota' \,.\, \iota''), \tag{84}$$

are found on trial to reduce* themselves to 24; which, after elimination of the 6 constants of the forms here denoted by a and b, or previously by (f) and (fg), furnish 18 equations of condition between the 9 other constants, of the forms here marked l, m, n, or previously (fgh); and these 18 equations may be thus arranged†:

$$\left.\begin{aligned} 0 &= l_1(n_1 - m_1) = l_2(n_2 - m_2) = l_3(n_3 - m_3), \\ 0 &= l_2(n_1 - m_1) = l_3(n_2 - m_2) = l_1(n_3 - m_3), \\ 0 &= l_3(n_1 - m_1) = l_1(n_2 - m_2) = l_2(n_3 - m_3); \end{aligned}\right\} \tag{85}$$

---

* The reason of this reduction is exhibited by the general analysis in [14.].

† For it is found that each of the three constants $(eff) + (fgg)$ must give a null product, when it is multiplied by any one of the constants $(e'f'g')$, or by any one of these other constants $(e''f''f'') - (e''g''g'')$; if each of the three systems, efg, $e'f'g'$, $e''f''g''$, represent, in some order or other, but not necessarily in one common order, the system of the three unequal indices, 1, 2, 3.

$$
\left.
\begin{aligned}
&0 = n_1^2 - m_1^2 = n_2^2 - m_2^2 = n_3^2 - m_3^2,\\
&0 = (n_2 + m_2)(n_1 - m_1) = (n_3 + m_3)(n_2 - m_2) = (n_1 + m_1)(n_3 - m_3),\\
&0 = (n_3 + m_3)(n_1 - m_1) = (n_1 + m_1)(n_2 - m_2) = (n_2 + m_2)(n_3 - m_3);
\end{aligned}
\right\}
\tag{86}
$$

they are therefore *satisfied*, without any restriction on $l_1\, l_2\, l_3$, by our supposing

$$
n_1 = m_1, \quad n_2 = m_2, \quad n_3 = m_3;
\tag{87}
$$

but if we do *not* adopt this supposition, they *require* us to admit this *other* system of equations,

$$
0 = l_1 = l_2 = l_3 = n_1 + m_1 = n_2 + m_2 = n_3 + m_3.
\tag{88}
$$

Whichever of these two suppositions, (87), (88), we adopt, there results a corresponding system of values of the six recently eliminated constants, of the forms a and b, or (f) and (fg); and it is found* that these values satisfy, without any new supposition being required, the $\frac{1}{3}(n^3 - n) = 8$ scalar equations, included in the general form

$$
\mathrm{S}(\iota\,.\,\iota'\iota'') = \mathrm{S}(\iota\iota'\,.\,\iota''),
\tag{89}
$$

which are required for the associative property.

[13.] In this manner I have been led to the *two* following systems of *associative quadrinomials*, which may be called systems (A) and (B); both possessing all those general properties of the polynomial expression P, which have been considered in the present paper; and one of them including the quaternions.

For the system (A), the quadrinomial being still of the form (81), or of the following equivalent form,

$$
\mathrm{Q} = w + \iota x + \kappa y + \lambda z,
\tag{90}
$$

where $w\,x\,y\,z$ are what were called in [1.] the *constituents*, the laws of the *vector-units* $\iota\,\kappa\,\lambda$ are all included in this formula of multiplication for *any two vectors*, such as

$$
\rho = \iota x + \kappa y + \lambda z, \quad \rho' = \iota x' + \kappa y' + \lambda z' :
\tag{91}
$$

$$
\begin{aligned}
(\mathrm{A})\ldots \qquad \rho\rho' = {} & (m_1^2 - l_2 l_3)xx' + (l_1 m_1 - m_2 m_3)(yz' + zy')\\
&+ (m_2^2 - l_3 l_1)yy' + (l_2 m_2 - m_3 m_1)(zx' + xz')\\
&+ (m_3^2 - l_1 l_2)zz' + (l_3 m_3 - m_1 m_2)(xy' + yx')\\
&+ (\iota l_1 + \kappa m_3 + \lambda m_2)(yz' - zy')\\
&+ (\kappa l_2 + \lambda m_1 + \iota m_3)(zx' - xz')\\
&+ (\lambda l_3 + \iota m_2 + \kappa m_1)(xy' - yx');
\end{aligned}
\tag{92}
$$

* This fact of calculation is explained by the general analysis of [15.]. The values of a and b may be deduced from the formulæ, $a_1 = m_1^2 - l_2 l_3$, $b_1 = l_1 m_1 - m_2 n_3$, with others cyclically formed from these.

and it is clear that *Quaternions** are simply that particular *case* of such QUADRINOMES (A), for which the *six* arbitrary constants $l_1 \dots m_3$ and the *three* vector-units $\iota\,\kappa\,\lambda$ receive the following values:

$$l_1 = l_2 = l_3 = 1, \quad m_1 = m_2 = m_3 = 0, \quad \iota = i, \quad \kappa = j, \quad \lambda = k. \tag{93}$$

For the other associative quadrinomial system (B), which we may call for distinction TETRADS, if we retain the expressions (90), (91), we must replace the formula of *vector-multiplication* (92) by one of the following form:

$$\text{(B)}\ldots \qquad \begin{aligned} \rho\rho' &= (lx + my + nz)(lx' + my' + nz') \\ &\quad + (\kappa n - \lambda m)(yz' - zy') + (\lambda l - \iota n)(zx' - xz') \\ &\quad + (\iota m - \kappa l)(xy' - yx'); \end{aligned} \tag{94}$$

involving thus only *three* arbitrary constants, $l\,m\,n$, besides the *three* vector-units, $\iota\,\kappa\,\lambda$; and apparently having no connexion with the *quaternions*, beyond the circumstance that one common analysis [12.] conducts to both the *quadrinomes* (A) and the *tetrads* (B).

As regards certain *modular properties* of these two quadrinomial systems, we shall shortly derive them as consequences of the general theory of polynomes of the form P, founded on the principles of the foregoing articles.

[14.] In general, the formula (59) gives, by [2.], the two following equations, which may in their turn replace it, and are, like it, derived from the comparison of the *vector parts* of the general associative formula, or from the supposition that $k > 0$ in (54):

$$(fg) = \Sigma(geh)(fhe), \quad \text{if} \quad e \gtrless g; \tag{95}$$

$$0 = \Sigma(geh)(fhk), \quad \text{if} \quad k \gtrless e, \;\; k \gtrless g; \tag{96}$$

the summation extending in each from $h = 1$ to $h = n$. Interchanging f and g in (95), we have

$$(gf) = \Sigma(feh)(ghe), \quad \text{if} \quad e \gtrless f; \tag{97}$$

and making $g = f$, in either (95) or (97), we obtain the equation,

$$(f) = \Sigma(feh)(fhe), \quad \text{if} \quad e \gtrless f. \tag{98}$$

For each of the n symbols (f), there are $n-1$ distinct expressions of this last form, obtained by assigning different values to e; and when these expressions are equated to each other, there result $n(n-2)$ equations between the symbols of the form (fgh). For each of the $\frac{1}{2}n(n-1)$ symbols of the form (fg), where f and g are unequal, there are $n-1$ expressions (95), and $n-1$ other expressions of the form (97), because, by (33) and (36), $(gf) = (fg)$; and thus it might seem that there should arise, by equating these $2n-2$ expressions for each symbol (fg),

* See the author's Lectures, or the Philosophical Magazine for July, 1844, in which the first printed account of quaternions was given.

as many as $2n-3$ equations from each, or $\frac{1}{2}n(n-1)(2n-3)$ equations in all between the symbols (fgh). But if we observe that the sums of the $n-1$ expressions (95) for (fg), and of the $n-1$ expressions (97) for (gf) are, respectively,

$$(n-1)(fg) = \Sigma_e \Sigma_h (geh)(fhe), \quad (n-1)(gf) = \Sigma_e \Sigma_h (feh)(ghe); \tag{99}$$

where the summations may all be extended from 1 to n, because (ffh) and (ggh) are each $= 0$, by (35), since $h > 0$; and that these two double sums (99) are equal; we shall see that the formula

$$(gf) = (fg), \tag{100}$$

though true, gives no information respecting the symbols (fgh): or is not to be counted as a new and distinct equation, in combination with the $n-1$ equations (95), and the $n-1$ equations (97). In other words, the comparison of the sums (99) shows that we may confine ourselves to equating separately to each other, for each pair of unequal indices f and g, the $n-1$ expressions (95) for the symbol (fg), and the $n-1$ other expressions (97) for the symbol (gf), without proceeding afterwards to equate an expression of the one set to an expression of the other set. We may therefore suppress, as unnecessary, an equation of the form (100), for each of the $\frac{1}{2}n(n-1)$ symbols of the form (fg), or for each pair of unequal indices f and g, as was stated by anticipation towards the close of paragraph [9.]. There remain, however, $2(n-2)$ equations of condition, between the symbols (fgh), derived from each of those $\frac{1}{2}n(n-1)$ pairs; or as many as $n(n-1)(n-2)$ equations in all, obtained in this manner from (95) and (97), regarded as separate formulæ. Thus, without yet having used the formula (96), we obtain, with the help of (98), by elimination of the symbols (f), (fg), (gf), through the comparison of $n-1$ expressions for each of those n^2 symbols, $n^2(n-2)$ equations of condition, homogeneous and of the second dimension, between the symbols of the form (fgh). And without any such elimination, the formula (96) gives immediately $\frac{1}{2}n^2(n-1)(n-2)$ other equations of the same kind between the same set of symbols; because after choosing any pair of unequal indices e and g, we may combine this pair with any one of the n values of the index f, and with any one of the $n-2$ values of k, which are unequal both to e and to g. There are therefore, altogether, $\frac{1}{2}n^2(n+1)(n-2)$ homogeneous equations of the second dimension, obtained by comparison of the *vector parts* of the general formula of association, to be satisfied by the $\frac{1}{2}n^2(n-1)$ symbols of the form (fgh).

[15.] To prove now, generally, that when the *vector parts* of the associative formula are thus equal, the *scalar parts* of the same formula are necesarily equal also, or that the system of conditions (56) in [9.] is included in the system (57) or (59); we may conveniently employ the notations S and V, and pursue the analysis of paragraph [10.], so as to show that the system of equations (65), including (68), results from the system (67), including (69); or that if the formula (84) be satisfied for every set of three unequal or equal vector-units, $\iota\,\iota'\,\iota''$, then, for every such set, the formula (89) is satisfied also. For this purpose, I remark that the formula of *vector-association* (67), when combined with the *distributive* principle of multiplication [1.], and of operation with S and V [5.], gives generally, as in quaternions, the transformation

$$\mathrm{V}\rho\,\mathrm{V}\sigma\tau = \tau\,\mathrm{S}\rho\sigma - \sigma\,\mathrm{S}\rho\tau; \tag{101}$$

where ρ, σ, τ may denote *any three vectors*, and the symbol $\mathrm{V}\rho\,\mathrm{V}\sigma\tau$ is used to signify concisely the vector part of the product $\rho\times\mathrm{V}(\sigma\tau)$; whence also we may derive by (41) this other general transformation,

$$\mathrm{V}(\mathrm{V}\sigma\tau\,.\,\rho) = \sigma\,\mathrm{S}\rho\tau - \tau\,\mathrm{S}\rho\sigma. \tag{102}$$

If then we write

$$\mathrm{V}\sigma\tau = \rho', \quad \mathrm{V}\tau\rho = \sigma', \quad \mathrm{V}\rho\sigma = \tau', \tag{103}$$

and introduce another arbitrary vector ϖ, we shall have

$$\mathrm{V}\rho'\varpi = \sigma\,\mathrm{S}\tau\varpi - \tau\,\mathrm{S}\sigma\varpi; \tag{104}$$

and therefore

$$\mathrm{V}\rho\,\mathrm{V}\rho'\varpi = \tau'\,\mathrm{S}\tau\varpi + \sigma'\,\mathrm{S}\sigma\varpi; \tag{105}$$

but also

$$\mathrm{V}\rho\,\mathrm{V}\rho'\varpi = \varpi\,\mathrm{S}\rho\rho' - \rho'\,\mathrm{S}\rho\varpi; \tag{106}$$

whence

$$\varpi\,\mathrm{S}\rho\rho' = \rho'\,\mathrm{S}\rho\varpi + \sigma'\,\mathrm{S}\sigma\varpi + \tau'\,\mathrm{S}\tau\varpi, \tag{107}$$

and consequently

$$\mathrm{S}\rho\rho' = \mathrm{S}\sigma\sigma' = \mathrm{S}\tau\tau'; \tag{108}$$

but this is precisely by (103) the formula of *scalar-association* (65), stated in its most general form. The general dependence of (65) on (67), or of (56) on (57), is therefore proved to exist; and the $\frac{1}{3}(n^3 - n)$ associative conditions, for which $k = 0$ in (54), are seen to be consequences of $\frac{1}{2}(n^4 - n^3)$ other conditions for which $k > 0$; or even of those conditions diminished in number by $\frac{1}{2}(n^2 - n)$, according to what was stated by anticipation in [9.], and has been proved by the analysis of [14.]. This result is the more satisfactory, because otherwise the *conditions of association* would essentially involve a system of homogeneous equations of the *third* dimension relatively to the symbols (fgh), obtained by substituting in (56) the expressions (95) or (97) for the symbols of the form (fg), including the values (98) of the symbols (f). But we see now (as above stated) that the *total* number of *distinct* conditions may be reduced to $\frac{1}{2}(n^4 - n^3) - \frac{1}{2}(n^2 - n)$, between the total number $\frac{1}{2}(n^3 + n)$ of constants of multiplication; or finally, after the *elimination* of the $\frac{1}{2}(n^2 + n)$ symbols of the forms (f) and (fg), to a *system of homogeneous equations of the second dimension*, namely, those determined in [14.], of which the number amounts (as in that paragraph) to

$$\tfrac{1}{2}(n^4 - n^3) - n^2 = \tfrac{1}{2}n^2(n + 1)(n - 2), \tag{109}$$

between the symbols (fgh), whereof the number is

$$\tfrac{1}{2}(n^3 + n) - \tfrac{1}{2}(n^2 + n) = \tfrac{1}{2}n^2(n - 1). \tag{110}$$

[16.] For example, when $n = 2$, the *two* constants (121) and (122) have been seen in [11.] to be unresticted by *any* condition. When $n = 3$, we have 9 constants, lately denoted by $l_1\,l_2\,l_3\ m_1\,m_2\,m_3\ n_1\,n_2\,n_3$, wherewith to satisfy 18 homogeneous equations of the second dimension, namely those marked (85) and (86) in [12.]; which it has been seen to be possible to do, in two distinct ways (A) and (B), and even so as to leave some of the constants arbitrary, in each of the two resulting systems, of *associative quadrinomes and tetrads*. A similar result has been found by me to hold good for the case $n = 4$, or for the case of *associative quines*, such as

$$\mathrm{P} = w + \iota x + \kappa y + \lambda z + \mu u, \tag{111}$$

involving *four* vector-units $\iota\,\kappa\,\lambda\,\mu$, which obey the laws of conjugation (32), and of association (51). For although there are in this case only $24 = \frac{1}{2}n^2(n-1)$ constants of the form (fgh), to satisfy $80 = \frac{1}{2}n^2(n+1)(n-2)$ homogeneous equations of the second dimension, yet I have found that the *forms** of these equations are such as to allow this to be done in various ways, and even without entirely determining the constants. And it appears not impossible that similar results may be obtained for higher values of n; or that *associative*† polynomes of higher orders than *quines* may be discovered.

Observatory of Trinity College, Dublin,
July 4, 1854.

* The subject may be illustrated by the very simple remark, that although the four equations $tx = 0$, $ty = 0$, $ux = 0$, $uy = 0$, are such that *no three* of them *include* the *fourth*, since we might (for example) satisfy the three first alone by supposing $t = 0$, $x = 0$, yet they can *all four* be *satisfied* together by supposing either $x = 0$, $y = 0$, or $t = 0$, $u = 0$. Compare the equations (85) or (86), which are of the forms $tx = 0$, $ty = 0$, $tz = 0$, $ux = 0$, $uy = 0$, $uz = 0$, $vx = 0$, $vy = 0$, $vz = 0$. In the theory of *quines*, however, the forms are not quite so simple.

† The *octaves*, or *octonomial expressions*, which Mr. Cayley published in the Philosophical Magazine for March 1845, and which had been previously but privately communicated to me by Mr. J. T. Graves about the end of 1843, after my communication to him of the quaternions, are *not associative polynomes*. Thus in Mr. Cayley's notation, the four following of his seven *types*, (123) (624) (176) (734), give $\iota_1 \,.\, \iota_2\iota_4 = \iota_1\iota_6 = -\iota_7$, but $\iota_1\iota_2 \,.\, \iota_4 = \iota_3\iota_4 = +\iota_7$; or with Mr. Graves's symbols, the triads $i\,j\,k$, $i\,o\,n$, $j\,l\,n$, $k\,l\,o$, give $i.jl = in = -o$, but $ij.l = kl = +o$. See note to page (61) of the Preface to my Lectures. It was my perceiving this latter property of Mr. Graves's symbols in 1844, which chiefly discouraged me from pursuing the study of those *octaves*, as a species of *extensions of the quaternions*, which Mr. Graves as well as Mr. Cayley had designed them to be, and which in one sense no doubt they are.

Section III.

[17.] The following remarks may be useful, as serving to illustrate and develope the general analysis contained in some of the preceding paragraphs, especially in [14.] and as adapted to give some assistance towards any future study of associative polynomes, such as quines, of an order higher than quadrinomes, but subject like them to the law of conjugation (32).

The expression (98) may be thus more fully written:

$$(f) = (fee)^2 + (feg)(fge) + \Sigma'(feh)(fhe); \tag{112}$$

where $e\,f\,g\,h$ are all supposed to be unequal; the summation Σ' being performed relatively to h, for all those $n-3$ values of the latter, which are distinct from each of the three former indices. Interchanging e and g, and subtracting, we eliminate the symbol (f), and obtain the following formula:

$$\text{I.}\ldots \qquad (fee)^2 - (fgg)^2 = \Sigma'\{(fgh)(fhg) - (feh)(fhe)\}; \tag{113}$$

which type I. includes generally $n(n-2)$ distinct and homogeneous equations, of the second dimension, with $2(n-2)$ terms in each, between the $\frac{1}{2}n^2(n-1)$ symbols of the form (fgh). Thus, for the case of *quadrinomials* $(n = 3)$, by writing, in agreement with (82) and (83),

$$a_1 = (1), \quad b_1 = (23), \quad l_1 = (231), \quad m_1 = (313), \quad n_1 = (122), \tag{114}$$

and suppressing the sum Σ', we have by (112) the two expressions (compare a note to [12.]):

$$a_1 = m_1^2 - l_2 l_3 = n_1^2 - l_2 l_3; \tag{115}$$

together with four others formed from these, by cyclical permutation of the indices 1, 2, 3; and we are thus conducted, by elimination of the three symbols a_1, a_2, a_3, to three equations of the form $n_1^2 = m_1^2$; that is, to the 3 equations on the first line of (86), involving each 2 terms. For *quines* $(n = 4)$, if we make also, with the same permitted permutations,

$$\left.\begin{array}{l} a_4 = (4), \quad c_1 = (14), \quad p_1 = (234), \\ r_1 = (141), \quad s_1 = (142), \quad t_1 = (143), \quad u_1 = (144), \end{array}\right\} \tag{116}$$

the index h receives one value under each sign of summation Σ', and the resulting formulæ may be thus written:

$$(a_1 + l_3 l_2 + p_2 t_1 - s_1 p_3 =)\, n_1^2 + p_2 t_1 = m_1^2 - s_1 p_3 = u_1^2 + l_3 l_2; \tag{117}$$

$$(a_4 - s_1 t_2 - s_2 t_3 - s_3 t_1 =)\, r_1^2 - s_2 t_3 = r_2^2 - s_3 t_1 = r_3^2 - s_1 t_2; \tag{118}$$

where the line (117) is equivalent to three lines of the same form: so that the elimination of $a_1 \ldots a_4$ conducts here to 8 equations, of 4 terms each, between the 24 symbols of the form (fgh), or $l_1 \ldots u_3$, as by the general theory it ought to do. For polynomes of *higher* orders $(n > 4)$, we have the analogous equations,

$$\begin{aligned} (f) - (feg)(fge) - (fgk)(fkg) - (fke)(fek) \\ &= (fee)^2 - (fgk)(fkg) + \Sigma'(feh)(fhe) \\ &= (fgg)^2 - (fke)(fek) + \Sigma''(fgh)(fhg) \\ &= (fkk)^2 - (feg)(fge) + \Sigma''(fkh)(fhk); \end{aligned} \tag{119}$$

where h, under Σ'', receives only $n-4$ values, being distinct from each of the four unequal indices, $e\,f\,g\,k$.

[18.] By changing e to f in (95), and attending to the properties of the symbols (fgh), we obtain the expression

$$(fg) = \Sigma(fgh)(hff); \tag{120}$$

where f and g are unequal, and the summation Σ extends from $h = 1$ to $h = n$. The term for which $h = f$ vanishes, and the formula (120) may be thus more fully written:

$$(fg) - (fge)(eff) = (fgg)(gff) + \Sigma'(fgh)(hff); \tag{121}$$

where the letters $e\,f\,g$ denote again some three unequal indices, and the summation Σ' is performed as in the foregoing paragraph. But also, by (97) and (100),

$$(fg) - (fge)(eff) = (fee)(gee) + \Sigma'(feh)(ghe); \tag{122}$$

subtracting, therefore, (122) from (121), we eliminate the symbol (fg), and obtain the type

$$\text{II.} \ldots \qquad (fee)(gee) - (fgg)(gff) = \Sigma'\{(fgh)(hff) - (feh)(ghe)\}; \tag{123}$$

which represents in general a system of $n(n-1)(n-2)$ distinct and homogeneous equations of the second dimension, containing each $2(n-2)$ terms, and derived by eliminations of the kind last mentioned, from the formulæ (95), (97), (100), in a manner agreeable to the analysis of paragraph [14.]. Indeed, it was shown in that paragraph, that the equation

$$(gf) = (fg), \qquad (100),$$

though known from earlier and simpler principles to be true, might be regarded as *included* in (95) and (97); but this need not prevent us from *using* that equation in combination with the others, whenever it may seem advantageous to do so: and other combinations of them may with its help be formed, which are occasionally convenient, or even sometimes necessary, although the *general* results of the elimination of the symbols (fg) are sufficiently represented by the recent type II., or by the formula (123). For example, a subordinate type, including only $\frac{1}{2}n(n-1)(n-2)$ distinct equations, of $2(n-2)$ terms each, between the symbols (fgh), may thus be formed, by subtracting (95) from (97), under the condition that $e\,f\,g$ shall still denote some three unequal indices; namely,

$$0 = \Sigma\{(feh)(ghe) - (geh)(fhe)\}; \tag{124}$$

or more fully, but at the same time with the suppression of a few parentheses, which do not appear to be at this stage essential to clearness,

$$(fge)(eff + egg) = \Sigma'(geh\,.\,fhe - feh\,.\,ghe): \tag{125}$$

this last formula admitting also of being obtained from (122), by interchanging f and g, and subtracting. Again, a type which is in general still more subordinate, as including only $\frac{1}{2}n(n-1)$ distinct equations, of $2(n-2)$ terms each, may be derived by the same process from (120); namely the type,

$$0 = \Sigma(fgh)(hff + hgg); \tag{126}$$

or in a slightly more expanded form,

$$(fge)(eff + egg) = \Sigma^{\backprime}(gfh)(hff + hgg); \tag{127}$$

which may also be easily derived, in the same way, from (121). It will, however, be found, by pursuing a little further the analysis of [14.], that the equations of this last type, (126) or (127), are always consequences of the equations of the intermediate type, (124) or (125); the sum of the $n - 2$ equations of the form (125), which answer to the various values of e that remain when f and g have been selected, being in fact equivalent to the formula (126). It will also be found, by the same kind of analysis, that the intermediate equations of the type (124) or (125) are *generally* deducible from those of the form (123). But on the subject of these *general reductions*, connected with the elimination of the symbols (fg) or (gf), it may be proper to add a few words.

[19.] Let us admit, at least as temporary abridgements, the notations

$$[fg] = \Sigma(fgh\,.\,hff); \quad [fge] = \Sigma(fhe\,.\,geh); \tag{128}$$

where e, f, g are any three unequal indices, and h varies under Σ, as before, from 1 to n. Then the formula (95) gives $n - 1$ distinct equivalents for the symbol (fg), of which one is by (120) of the form $[fg]$, and the $n - 2$ others are each of the form $[fge]$; in such a manner that we may write, instead of (95), with these last notations, the system of the two formulæ,

$$(fg) = [fg], \quad (fg) = [fge]; \tag{129}$$

whereof the latter is equivalent to a system of $n - 2$ equations: and of course, instead of (97), we may in like manner write

$$(gf) = [gf], \quad (gf) = [gfe]. \tag{130}$$

The equations (99) may now be thus presented:

$$\left.\begin{aligned} (n-1)(fg) &= [fg] + \Sigma^{\backprime}[fge] = \Sigma\Sigma(fhe\,.\,geh); \\ (n-1)(gf) &= [gf] + \Sigma^{\backprime}[gfe] = \Sigma\Sigma(feh\,.\,ghe); \end{aligned}\right\} \tag{131}$$

where e under the sign Σ' is distinct from each of the two indices f and g; but, under the double sign $\Sigma\Sigma$, both e and h may each receive any one of the values from 1 to n. The two double sums are equal, as in [14.], and therefore we must have, *identically*,

$$[fg] + \Sigma'[fge] = [gf] + \Sigma'[gfe]; \tag{132}$$

the equation (100) being at the same time seen again to be a consequence, by simple additions, of the formulæ (95) and (97). Thus, after assigning any two unequal values to the indices f and g, we see that the two symbols, (fg), (gf); the two others, $[fg]$, $[gf]$; the $n - 2$ symbols, $[fge]$; and the $n - 2$ symbols, $[gfe]$, are indeed all equal to each other: but that the $2n - 1$ equations between these $2n$ equal symbols are connected by a *relation*, such that any $2n-2$ of them, which are distinct among themselves, *include* the remaining one: and that therefore, after the elimination of (fg) and (gf), there remain only $2(n-2)$ *distinct* equations of condition, as was otherwise shown in [14.]. But, in that paragraph, we proposed to form those resulting conditions on a plan which may now be represented by the formulæ

$$[fg] = [fge], \quad [gf] = [gfe]; \tag{133}$$

whereas we now prefer, for the sake of the convenience gained by the disappearance of certain terms in the subtractions, to employ that other mode of combination, which conducted in [18.] to the formula (123), and may now be denoted as follows:

$$[fg] = [gfe], \quad [gf] = [fge]. \tag{134}$$

Summing these last with respect to e, we find

$$(n-2)[fg] = \Sigma'[gfe], \quad (n-2)[gf] = \Sigma'[fge]; \tag{135}$$

and therefore, by the identity (132),

$$(n-3)[gf] = (n-3)[fg]. \tag{136}$$

If, then, $n > 3$, we are entitled to infer, from (123) or (134), the following formula, which is equivalent to (126),

$$[gf] = [fg]; \tag{137}$$

and therefore also by (134) this other type, equivalent to (124),

$$[fge] = [gfe], \tag{138}$$

which includes $n-2$ equations, when f and g are given, and conducts, reciprocally, by (132), to (137). *In general*, therefore, if we adopt the type (134), we need not retain *also* either of these two latter types, (137), (138). But in the particular *case* where $n = 3$, that is, in the case of quadrinomes, the identity (132) reduces the two equations (134) to one, after f and g have been selected; and with this one we must then combine either of the two equations (137) or (138), which in this case become identical with each other.

[20.] In particular, for this case of *quadrinomials* ($n = 3$), we have with the notations (114), (128), the four following values for (23), or for b_1 (compare again a note to [12.]):

$$\left.\begin{aligned}
[23] &= 231\,.\,122 + 233\,.\,322 = l_1 n_1 - n_2 m_3;\\
[231] &= 211\,.\,311 + 231\,.\,313 = -m_2 n_3 + l_1 m_1;\\
[32] &= 321\,.\,133 + 322\,.\,233 = l_1 m_1 - m_3 n_2;\\
[321] &= 311\,.\,211 + 321\,.\,212 = -n_3 m_2 + l_1 n_1;
\end{aligned}\right\} \tag{139}$$

but, whether we equate the first to the fourth, or the second to the third of these expressions for b_1, in conformity with the type (134), we obtain only one common equation of condition, $n_2 m_3 = n_3 m_2$, equivalent indeed by cyclical permutation to three, namely to the following,

$$0 = n_2 m_3 - n_3 m_2 = n_3 m_1 - n_1 m_3 = n_1 m_2 - n_2 m_1; \tag{140}$$

which evidently agree with certain simple combinations of the six equations on the two last lines of (86). If however we compare either the first value (139) with the third, or the second of those values with the fourth, according to the type (137) or (138), we find by each comparison the common condition $l_1n_1 = l_1m_1$, and thus recover the equations of the first line of (85). In this way then we may obtain the required number of six distinct equations, with two terms each, between the nine symbols (fgh), or $l_1 \ldots n_3$, for the case of quadrinomes, by elimination of the three symbols (fg), or of b_1, b_2, b_3.

[21.] For the case of *quines* ($n = 4$), the general theory requires that the corresponding elimination of the $6 = \frac{1}{2}n(n-1)$ symbols of this form (fg), or $b_1 \ldots c_3$, should conduct to $24 = n(n-1)(n-2)$ distinct equations of condition, with $4 = 2(n-2)$ terms each, between the $\frac{1}{2}n^2(n-1) = 24$ symbols of the form (fgh), or $l_1 \ldots u_3$, each equation thus obtained being homogeneous, and of the second dimension; and that all these 24 conditions should be included in the formula (134), or in the single type (123). And in fact we thus obtain, by comparison of the six expressions for b_1, of which one is

$$b_1 = (23) = [23] = \Sigma(23h \,.\, h22) = l_1n_1 - n_2m_3 - p_1r_2, \tag{141}$$

the four following equations of condition, included in that type or formula:

$$\left.\begin{aligned} 0 = [23] - [321] = [32] - [231]; \\ 0 = [23] - [324] = [32] - [234]; \end{aligned}\right\} \tag{142}$$

that is, with the notations $l_1 \ldots u_3$,

$$\left.\begin{aligned} n_2m_3 - m_2n_3 = p_3s_3 - p_1r_2 = p_1r_3 - p_2t_2; \\ n_2m_3 + u_2u_3 = p_3s_3 + l_1m_1 = l_1n_1 - p_2t_2; \end{aligned}\right\} \tag{143}$$

while we have in like manner six expressions for c_1, of which one is

$$c_1 = [41] = \Sigma(41h \,.\, h44) = -(r_1u_1 + s_1u_2 + t_1u_3), \tag{144}$$

and of which the comparison conducts to the four other distinct conditions:

$$\left.\begin{aligned} r_1u_1 - n_1r_2 = l_3t_3 + n_3t_1 = -l_2s_2 - t_1u_3; \\ r_1u_1 + m_1r_3 = l_3t_3 - s_1u_2 = -l_2s_2 - m_2s_1; \end{aligned}\right\} \tag{145}$$

where cyclical permutation of indices is still allowed. The equations obtained from the types (137), (138) would be found (as the theory requires) to be merely consequences of these; for example, by making $e = 1$, $f = 2$, $g = 3$, $h = 4$, those two types give only the conditions,

$$l_1(n_1 - m_1) = p_1(r_2 + r_3) = p_2t_2 + p_3s_3, \tag{146}$$

which are obviously included in (143).

[22.] With respect to those other homogeneous equations of the second dimension, between the symbols (fgh), which are obtained immediately, or without any elimination of the symbols (f), (fg), from the general conditions of association, and are included in the formula (96), they may now be developed as follows.

Making $k = f$ in (96), and then interchanging f and e, for the sake of comparison with (123), we obtain the type

$$\text{III.}\ldots \qquad fgg\,.\,gee - gff\,.\,fee = \Sigma'(gfh\,.\,hee); \tag{147}$$

which includes generally $\frac{1}{2}n(n-1)(n-2)$ distinct equations, of $n-1$ terms each. For quines, we have thus 12 equations of 3 terms sufficiently represented by the following:

$$\left.\begin{aligned} n_1n_2 - m_2m_1 &= p_3r_3, \quad n_1u_2 + m_2u_1 = -l_3u_3; \\ r_2m_2 + u_2r_1 &= s_2n_3, \quad\ \ r_2n_2 - u_2r_3 = +t_2m_1; \end{aligned}\right\} \tag{148}$$

the value 4 being attributed to the index h or e, in forming the equations on the first line, but to f or g for the second line. For quadrinomes, the corresponding equations are only three, namely

$$0 = n_1n_2 - m_1m_2 = n_2n_3 - m_2m_3 = n_3n_1 - m_3m_1; \tag{149}$$

which however are sufficient, in conjunction with the three lately marked as (140), to reproduce the six equations of the two last lines of (86). In general, by adding and subtracting the two types (123), (147), we obtain the formula,

$$(fee \pm fgg)(gee \mp gff) = \Sigma'(fgh)(hff \mp hee) - \Sigma'(feh\,.\,ghe); \tag{150}$$

where, as a verification, if we take the lower signs, and interchange f and g, so as to recover the first member with the upper signs, the comparison of the two expressions for that member conducts to an equation between the two second members, which may also be obtained by the comparison of (125) and (127).

[23.] Again, making $f = e$ in (96), and then changing k to f, we obtain the formula,

$$\text{IV.}\ldots \qquad 0 = (egf)(eff + egg) + \Sigma'(egh\,.\,ehf); \tag{151}$$

where $e\,f\,g\,h$ are again unequal indices. This IVth type includes generally $n(n-1)(n-2)$ distinct equations, with $n-1$ terms each. For the case $n = 3$ there arise thus 6 equations of 2 terms, namely the six on the two last lines of (85); so that the 18 equations (85), (86), for associative quadrinomials, have thus been completely reproduced, as consequences of the general theory. For the case of quines, the type (151) gives 24 equations of 3 terms, which may be represented as follows:

$$\left.\begin{aligned} 0 &= l_2(n_1 - m_1) + p_2s_1 = l_3(n_1 - m_1) + p_3t_1; \\ 0 &= s_2(r_2 + r_3) + t_1t_2 = t_3(r_2 + r_3) + s_3s_1; \end{aligned}\right\} \tag{152}$$

$$\left.\begin{aligned} 0 &= t_1(u_1 - m_1) + l_3s_1 = p_2(u_1 - m_1) + l_2p_3; \\ 0 &= s_1(u_1 + n_1) - l_2t_1 = p_3(u_1 + n_1) - l_3p_2; \end{aligned}\right\} \tag{153}$$

either h or e being $= 4$ in (152), and either f or g having that value in (153), while 1, 2, 3 may still be cyclically permuted.

[24.] Finally, by supposing, as in (119), that $e\,f\,g\,k$ are four unequal indices, and that h under Σ'' is unequal to each of them, we obtain from (96) one other type, including *generally* $\frac{1}{2}n(n-1)(n-2)(n-3)$ equations, of $n-1$ terms each, but furnishing no new conditions of association for quadrinomials: namely,

$$\mathrm{V}\ldots\quad efk\,.\,gee + fgk\,.\,egg + egk\,.\,fkk = \Sigma''(fhk\,.\,geh). \tag{154}$$

For quines, the sum Σ'' vanishes, and we obtain twelve equations of three terms each, which may (with the help of permutations) be all represented by the four following:

$$\left.\begin{aligned} 0 &= l_1r_2 - u_2s_3 - n_3t_2 = l_1r_3 - m_2s_3 + u_3t_2, \\ 0 &= l_1r_1 + n_2s_3 + m_3t_2 = n_1p_1 - m_2p_2 - u_3p_3; \end{aligned}\right\} \tag{155}$$

where the index 4 has been made to coincide with e or with g in the first line, but with f or k in the second.

[25.] In general, the number of distinct associative equations, included in the three last types (147), (151), (154), or III., IV., V., which have been all derived from the formula (96), and have been obtained without elimination of (f) or (fg), amounts in the aggregate to

$$\tfrac{1}{2}n(n-1)(n-2) + n(n-1)(n-2) + \tfrac{1}{2}n(n-1)(n-2)(n-3) = \tfrac{1}{2}n^2(n-1)(n-2); \tag{156}$$

as, by the analysis of [14.], it ought to do. And when we add this number to the $n(n-2)$ of the type I., or (113), and to the $n(n-1)(n-2)$ of the type II., or (123), obtained by such elimination, we have in all this other number,

$$\tfrac{1}{2}n^2(n-1)(n-2) + n^2(n-2) = \tfrac{1}{2}n^2(n+1)(n-2), \tag{157}$$

of distinct and homogeneous equations of the second dimension, between the $\frac{1}{2}n^2(n-1)$ symbols of the form (fgh): as, by the formulæ (109), (110) of [15.], we ought to have. As regards the *signification* of the five foregoing principal types, which it has been thought convenient to distinguish among themselves, and to arrange according to the various ways in which they involve the symbols of the form (eff), it will be found, on reviewing the analysis employed, that they all express *ultimately* only consequences of that *one* very simple and useful formula,

$$\mathrm{V}\iota\,\mathrm{V}\iota'\iota'' = \iota''\,\mathrm{S}\iota\iota' - \iota'\,\mathrm{S}\iota''\iota, \tag{67}$$

which, with a slightly different notation, has been elsewhere shown to be so important in the Calculus of Quaternions*. In fact, the equations (95) ... (98), on which those five separate types have been founded, may all be deduced from (67) and (69), whereof the latter is a consequence of the former.

Observatory of Trinity College, Dublin,
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* See the author's Lectures on this subject, cited in former notes to this paper.

Section IV.

[26.] For quines, the equations of condition between the 24 symbols $l_1 \ldots u_3$ amount (as has already been remarked) to 80 in all; namely to 8, 24, 12, 24, and 12 equations, included respectively in the five types last mentioned, and sufficiently developed above, by the formulæ (117) (118) (143) (145) (148) (152) (153) (155): which also enable us, with the help of (141), (144), to determine the values of the four symbols $a_1 \ldots a_4$, and of the six other symbols $b_1 \ldots c_3$, when values of $l_1 \ldots u_3$ have been found, which satisfy the eighty conditions. And then, if we denote the quine itself by the following expression (compare [1.]),

$$\mathrm{P} = x_0 + \iota_1 x_1 + \iota_2 x_2 + \iota_3 x_3 + \iota_4 x_4, \tag{158}$$

which is a little more symmetric than the form (111), the *laws of multiplication of any two such quines*, P, P′, will be sufficiently expressed by the formulæ

$$\left.\begin{array}{l} \iota_1^2 = a_1, \quad \iota_4^2 = a_4, \quad \mathrm{S}\iota_2\iota_3 = b_1, \quad \mathrm{S}\iota_1\iota_4 = c_1, \\ \mathrm{V}\iota_2\iota_3 = \iota_1 l_1 + \iota_2 m_3 + \iota_3 n_2 + \iota_4 p_1, \\ \mathrm{V}\iota_1\iota_4 = \iota_1 r_1 + \iota_2 s_1 + \iota_3 t_1 + \iota_4 u_1; \end{array}\right\} \tag{159}$$

if we remember that 1, 2, 3 may still be cyclically permuted, and that the law of conjugation (32) gives

$$\mathrm{K}\iota'\iota = \iota\iota', \quad \mathrm{S}\iota'\iota = \mathrm{S}\iota\iota', \quad \mathrm{V}\iota'\iota = -\mathrm{V}\iota\iota'. \tag{160}$$

For in this manner, by (41), if ϖ denote, as in (14), the vector part of P, so that

$$\varpi = \iota_1 x_1 + \iota_2 x_2 + \iota_3 x_3 + \iota_4 x_4, \tag{161}$$

we shall have

$$\begin{aligned} \mathrm{S}\varpi\varpi' = {} & a_1 x_1 x_1' + a_2 x_2 x_2' + a_3 x_3 x_3' + a_4 x_4 x_4' \\ & + b_1(x_2 x_3' + x_3 x_2') + \&\mathrm{c.} + c_1(x_1 x_4' + x_4 x_1') + \&\mathrm{c.}, \end{aligned} \tag{162}$$

$$\begin{aligned} \mathrm{V}\varpi\varpi' = {} & (\iota_1 l_1 + \iota_2 m_3 + \iota_3 n_2 + \iota_4 p_1)(x_2 x_3' - x_3 x_2') + \&\mathrm{c.} \\ & + (\iota_1 r_1 + \iota_2 s_1 + \iota_3 t_1 + \iota_4 u_1)(x_1 x_4' - x_4 x_1') + \&\mathrm{c.}, \end{aligned} \tag{163}$$

each "&c." representing terms obtained by the permutations already mentioned; and if the constants $a\,b\,c\,l\,m\,n\,p\,r\,s\,t\,u$ have been chosen so as to fulfil the conditions above developed, we may then conclude (compare (51)) that the following EQUATIONS OF ASSOCIATION hold good, for the multiplication of any *three* such *vector-units* ι, or *quadrinomial vectors* ϖ, or *quinquinomial expressions* P, whether equal or unequal among themselves:

$$\iota \,.\, \iota'\iota'' = \iota\iota' \,.\, \iota''; \quad \varpi \,.\, \varpi'\varpi'' = \varpi\varpi' \,.\, \varpi''; \quad \mathrm{P} \,.\, \mathrm{P}'\mathrm{P}'' = \mathrm{P}\mathrm{P}' \,.\, \mathrm{P}''; \tag{164}$$

which it has been the main object of our recent investigations to establish.

[27.] Without pretending to do more, on the present occasion, than merely to *exemplify the possibility of satisfying*, for *quines*, the foregoing equations of association, I may here remark that if we restrict the question by assuming (with the usual permutations),

$$(\mathrm{A},\ \mathrm{B})^{*} \ldots \qquad n_1 = m_1, \quad p_1 = 0, \quad u_1 = 0, \tag{165}$$

then numerous simplifications take place, and the 80 equations between the 24 symbols $l\,m\,n\,p\,r\,s\,t\,u$ are found to reduce themselves to 44 equations between the 15 symbols $l\,m\,r\,s\,t$, obtained from the five types I. to V. of recent paragraphs, which may be thus denoted and arranged:

from type I.	$m_1^2 = l_2 l_3, \quad r_2^2 - r_1^2 = s_3 t_1 - s_2 t_3;$	(166)
from II. and III.	$m_1 l_1 = m_2 m_3, \quad m_1 r_1 = m_2 s_1 = m_3 t_1,$	(167)
and	$m_1 r_2 = l_2 s_2, \quad m_1 r_3 = l_3 t_3, \quad m_1(r_1 + r_2 + r_3) = 0;$	(168)
from IV.	$s_2(r_2 + r_3) = -t_1 t_2, \quad t_3(r_2 + r_3) = -s_3 s_1,$	(169)
and	$m_1 s_1 = l_2 t_1, \quad m_1 t_1 = l_3 s_1;$	(170)
and from V.	$m_1 s_2 = l_3 r_2, \quad m_1 t_3 = l_2 r_3, \quad l_1(r_1 + r_2 + r_3) = 0.$	(171)

Now these conditions may all be satisfied in each of two principal ways, conducting to *two distinct systems of associative quines*, which may be called Systems (A) and (B), but which are *not the only possible systems* of such quines, because we *need* not have commenced by assuming the equations (165), although that assumption has simplified the problem. For first we may suppose that the constants l and m are different from zero, but that the constants r are connected by the relation

$$(\mathrm{A}) \ldots \qquad r_1 + r_2 + r_3 = 0; \tag{172}$$

or secondly, we may reject this relation between the constants r, and suppose instead that the six constants l and m all vanish, so that

$$(\mathrm{B}) \ldots \qquad l_1 = l_2 = l_3 = m_1 = m_2 = m_3 = 0. \tag{173}$$

With the first supposition, (172), we are to combine the nine relations between the fifteen constants $l\,m\,r\,s\,t$, which are sufficiently expressed by the formula (167), or by the following:

$$(\mathrm{A}_1) \ldots \qquad l_1 = m_1^{-1} m_2 m_3, \quad s_1 = m_2^{-1} m_1 r_1, \quad t_1 = m_3^{-1} m_1 r_1; \tag{174}$$

and then all the other conditions of association will be found to be satisfied, if we equate each of the ten symbols $a\,b\,c$ to zero, or if we establish this other formula,

$$(\mathrm{A}_2) \ldots \qquad a_1 = 0, \quad b_1 = 0, \quad c_1 = 0, \quad a_4 = 0 : \tag{175}$$

* This line is lettered thus, because it contains the conditions common to the two systems (A) and (B) of associative quines, which are deduced a little further on.

while there will still remain *five arbitrary constants* of the system, for instance $r_1\, r_2\, m_1\, m_2\, m_3$. With the second supposition, (173), we are to combine four distinct relations between the nine constants $r\, s\, t$, contained in the formula (169), or in the following:

$$(\mathrm{B}_1)^* \ldots \qquad r_1 + r_2 = -s_1^{-1} t_1 t_3, \quad s_1 s_2 s_3 = t_1 t_2 t_3; \tag{176}$$

which give also, as a consequence, this other relation:

$$(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = -s_1 s_2 s_3; \tag{177}$$

and then the other conditions of association will all be satisfied, if we make, instead of (175),

$$(\mathrm{B}_2) \ldots \qquad a_1 = b_1 = c_1 = 0; \quad a_4 = (r_1 + r_2 + r_3)^2 : \tag{178}$$

this system also involving five arbitrary constants, for example $s_1\, s_2\, t_1\, t_2\, t_3$. The assertion respecting *quines*, which was made near the end of [16.], has therefore been fully justified.

[28.] Finally, as regards the system (A) of quines, it may be observed, 1st, that *in this system*, by (162) and (175), we have generally,

$$(\mathrm{A}_3) \ldots \qquad \mathrm{S}\varpi\varpi' = 0; \tag{179}$$

or that "the product of any two quadrinomial vectors ϖ, ϖ', reduces itself to a pure vector;" and 2nd, that, by (163), (165), "this vector product, $\varpi\varpi'$, is of trinomial form, involving no part with ι_4 for a factor." This product is therefore already seen to be of the form

$$\varpi\varpi' = \iota_1 X_1 + \iota_2 X_2 + \iota_3 X_3; \tag{180}$$

but I say, 3rd, that "its three coefficients, or *coordinates*, X_1, X_2, X_3 have constant ratios," or that "the product $\varpi\varpi'$ may be constructed by a right line in space of which the direction though not the length is fixed," and which may therefore be conceived to "coincide in position with one fixed axis (ξ) *of the system*." In fact, by (163), (165), (174), we have

$$m_1 X_1 = m_2 X_2 = m_3 X_3 = X, \tag{181}$$

and therefore

$$(\mathrm{A}_4) \ldots \qquad \varpi\varpi' = X\xi, \tag{182}$$

if we make for abridgement

$$(\mathrm{A}_5) \ldots \quad \left\{ \begin{array}{ll} X = m_2 m_3 (x_2 x_3' - x_3 x_2') + m_3 m_1 (x_3 x_1' - x_1 x_3') + m_1 m_2 (x_1 x_2' - x_2 x_1') & \\ \qquad + m_1 r_1 (x_1 x_4' - x_4 x_1') + m_2 r_2 (x_2 x_4' - x_4 x_2') + m_3 r_3 (x_3 x_4' - x_4 x_3'), & (183) \\ \text{and } \xi = m_1^{-1}\iota_1 + m_2^{-1}\iota_2 + m_3^{-1}\iota_3. & (184) \end{array} \right.$$

* It must be observed that these equations (176), which are part of the *basis* of the system (B), are *true* in the system (A) also, as corollaries from (174) and (172) which last equation does not hold in (B); and which allows us to reduce, for (A) but not for (B), the relation (177) to the simpler form $r_1 r_2 r_3 = s_1 s_2 s_3$.

In the 4th place, "if any quadrinomial vector ϖ be multiplied by or into the axis ξ, the product vanishes;" or in symbols,

$$(\mathrm{A}_6)\ldots \qquad \xi\varpi = 0, \quad \varpi\xi = 0; \tag{185}$$

because by (172) the scalar coefficient X becomes $= 0$, if we change either x_1, x_2, x_3, and x_4, or x'_1, x'_2, x'_3, and x'_4, to m_1^{-1}, m_2^{-1}, m_3^{-1}, and 0, respectively. This coefficient X vanishes also, when we equate $x'_1\,x'_2\,x'_3\,x'_4$ to $x_1\,x_2\,x_3\,x_4$ respectively; and hence, or from (179), we may infer, 5th, that "in this system of quines (A), the square of every quadrinomial vector vanishes." And finally, by an easy combination of the formulæ (182), (185), or of the 3rd and 4th of the foregoing properties of this system, we see, 6th, that, in it, "every product of three quadrinomial vectors vanishes;" or that

$$(\mathrm{A}_7)\ldots \qquad \varpi\varpi' \,.\, \varpi'' = 0, \quad \varpi \,.\, \varpi'\varpi'' = 0. \tag{186}$$

[29.] The *associative property* (164) is therefore verified for the system (A), by showing that, *in it*, each of the two ternary products of vectors, which ought to be equal, vanishes. In the system (B), it is easy to see that any such ternary product must be itself a vector; because, in (B), no binary product of vectors involves ι_4, nor does any such product involve a scalar part, except what arises from ι_4^2. We have, therefore, *here*, this new result,

$$(\mathrm{B}_3)\ldots \qquad \mathrm{S}(\varpi\varpi' \,.\, \varpi'') = \mathrm{S}(\varpi \,.\, \varpi'\varpi'') = 0. \tag{187}$$

And when we proceed to develop these two ternary products, the associative property of multiplication is again found to be verified, under the form,

$$(\mathrm{B}_4)\ldots \qquad \varpi\varpi' \,.\, \varpi'' = \varpi \,.\, \varpi'\varpi'' = a_4(\varpi x'_4 x''_4 - \varpi' x''_4 x_4 + \varpi'' x_4 x'_4); \tag{188}$$

where it is worth observing that, by the laws of the system in question, the result may be put under this other and somewhat simpler form:

$$(\mathrm{B}_5)\ldots \qquad \varpi\varpi' \,.\, \varpi'' = \varpi \,.\, \varpi'\varpi'' = \varpi\,\mathrm{S}\varpi'\varpi'' - \varpi'\,\mathrm{S}\varpi''\varpi + \varpi''\,\mathrm{S}\varpi\varpi'. \tag{189}$$

Indeed, this last expression might have been *foreseen*, as a consequence from the *general principles* of this whole theory of *associative polynomes**, combined with the particular property (187) of the quines (B). For, by that property, each of the two ternary products is equal to its own vector part; but by (101) we have, *generally*, in the present theory, as in the calculus of quaternions, the following expression for the *vector part of the product of any three vectors*, of any such associative polynomes as we are considering:

$$\mathrm{V} \,.\, \rho\sigma\tau = \rho\,\mathrm{S}\sigma\tau - \sigma\,\mathrm{S}\tau\rho + \tau\,\mathrm{S}\rho\sigma; \tag{190}$$

which is a formula of continual application in Quaternions, and in these extensions also is important.

Observatory of Trinity College, Dublin,
August 16, 1854.

* It will hereafter be proved *generally* that for *all* associative polynomes which satisfy the law of conjugation (though *not exclusively* for such *associative* polynomes), the *tensor*, as defined in [6.], is also a *modulus*; which theorem can be verified for the quines (A) and (B), and for the quadrinomes and tetrads so lettered in [13.], as well as for the trinomes [11.].

Section V.

[30.] In applying to associative *quines* the general theory of the Third Section*, we may (as has been seen) omit each of the signs Σ' as unnecessary, the index h receiving only one value in the sum thereby indicated; and may suppress each sum Σ'' as vanishing. In this manner the type IV., or the formula (151), becomes,

$$\text{IV.} \ldots \qquad (egf)(eff + egg) = geh \,.\, ehf; \tag{191}$$

while the equation (127), already derived as a sub-type from II., gives, by interchanging e and f,

$$(egf)(fee + fgg) = (geh)(hee + hgg). \tag{192}$$

Multiplying the latter of these two equations by $eff + egg$, and the former by $fee + fgg$, and subtracting, we eliminate the symbol (egf), and find that

$$(geh)\{(ehf)(fee + fgg) - (eff + egg)(hee + hgg)\} = 0; \tag{193}$$

and a similar elimination of (geh) gives the equation,

$$(egf)\{(ehf)(fee + fgg) - (eff + egg)(hee + hgg)\} = 0. \tag{194}$$

And because $(geh) = -(egh)$, by (34), we may make any separate or combined interchanges, of e with g, and of f with h, and so vary the expression within the $\{\,\}$, without introducing any new factor, distinct from (egf) and (egh), outside them. If, then, for any particular arrangement of the four unequal indices, e, f, g, h, as chosen from among the four numbers 1, 2, 3, 4, the *two* following conditions are not *both* satisfied,

$$(egf) = 0, \quad (egh) = 0, \tag{195}$$

we must have, for that arrangement of the indices, a system of *four* other equations, whereof one is

$$\text{VI.} \ldots \qquad (ehf)(fee + fgg) = (eff + egg)(hee + hgg), \tag{196}$$

while the three others are formed from it, by the interchanges just now mentioned. And if we further suppose that the two sums, $fee + fgg$ and $hee + hgg$, are for the same arrangement different from zero, and write for abridgement, *as a definition*,

$$(ehf)_0 = (fee + fgg)^{-1}(eff + egg)(hee + hgg), \tag{197}$$

the four equations furnished by the formula (196), which may be regarded as a *sixth type for quines*, may be concisely expressed as follows:

$$\left.\begin{array}{ll} (ehf) = (ehf)_0, & (ghf) = (ghf)_0; \\ (efh) = (efh)_0, & (gfh) = (gfh)_0. \end{array}\right\} \tag{198}$$

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* Comprising paragraphs [17.] to [25.], and published in the Phil. Mag. for October, 1854.

With the notations $l_1 \ldots u_3$, for the symbols (efg), (eff), we find thus that unless l_1 and p_1 both vanish, we must have the four equations,

$$\left.\begin{aligned} t_2(n_1 - m_1) &= (m_2 - n_2)(r_2 + r_3); \quad & s_3(n_1 - m_1) &= (m_3 - n_3)(r_2 + r_3); \\ p_3(r_2 + r_3) &= (u_2 + n_2)(n_1 - m_1); \quad & p_2(r_2 + r_3) &= (m_3 - u_3)(n_1 - m_1); \end{aligned}\right\} \tag{199}$$

and that unless $s_1 = 0$, $t_1 = 0$, then

$$\left.\begin{aligned} -l_2(u_2 - m_2) &= (n_1 + u_1)(n_3 + u_3); \quad & t_3(u_2 - m_2) &= (r_1 + r_2)(n_3 + u_3); \\ l_3(n_3 + u_3) &= (u_1 - m_1)(u_2 - m_2); \quad & s_2(n_3 + u_3) &= (r_1 + r_3)(u_2 - m_2). \end{aligned}\right\} \tag{200}$$

[31.] Supposing then that no one of the twelve symbols (efg) vanishes, and that each of the twelve sums $eff + egg$ is also different from zero, the various arrangements of the four indices $e\,f\,g\,h$ give us a system of twenty-four equations, included in the new type VI., or in any one of the four formulæ (198); which equations may, by (34), be arranged in twelve pairs, as follows:

$$(ehf) = (ehf)_0 = -(hef)_0. \tag{201}$$

It might seem that *twelve* equations between the twelve symbols of the form (eff) should thus arise, by the comparison of two expressions for each of the twelve symbols of the form (efg); but if we write for abridgement

$$[g] = (fee + fgg)(fhh + fgg)\{(ehf)_0 + (hef)_0\}, \tag{202}$$

and observe that by the definition (197) of the symbol $(ehf)_0$, we have then

$$[g] = (eff + egg)(fhh + fgg)(hee + hgg) + (hff + hgg)(fee + fgg)(ehh + egg), \tag{203}$$

we shall see that this quantity $[g]$ is independent of the arrangement of the three indices e, f, h; and therefore that the twelve equations between the twelve symbols (eff), obtained through (201), reduce themselves to the *four* following relations,

$$[e] = 0, \quad [f] = 0, \quad [g] = 0, \quad [h] = 0; \tag{204}$$

which are not even all distinct among themselves, since any *three* of them include the fourth. An easy combination of the two first or of the two last of these four relations (204) conducts to this other formula, which is equivalent to three distinct equations:

$$\begin{aligned} &(eff + ehh)(fee + fgg)(ghh + gee)(hgg + hff) \\ &\qquad = (eff + egg)(fee + fhh)(ghh + gff)(hgg + hee); \end{aligned} \tag{205}$$

and which may also be thus written,

$$(hef)_0(egf)_0 = (ehf)_0(gef)_0. \tag{206}$$

With the notations $l_1 \ldots u_3$, the twenty-four equations (201) are sufficiently represented by the formulæ (199) and (200), if cyclical permutation of the indices be employed; the four equations (204) take the forms,

$$\left.\begin{aligned}(r_1+r_2)(n_3+u_3)(m_2-n_2)&=(r_1+r_3)(m_2-u_2)(m_3-n_3),\\(n_1+u_1)(n_2+u_2)(n_3+u_3)&=(m_1-u_1)(m_2-u_2)(m_3-u_3);\end{aligned}\right\}\tag{207}$$

whereof the equation on the second line may be obtained from the product of the three represented by the first line: and the three equations (205) or (206) are included in the following, which is evidently a consequence of (207),

$$(r_1+r_3)(n_1+u_1)(n_2+u_2)(n_3-m_3)=(r_1+r_2)(u_1-m_1)(n_2-m_2)(u_3-m_3).\tag{208}$$

[32.] As regards the quotients and products of the symbols (efg), which we shall continue to write occasionally without parentheses, we have by type VI., or by (197), (198),

$$\frac{ehf}{ghf}=\frac{eff+egg}{gff+gee};\tag{209}$$

$$ehf\,.\,efh=(eff+egg)(ehh+egg);\tag{210}$$

$$ehf\,.\,gfh=(eff+egg)(ghh+gee);\tag{211}$$

eliminating (ehf) between the two last of which three equations, we obtain a relation of the same form as the first. Interchanging g and h in (210), and subtracting, we find that

$$\text{I.}\ldots\qquad ehf\,.\,efh-egf\,.\,efg=(egg)^2-(ehh)^2;\tag{212}$$

but this is precisely what the type I., or the formula (113), becomes for quines, when we cyclically advance the four indices in the order $f\,e\,g\,h$; the conditions (177), (118) of that *first type* will therefore be satisfied, if we satisfy those of the *sixth.* Had we divided instead of subtracting, we should have found

$$\frac{ehf\,.\,efh}{egf\,.\,efg}=\frac{eff+egg}{eff+ehh}.\tag{213}$$

To interchange f, g, and divide, would only lead by (210) to another equation of this last form; but the same operations performed on (211) conduct to the equation

$$\frac{ehf}{heg}=\frac{ghh+gee}{fhh+fee};\tag{214}$$

which, when we interchange g and h, reproduces the formula (192); and shows thereby that the sub-type (127), included under type II., is satisfied by our new type VI., which indeed it had assisted to discover. The same equation (192) may also be derived from the formula (205), by dividing each member of that formula by $(fee+fhh)(hff+hgg)$, and attending to the expressions given by type VI., for (egf) and (geh) respectively. To interchange e h, in (211) and divide, would only conduct to another equation of the same form as (214). Permuting cyclically the three indices e, f, g in (209), and multiplying together the two equations so obtained therefrom, the product gives

$$\frac{fhg \,.\, ghe}{ehg \,.\, fhe} = \frac{gff + gee}{eff + egg}; \tag{215}$$

and if we multiply this equation by (209) itself, we find that

$$ehf \,.\, fhg \,.\, ghe = ghf \,.\, ehg \,.\, fhe. \tag{216}$$

In fact if we operate thus on the expression (197) for $(efh)_0$, or for its equal (ehf), or on the formula (196), we are led to this new equation,

$$ehf \,.\, fhg \,.\, ghe = (hee + hgg)(hff + hee)(hgg + hff), \tag{217}$$

of which the second member does not alter, when we interchange any two of the three indices e, f, g. Another multiplication of three equations of the form (209), with the cycle $e\,g\,h$, conducts to the equation $[f] = 0$ of (204). Interchanging e, h in (210), and substituting the value so obtained for the product of the two extreme factors of the second member of (217), we find this other expression,

$$ehf \,.\, fhg \,.\, ghe = hef \,.\, hfe \,.\, (hee + hff); \tag{218}$$

which is still seen to remain unaltered, by an interchange of e and f. Interchanging f, g, and dividing, we obtain by (216) an equation of the same form as (213); and if we divide each member of (218) by (hef), we are conducted to the formula

$$\text{IV.} \ldots \qquad fhg \,.\, hge = hfe \,.\, (hee + hff), \tag{219}$$

which is of the same form as the equation (191), or as the type IV., and may be changed thereto by cyclical permutation of the four indices $e\,f\,g\,h$. The same relation (219) may also be derived more directly from type VI., by substitutions of the values (198); for it will be found that the definition (197) gives this *identity*,

$$(fhg)_0(ghe)_0 = (hee + hff)(fhe)_0. \tag{220}$$

The conditions of type IV., like those of type I., and of the subtype (127) of II., are therefore all included in those of the new type VI.; which gives also in various ways this other formula respecting products of four symbols of the form (efg),

$$egh \,.\, fhg \,.\, gfe \,.\, hef = ehg \,.\, fgh \,.\, gef \,.\, hfe: \tag{221}$$

indeed it will be found that the members of this last equation, taken in their order, are respectively equal by (196) to the members of the equation (205).

With the notations $l_1 \ldots u_3$, supposing that none of the twelve constants $l\,p\,s\,t$ vanish, and that the twelve combinations of the forms $n_1 - m_1$, $n_1 + u_1$, $u_1 - m_1$, $r_1 + r_2$, are in like manner different from zero, we find thus, or from the equations (199) (200), combined with their consequences (207), the following among other relations, in which cylical permutation of the indices is still allowed:

$$\left.\begin{aligned} \frac{l_1}{t_2} &= -\frac{n_3+u_3}{r_1+r_3}, & \frac{t_2}{s_3} &= \frac{m_2-n_2}{m_3-n_3}, & \frac{s_3}{l_1} &= \frac{r_1+r_2}{u_2-m_2}, \\ \frac{p_1}{p_2} &= \frac{m_2-u_2}{n_1+u_1}, & \frac{l_1}{p_1} &= \frac{r_2+r_3}{n_1-m_1}, & \frac{s_1}{t_1} &= \frac{n_3+u_3}{m_2-u_2}; \end{aligned}\right\} \tag{222}$$

$$\left.\begin{aligned} l_2l_3 &= (n_1+u_1)(m_1-u_1), & s_2t_3 &= (r_1+r_2)(r_1+r_3), \\ p_1s_2 &= (m_2-n_2)(m_2-u_2), & p_1t_3 &= (m_3-n_3)(n_3+u_3), \\ -l_2s_2 &= (n_1+u_1)(r_1+r_3), & l_3t_3 &= (u_1-m_1)(r_1+r_2), \\ p_1s_1 &= (m_1-n_1)(n_3+u_3), & p_1t_1 &= (m_1-n_1)(m_2-u_2). \end{aligned}\right\} \tag{223}$$

The conditions (152), (153), of the *fourth type*, are satisfied; and we have these other products, of which some have occurred already, in (176), (177), in connexion with the particular systems (A) and (B) of quines:

$$\left.\begin{aligned} s_1s_2s_3 = t_1t_2t_3 &= -(r_1+r_2)(r_2+r_3)(r_3+r_1); \\ s_1p_2l_3 = t_1l_2p_3 &= (n_1+u_1)(m_1-u_1)(m_1-n_1); \\ s_2s_3l_2p_3 &= t_2t_3p_2l_3; \end{aligned}\right\} \tag{224}$$

where the two members of the equation on the last line are easily proved by (223) to be respectively equal to those of (208).

[33.] As yet we have only partially satisfied the conditions of type II., or of the formula (123), which may for quines be written thus:

$$\text{II.} \ldots \qquad fgh\,.\,hff = fee\,.\,gee - fgg\,.\,gff + feh\,.\,ghe. \tag{225}$$

Substituting for the last product in this formula its value given by (211), namely

$$feh\,.\,ghe = (fgg + fhh)(gee + gff), \tag{226}$$

and writing for abridgement

$$v_f = fee + fgg + fhh, \tag{227}$$

we are in this way led to establish the following *seventh type** for quines:

$$\text{VII.} \ldots \qquad fgh\,.\,hff = v_f\,.\,gee + fhh\,.\,gff. \tag{228}$$

* It will be shown that this *single type* (the seventh) *includes all the others*, or is sufficient to express *all* the *general conditions* of association, between the 24 symbols of the forms (efg) and (eff). But the eliminations required for this deduction cannot be conveniently described at this stage.

Or since, by the sixth type, we have already

$$\text{VI.}\dots \qquad fgh\,.\,(hee+hff) = (fee+fhh)(gee+gff), \tag{229}$$

it is only necessary, for the purpose of satisfying the conditions of type II., or the equations (143), (145), to suppose besides that

$$\text{III.}\dots \qquad fgh\,.\,hee = fee\,.\,gff - fgg\,.\,gee; \tag{230}$$

such being the expression which remains, when we subtract (228) from (229). But this last equation (230) is precisely what the type III., or the formula (147), becomes for quines; it reproduces therefore the equations (148): and conversely, if we retain that old type III., it will not be *necessary*, although it may be convenient, to introduce the new type VII., in combination with type VI. And if in (230) we substitute for the symbol (fgh) its value given by (229), and so combine types III. and VI., we obtain the equation

$$\frac{fee+fhh}{hee+hff} = \frac{gff\,.\,(fee+fgg) - fgg\,.\,(gee+gff)}{hee\,.\,(gee+gff)}; \tag{231}$$

that is, by (209),

$$\frac{fge}{hge} + \frac{fgg}{hee} = \frac{gff}{hee}\,.\,\frac{fhe}{ghe}; \tag{232}$$

or finally,

$$\text{V.}\dots \qquad hee\,.\,fge = fgg\,.\,ghe - gff\,.\,fhe. \tag{233}$$

But this is exactly what the type V., or the formula (154), becomes for quines, when we suppress the sum Σ''', change k to h, and advance cyclically the three indices $e\,f\,h$; it includes therefore the equations (155), which were the only remaining conditions of association to be fulfilled. If then we satisfy the *two types*, III. and VI., we shall satisfy *all the conditions of association for quines*: since we shall thereby have satisfied *also* the *four other earlier types*, namely those numbered as I. II. IV. V. It only remains then to consider *what new restrictions* on the constants (eff) are introduced by the comparison of the values which type III. gives for the other constants (efg), as expressed in terms of them, with the values furnished by type VI.; or to discuss the consequences of the following general formula, obtained by eliminating the symbol (fgh) between (229) (230), and not essentially distinct from the recent equations (231):

$$\text{VIII.}\dots \qquad hee\,.\,(fee+fhh)(gee+gff) = (hee+hff)(fee\,.\,gff - gee\,.\,fgg); \tag{234}$$

which contains all the old and new relations, subsisting between the twelve constants of the form (eff), and may be regarded as an *eighth type* for quines.

[34.] Denoting the first minus the second member of (234) by the symbol $[efgh]$, we easily see that

$$\begin{aligned}[efgh] &= gee\,.\,(v_f\,.\,hee + fgg\,.\,hff) + gff\,.\,(fhh\,.\,hee - hff\,.\,fee)\\ &= hee\,.\,(v_f\,.\,gee + fhh\,.\,gff) + hff\,.\,(fgg\,.\,gee - gff\,.\,fee);\end{aligned} \tag{235}$$

and therefore that we have, *identically*,

$$[efgh] = [efhg]; \tag{236}$$

this last or eighth type (234) contains therefore, at most, only a system of twelve equations. Interchanging f and g, and attending to the notation (202), we see, by (203), (234), that of the three equations

$$[efgh] = 0, \quad [egfh] = 0, \quad [e] = 0, \tag{237}$$

any two include the third; if then we only seek what *new* conditions, additional to those marked (204), are to be satisfied by the symbols (eff), or rather by the eight following ratios of those symbols,

$$eff : egg : ehh; \quad fee : fgg : fhh; \quad gee : gff : ghh; \quad hee : hff : hgg, \tag{238}$$

we need only retain at most four new equations, suitably selected from among those furnished by type VIII., such as the four following, which differ among themselves by the initial letters within the brackets, and so belong to different groups,

$$[efgh] = 0, \quad [fghe] = 0, \quad [ghef] = 0, \quad [hefg] = 0; \tag{239}$$

and then to combine these with any three of the four former relations (204), for example with the three first, namely

$$[e] = 0, \quad [f] = 0, \quad [g] = 0; \tag{240}$$

from which the fourth equation $[h] = 0$ would follow, by means of the *identity*,

$$\begin{aligned}(ehf)_0(fge)_0(gfh)_0(heg)_0 &= (ehg)_0(fgh)_0(gfe)_0(hef)_0 \\ &= (eff + egg)(fee + fhh)(gee + ghh)(hff + hgg).\end{aligned} \tag{241}$$

It might seem however that the seven equations (239) and (240), thus remaining, should suffice to determine seven of the eight ratios (238): whereas I have found that it is allowed to *assume two pairs of ratios* arbitrarily, out of the four pairs (238), and then to deduce the two other pairs from them. For I find that it is sufficient to retain, instead of the *twelve* equations included under type VIII., or the *seven* equations (239), (240), a system of only *four* equations of the type just mentioned; namely *two pairs*, selected from any two of the four groups, which have (for each group, and also for each pair) a common initial letter within the brackets; for instance, these two pairs of equations:

$$[efgh] = 0, \quad [egfh] = 0, \quad [fegh] = 0, \quad [fgeh] = 0; \tag{242}$$

which leave as many as *eight arbitrary constants* (for example these eight, eff, egg, ehh, fhh, ghh, hee, hff, hgg, from which all the rest can be determined) *in the resulting system of associative quines*. An outline of the investigation by which this important reduction is effected, may be presented in the following way.

[35.] The two first equations (242) connect the three last pairs of ratios (238), in such a manner that when any two of those three pairs are assumed, or known, the third can be determined. Hence, with the interpretation (197) of the symbol $(ghf)_0$, we easily find that those two equations (242) give,

$$(e)\ldots \qquad \left\{ \begin{aligned} gee\,.\,hgg - ghh\,.\,hee &= (ghf)_0\,.\,fee; \\ v_g\,.\,hee + gff\,.\,hgg &= (ghf)_0\,.\,fgg; \\ -(v_h\,.\,gee + hff\,.\,ghh) &= (ghf)_0\,.\,fhh; \end{aligned} \right\} \tag{243}$$

because we find that fee, fgg, ghh are proportional to the left-hand members of these last equations (243); and that the sum of the two first of those left-hand members is identically equal to the product $(gee + gff)(hee + hgg)$. For the same reason, the two first of these three equations (243) express really only one relation, namely that which is contained in the second equation (242), although they do so under different forms, both of which it is useful to know; and it is convenient to have ready also this other combination, obtained by adding the three equations (243) together,

$$(e)\ldots \qquad v_h\,.\,gff - v_g\,.\,hff = v_f\,.\,(ghf)_0; \tag{244}$$

which, like those equations (243) themselves, we shall consider as belonging to the group (e), because they are all derived from two of the three equations of that group, included under type VIII., which in the recent notation $[efgh]$ have e for their initial letter; and because the third equation of that group, included under the same type, namely

$$(e)\ldots \qquad [ehfg] = 0, \tag{245}$$

may be derived from them, by the elimination of the symbol $(ghf)_0$ between the first and third of the equations (243). In like manner the two last equations (242) include a third of the same type VIII., and belonging to the same group (*f*), namely

$$(f)\ldots \qquad [fheg] = 0; \tag{246}$$

because they conduct to the following system of expressions, which may be formed from (243) (244) by cyclical permutation of efg:

$$(f)\ldots \qquad \left\{ \begin{aligned} eff\,.\,hee - ehh\,.\,hff &= gff\,.\,(ehg)_0; \\ v_e\,.\,hff + egg\,.\,hee &= gee\,.\,(ehg)_0; \\ -(v_h\,.\,eff + hgg\,.\,ehh) &= ghh\,.\,(ehg)_0; \\ v_h\,.\,egg - v_e\,.\,hgg &= v_g\,.\,(ehg)_0. \end{aligned} \right\} \tag{247}$$

Multiplying then the equations (243) and (244) by $(ehg)_0$, and observing that the identity (220) gives

$$(ehg)_0\,.\,(ghf)_0 = (hee + hff)(ehf)_0, \tag{248}$$

we find, on substitution of the first for the second members of (247), that the results are divisible by $hee+hff$; and that thus the elimination of the third pair of ratios (238), between

(243), (244), (247), or between the four equations (242), conducts to expressions of the recent forms, namely,

$$(g)\ldots \qquad \left\{\begin{array}{r} hee\,.\,egg - ehh\,.\,hgg = fgg\,.\,(ehf)_0; \\ v_e\,.\,hgg + eff\,.\,hee = fee\,.\,(ehf)_0; \\ -(v_h\,.\,egg + hff\,.\,ehh) = fhh\,.\,(ehf)_0; \\ v_h\,.\,eff - v_e\,.\,hff = v_f\,.\,(ehf)_0. \end{array}\right\} \tag{249}$$

A similar analysis may be applied to effect the elimination of the fourth pair of ratios (238), with results entirely analogous. On the whole then it is found, that the four equations (242) express such connexions between the four pairs of ratios (238), as to satisfy not only the two remaining equations, (245) and (246), of their *own* groups, (e) and (f), but also the six other equations of the two *other* groups, (g) and (h), included under type VIII.; namely

$$\left.\begin{array}{llll} (g)\ldots & [gefh] = 0, & [gfeh] = 0, & [ghef] = 0; \\ (h)\ldots & [hefg] = 0, & [hfeg] = 0, & [hgef] = 0; \end{array}\right\} \tag{250}$$

for the first line is satisfied by the ratios (249), and the second by the analogous ratios, which are found in a similar way. Thus all the *twelve* equations of type VIII. are satisfied, if we satisfy only *four* suitably selected equations of that type; for example, the equations (242): which was what we proposed to demonstrate.

[36.] The *eighty* equations of association, assigned in the Third Section, between the *twenty-four* constants $l_1 \ldots u_3$, or (efg), (eff), have therefore, by the recent analysis, been ultimately *reduced to sixteen*; namely the *four* equations which thus remain from the last type VIII.; and the *twelve* others which were contained in the type III., established in that earlier Section: and which (as was lately remarked) leave still no fewer than EIGHT CONSTANTS ARBITRARY in this theory of ASSOCIATIVE QUINES. We may indeed vary in many ways, consistently with the same general theory, and by the assistance of the other recent types VI. and VII., the system of the sixteen equations of condition which are to be satisfied, and the choice of the eight constants which are to be regarded as still remaining arbitrary and undetermined: and it may not be useless, nor uninteresting, to make some remarks hereafter, upon the subject of such selections. But in the mean time it appears to be important to observe, that if some of the recent results, especially the formulæ (210), (228), be combined with some of those previously obtained, and more particularly with the equations (112), (121), of Section III., the following very simple expressions are found, for the ten remaining *constants of multplication*, the discussion of which had been reserved:

$$(f) = v_f^2; \quad (fg) = v_f\,.\,v_g; \tag{251}$$

or, with the notations abc, and with the usual cyclical permutation of the indices 1, 2, 3,

$$a_1 = v_1^2, \quad a_4 = v_4^2, \quad b_1 = v_2v_3, \quad c_1 = v_1v_4. \tag{252}$$

If then we write for abridgement,

$$\left.\begin{aligned} v &= v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4, \\ v' &= v_1x'_1 + v_2x'_2 + v_3x'_3 + v_4x'_4, \end{aligned}\right\} \tag{253}$$

the square of any quadrinomial vector ϖ, and the scalar of the product of two such vectors, will take these remarkably simple forms:

$$\varpi^2 = v^2; \quad \mathrm{S}\varpi\varpi' = v\,.\,v'; \tag{254}$$

this latter scalar thus decomposing itself into a product* of two linear functions of the constituents, namely those here denoted by v and v'. And because it is easy to prove, from what has been already shown, compare (244), that in the present theory the constants v_e are connected by relations of the form

$$-v_e\,.\,efe = v_e\,.\,fee = v_f\,.\,eff + v_g\,.\,efg + v_h\,.\,efh, \tag{255}$$

we find, by multiplying this equation by v_g, and attending to (251), the following theorems for those *general associative quines* which have been in this section considered:

$$\left.\begin{aligned} &0 = \mathrm{S}\iota_g\,\mathrm{V}\iota_e\iota_f = \mathrm{S}\iota_g\iota_e\iota_f; \\ &0 = \mathrm{S}\varpi\varpi'\varpi''; \quad 0 = (\mathrm{V}\varpi\varpi')^2; \end{aligned}\right\} \tag{256}$$

results which may be compared with some of those obtained in Section IV., for the two particular quine-systems, (A) and (B).

Observatory of Trinity College, Dublin,
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[To be continued.]

* A similar decomposition into linear factors takes place for the *quadrinomes* (A) of par. [13.], but at the expense of one of the six arbitrary constants $l_1\,l_2\,l_3\,m_1\,m_2\,m_3$, when we establish between those symbols the relation,

$$l_1m_1^2 + l_2m_2^2 + l_3m_3^2 = l_1l_2l_3 + 2m_1m_2m_3.$$

In general, I find that it is possible to satisfy all the conditions of association for *polynomes*, and at the same time to secure a decomposition of $\mathrm{S}\varpi\varpi'$ into linear factors, while yet preserving so many as $3n-4$ constants of multiplication arbitrary. (For quadrinomes, $3n-4 = 9-4 = 5$; for quines $3n-4 = 12-4 = 8$.)