ON A NEW SYSTEM OF TWO GENERAL EQUATIONS OF CURVATURE

By

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NOTE ON THE TEXT

This edition is based on the original text published posthomously in volume 9 of the *Proceedings of the Royal Irish Academy*.

The following obvious typographical errors have been corrected:—

before equation (a), a full stop (period) has been changed to a colon;

in equation (k), '$(Z - x)$' has been corrected to '$(Z - z)$';

in equation (o), '$C = eE'' - e'E$' has been corrected to '$C = eE' - e'E$';

equation (q) in the original text was given as

$$(eR^{-1} - eK^{-1})(e''R^{-1} - e''K^{-1}) = (e'R^{-1} - eK^{-1})^2;$$

in equation (r), '$= 0$' has been appended to the polynomial '$R^{-2} - FR^{-1} + G$';

in equation (w), the equality sign $=$ has been added.

David R. Wilkins
Dublin, March 2000

On a New System of Two General Equations of Curvature,

Including as easy consequences a new form of the Joint Differential Equation of the Two Lines of Curvature, with a new Proof of their General Rectangularity; and also a new Quadratic for the Joint Determination of the Two Radii of Curvature: all deduced by Gauss's Second Method, for discussing generally the Properties of a Surface; and the latter being verified by a Comparison of Expressions, for what is called by him the Measure of Curvature.

Sir William Rowan Hamilton

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1. Notwithstanding the great beauty and importance of the investigations of the illustrious Gauss, contained in his *Disquisitiones Generales circà Superficies Curvas*, a Memoir which was communicated to the *Royal Society of Göttingen* in October, 1827, and was printed in Tom. vi. of the *Commentationes Recentiores*, but of which a Latin reprint has been since very judiciously given, near the beginning of the Second Part (Deuxième Partie, Paris, 1850) of Liouville's *Edition* of* Monge, it still appears that there is room for some not useless Additions to the Theory of *Lines* and *Radii of Curvature*, for *any given Curved Surface*, when treated by what Gauss calls the *Second Method* of discussing the *General Properties of Surfaces*. In fact, the *Method* here alluded to, and which consists chiefly in treating the *three* co-ordinates of the *surface* as being so many *functions* of *two* independent variables, does not seem to have been used *at all* by Gauss, for the determination of the *Directions of the Lines of Curvature*; and as regards the *Radii of Curvature* of the *Normal Sections* which *touch* these *Lines* of Curvature, he appears to have employed the *Method, only for the Product*, and *not also* for the *Sum*, of the *Reciprocals*, of those *Two Radii.*

2. As regards the *notations*, let x, y, z be the rectangular co-ordinates of a point P upon a surface (S), considered as *three* functions of *two* independent variables, t and u; and let the 15 partial derivatives, or 15 partial differential coefficients, of x, y, z taken with respect to t and u, be given by the nine differential expressions:

$$(\text{a})\ldots\qquad \begin{cases} dx = x'\,dt + x_{\prime}\,du; & dx' = x''\,dt + x'_{\prime}\,du; & dx_{\prime} = x'_{\prime}\,dt + x_{\prime\prime}\,du;\\ dy = y'\,dt + y_{\prime}\,du; & dy' = y''\,dt + y'_{\prime}\,du; & dy_{\prime} = y'_{\prime}\,dt + y_{\prime\prime}\,du;\\ dz = z'\,dt + z_{\prime}\,du; & dz' = z''\,dt + z'_{\prime}\,du; & dz_{\prime} = z'_{\prime}\,dt + z_{\prime\prime}\,du. \end{cases}$$

* The foregoing dates, or references, are taken from a note to page 505 of that Edition.

3. Writing also, for abridgment,

(b) . . $$e = x'^2 + y'^2 + z'^2; \quad e' = x'x_\prime + y'y_\prime + z'z_\prime; \quad e'' = x_\prime^2 + y_\prime^2 + z_\prime^2$$

we shall have

(c) . . $$ee'' - e'^2 = K^2,$$

if

(d) . . $$K^2 = L^2 + M^2 + N^2,$$

and

(e) . . $$L = y'z_\prime - z'y_\prime; \quad M = z'x_\prime - x'z_\prime; \quad N = x'y_\prime - y'x_\prime;$$

so that

(f) . . $$Lx' + My' + Nz' = 0, \quad Lx_\prime + My_\prime + Nz_\prime = 0.$$

Hence $K^{-1}L$, $K^{-1}M$, $K^{-1}N$ are the *direction-cosines* of the *normal* to the surface (S) at P; and if x, y, z be the co-ordinates of any *other* point Q of the same normal,we shall have the equations

(g) . . $$K(X - x) = LR; \quad K(Y - y) = MR; \quad K(Z - z) = NR;$$

with

(h) . . $$R^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2;$$

where R denotes the normal line PQ, considered as changing sign in passing through zero.

4. The following, however, is for some purposes a more convenient *form* (comp. (f)) of the *Equations of the Normal*;

(i) . . $$(X - x)x' + (Y - y)y' + (Z - z)z' = 0;$$

(j) . . $$(X - x)x_\prime + (Y - y)y_\prime + (Z - z)z_\prime = 0.$$

Differentiating these, as if X, Y, Z were constant, that is, treating the point Q as an intersection of two consecutive normals, we obtain these two other equations,

(k) . . $$\begin{cases} (X - x)\,dx' + (Y - y)\,dy' + (Z - z)\,dz' = x'\,dx + y'\,dy + z'\,dz; \\ (X - x)\,dx_\prime + (Y - y)\,dy_\prime + (Z - z)\,dz_\prime = x_\prime\,dx + y_\prime\,dy + z_\prime dz. \end{cases}$$

If, then, we write, for abridgment,

(l) . . $$\begin{cases} v = du : dt; & E = Lx'' + My'' + Nz''; \\ E' = Lx_\prime' + My_\prime' + Nz_\prime'; & E'' = Lx_{\prime\prime} + My_{\prime\prime} + Nz_{\prime\prime}; \end{cases}$$

we shall have, by (a) (b) (g), the two important formulæ:

(m) . . $$R(E + E'v) = K(e + e'v); \quad R(E' + E''v) = K(e' + e''v);$$

which we propose to call the two general *Equations of Curvature*.

5. In fact, by elimination of R, these equations (m) conduct to a *quadratic in* v, of which the roots may be denoted by v_1 and v_2, which first presents itself under the form,

$$\text{(n)}\,.\,.\qquad (e + e'v)(E' + E''v) = (e' + e''v)(E + E'v),$$

but may easily be thus transformed,

$$\text{(o)}\,.\,.\qquad \begin{cases} Av^2 - Bv + C = 0, \text{ or } A\,du^2 - B\,dt\,du + C\,dt^2 = 0, \\ \text{with } A = e'E'' - e''E', \quad B = e''E - eE'', \quad C = eE' - e'E; \end{cases}$$

so that we have the following *general relation*,

$$\text{(p)}\,.\,.\qquad eA + e'B + e''C = 0,$$

(of which we shall shortly see the geometrical signification), between the *coefficients*, A, B, C, of the *joint differential equation* of the system of the two *Lines of Curvature* on the surface.

6. The root v_1 of the quadratic (o) determines the *direction* of what may be called the *First Line of Curvature*, through the point P of that surface; and the *First Radius of Curvature*, for the same point P, or the radius R_1 of curvature of the *normal section* of the surface which *touches* that *first line*, may be obtained from *either* of the two equations (m), as the value of R which corresponds in that equation to the value v_1 of v. And in like manner, the *Second Radius of Curvature* of the same surface at the same point has the value R_2, which answers to the value v_2 of v, in each of the same two *Equations of Curvature* (m). We see, then, that this *name* for those two equations is justified by observing that when the two independent variables t and u are given or known; and therefore also the seven functions of them, above denoted by e, e', e'', E, E', E'', and K. The equations (m) are satisfied by *two* (but *only two*) *systems of values*, v_1, R_1, and v_2, R_2, of (I.) the *differential quotient* v, or $\dfrac{du}{dt}$, which determines the *direction* of a *line of curvature* on the surface; and (II.) the symbol R, which determines (comp. No. 4) at once the *length* and the *direction*, of the *radius of curvature*, corresponding to that *line*.

7. Instead of eliminating R between the two equations (m), we may *begin* by eliminating v; a process which gives the following quadratic in R^{-1} (the curvature):—

$$\text{(q)}\,.\,.\qquad (eR^{-1} - EK^{-1})(e''R^{-1} - E''K^{-1}) = (e'R^{-1} - E'K^{-1})^2;$$

$$\text{or}\quad \text{(r)}\,.\,.\qquad R^{-2} - FR^{-1} + G = 0; \text{ where (because } ee'' - e'^2 = K^2),$$

$$\text{(s)}\,.\,.\qquad F = R_1^{-1} + R_2^{-1} = (eE'' - 2e'E' + e''E)K^{-3}, \text{ and}$$

$$\text{(t)}\,.\,.\qquad G = R_1^{-1}R_2^{-1} = (EE'' - E'^2)K^{-4}.$$

We ought, therefore, as a *First General Verification*, to find that this last expression, which may be thus written,

(u) . .
$$G = R_1^{-1} R_2^{-1} = \frac{EE'' - E'E'}{(L^2 + M^2 + N^2)^2},$$

agrees with that reprinted in page 521 of Liouville's Monge, for what Gauss calls the *Measure of Curvature* (k) of a *Surface*; namely,

(v) . .
$$k = \frac{DD'' - D'D'}{(AA + BB + CC)^2};$$

which accordingly it evidently does, because our symbols $L\ M\ N\ A\ B\ C$ represent the combinations which he denotes by $A\ B\ C\ D\ D'\ D''$.

8. As a *Second General Verification*, we may observe that if I be the *inclination* of any *linear element*, $du = v\,dt$, to the *element* $du = 0$, at the point P, then

(w) . .
$$\tan I = \frac{Kv}{e + e'v};$$

and therefore, that if H be the *angle* at which the *second crosses the first*, of *any two lines* represented *jointly* by such an equation as

(x) . .
$$Av^2 - Bv + C = 0, \text{ with } v_1 \text{ and } v_2 \text{ for roots, then}$$

(y) . .
$$\tan H = \tan(I_2 - I_1) = \frac{K(B^2 - 4AC)^{\frac{1}{2}}}{eA + e'B + e''C};$$

so that the *Condition of Rectangularity* ($\cos H = 0$), for any *two* such lines, may be thus written:

(z) . .
$$eA + e'B + e''C = 0.$$

But this *condition* (z) had already occurred in No. 5, as an equation (p) which is satisfied generally by the *Lines of Curvature*; we see therefore anew, by this analysis, that those *lines* on *any surface* are in general (as is indeed well known) *orthogonal* to each other.

9. Finally, as a *Third General Verification*, we may assume x and y *themselves* (instead of t and u), as the two independent variables of the problem, and then, if we use *Monge's Notation* of p, q, r, s, t, we shall easily recover all his leading results respecting *Curvatures of Surfaces*, but by transformations on which we cannot here delay.