ON CONTINUED FRACTIONS IN QUATERNIONS

By

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On Continued Fractions in Quaternions. By Sir WILLIAM ROWAN HAMILTON, *LL.D., M.R.I.A., F.R.A.S., &c., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland**.

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1. It is required to integrate the equation in differences,

$$u_{x+1}(u_x + a) = b,$$

where x is a variable whole number, but a, b, u are quaternions†. Let q_1 and q_2 be any two assumed quaternions; then

$$u_{x+1} + q_1 = b(a + u_x)^{-1} + q_1 = (b + q_1 a + q_1 u_x)(a + u_x)^{-1},$$

$$u_{x+1} + q_2 = b(a + u_x)^{-1} + q_2 = (b + q_2 a + q_2 u_x)(a + u_x)^{-1},$$

$$\frac{u_{x+1} + q_2}{u_{x+1} + q_1} = \frac{b + q_2 a + q_2 u_x}{b + q_1 a + q_1 u_x} = q_2 \frac{q_2^{-1} b + a + u_x}{q_1^{-1} b + a + u_x} q_1^{-1}.$$

If therefore we suppose that q_1, q_2 are roots of the quadratic equation

$$q^2 = qa + b,$$

which gives

$$q^{-1} b + a = q,$$

we shall have

$$\frac{u_{x+1} + q_2}{u_{x+1} + q_1} = q_2 \frac{u_x + q_2}{u_x + q_1} q_1^{-1},$$

and finally,

$$\frac{u_x + q_2}{u_x + q_1} = q_2^x \frac{u_0 + q_2}{u_0 + q_1} q_1^{-x}.$$

* Communicated by the Author.

† The advertisement, that the present writer's Lectures on Quaternions were to be ready in January last, was inserted, contrary to his wishes, through the over-zeal of an agent: but the work in question is now nearly all in type.

2. It was in a less simple way that I was led to the last written result. I assumed

$$u_x = \left(\frac{b}{a+}\right)^x c,$$

and treated this continued fraction as a particular case of the following,

$$u_x = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_x}{a_x+c} = \frac{N_x}{D_x} = \frac{N'_x(a_x+c)+N''_x b_x}{D'_x(a_x+c)+D''_x b_x}.$$

By changing c to $\dfrac{b_{x+1}}{a_{x+1}+c}$, I obtained the equations,

$$N'_{x+1} = N'_x a_x + N''_x b_x, \quad N''_{x+1} = N'_x,$$

$$D'_{x+1} = D'_x a_x + D''_x b_x, \quad D''_{x+1} = D'_x,$$

with the initial conditions

$$N'_1 = 0, \quad N''_1 = 1, \quad D'_1 = 1, \quad D''_1 = 0,$$

which allowed me to assume

$$N'_0 = 1, \quad D'_0 = 0.$$

Making next

$$a_x = a, \quad b_x = b,$$

there resulted

$$N_x = N'_x(a+c) + N'_{x-1}b, \quad D_x = D'_x(a+c) + D'_{x-1}b,$$
$$N'_{x+1} = N'_x a + N'_{x-1}b, \quad D'_{x+1} = D'_x a + D'_{x-1}b.$$

This led me to assume

$$N'_x = lq_1^x + mq_2^x, \quad D'_x = l'q_1^x + m'q_2^x,$$
$$q_1 = a + q_1^{-1}b, \quad q_2 = a + q_2^{-1}b,$$
$$l+m=1, \quad lq_1 + mq_2 = 0, \quad l'+m'=0, \quad l'q_1 + m'q_2 = 1;$$

whence there followed,

$$\begin{aligned} l &= (q_1^{-1} - q_2^{-1})^{-1}q_1^{-1} = -q_2(q_1-q_2)^{-1}, \\ m &= -(q_1^{-1} - q_2^{-1})^{-1}q_2^{-1} = +q_1(q_1-q_2)^{-1}, \\ l' &= -m' = (q_1-q_2)^{-1}. \end{aligned}$$

Hence

$$N_x = lq_1^x(q_1+c) + mq_2^x(q_2+c), \quad D_x = l'q_1^x(q_1+c) + m'q_2^x(q_2+c);$$

and making, for conciseness,

$$v_x = \frac{q_2^x(q_2+c)}{q_1^x(q_1+c)} = q_2^x \frac{q_2+c}{q_1+c} q_1^{-x},$$

it was found that

$$u_x = \left(\frac{b}{a+}\right)^x c = \frac{N_x}{D_x} = \frac{l+mv_x}{l'+m'v_x} = \frac{-q_2(q_1-q_2)^{-1} + q_1(q_1-q_2)^{-1}v_x}{(q_1-q_2)^{-1}(1-v_x)}.$$

Thus

$$u_x + q_1 = \frac{1}{(q_1-q_2)^{-1}(1-v_x)} = (1-v_x)^{-1}(q_1-q_2);$$
$$u_x + q_2 = v_x(1-v_x)^{-1}(q_1-q_2);$$

and finally,

$$\frac{u_x+q_2}{u_x+q_1} = v_x,$$

as before.

And because in no one stage of the foregoing process has the commutative principle of multiplication been employed, the results hold good for quaternions, and admit of interesting interpretations.

Observatory, March 20, 1852.

3. It results from what has been shown in the two former articles of this paper, that, whether in quaternions* or in ordinary algebra, the value of the continued fraction,

$$u_x = \left(\frac{b}{a+}\right)^x c, \tag{1}$$

may be found from the equation

$$\frac{u_x - u''}{u_x - u'} = v_x, \tag{2}$$

where

$$v_x = u''^x \frac{c-u''}{c-u'} u'^{-x}, \tag{3}$$

or from the expression

$$u_x = (1-v_x)^{-1}(u'' - v_x u'), \tag{4}$$

if u' and u'' be two unequal roots of the quadratic,

$$u^2 + ua = b. \tag{5}$$

* The writer has again to regret that unforeseen causes of delay have occurred to retard the publication of his Volume of Lectures on Quaternions, of which, however, he hopes that the value will be found to have been increased, by the additions which he has inserted.

If, then, $a\,b\,c\,u'\,u''$ be five real quaternions, of which the three last are unequal among themselves, and the two latter have unequal tensors,

$$Tu' > Tu'', \tag{6}$$

we shall have the following limiting values:

$$Tv_\infty = 0, \quad v_\infty = 0, \quad u_\infty = u''. \tag{7}$$

We may then enunciate this Theorem:—If the real quaternion c be *not a root* of the quadratic equation (5) in u, *the value of the continued fraction* (1) *will converge indefinitely towards that one of the real quaternion roots of that quadratic, which has the lesser tensor.* If the quaternion c or u_0 be a root of that equation, it is clear that the fraction will be constant.

September 21, 1852.

4. Those who have acquired some familiarity with the interpretation of the results obtained by the Calculus of Quaternions, will have now little difficulty in seeing that the following geometrical theorems* are obtained from the consideration of the continued fraction $\rho_x = \left(\dfrac{\beta}{\alpha +}\right)^x \rho_0$, where α, β, ρ_0, ρ_x are real vectors, β being perpendicular to the other three, and the condition $\alpha^4 + 4\beta^2 > 0$ being satisfied.

Let C and D be two given points, and P an assumed point. Join DP, and draw CQ perpendicular thereto, and towards a given hand, in the assumed plane CDP, so that the rectangle CQ . DP may be equal to a given area. From the derived point Q, as from a new assumed point, derive a new point R, by the same rule of construction. Again conceive that S is derived from R, and T from S, &c., by an indefinite repetition of the process. Then, *if the given area be less than half the square of the given line* CD, and if a semicircle (towards the proper hand) be constructed on that line as diameter, it will be possible to inscribe a parallel chord AB, such that the given area shall be represented by the product of the diameter CD, and the distance of this chord therefrom. We may also conceive that B is nearer than A to C, so that ABCD is an uncrossed trapezium inscribed in a circle, and the angle ABC is obtuse. This construction being clearly understood, it becomes obvious, Ist, that because the given area is equal to each of the two rectangles, CA . DA and CB . DB while the angles in the semicircle are right, then, whether we begin by assuming the position of the point P to be at the corner A, or at the corner B, of the trapezium, every one of the derived points, Q, R, S, T, &c., will *coincide* with the position so assumed for P, however far the process of derivation may be continued. But I also say, IInd, that if *any other point* in the plane, *except these two fixed points*, A, B, be assumed for P, then not only will its *successive derivatives*, Q, R, S, T, ... be all *distinct* from it, and from each other, but they will *tend* successively and indefinitely to *coincide with that one of the two fixed points* which has been above named B. I add, IIIrd, that if, from any point T, distinct from A and from B, we go *back*, by an *inverse* process of

* These theorems are taken from art. 665 of the author's (as yet unpublished) Lectures on Quaternions.

derivation, to the *next preceding point* S of the recently constructed series, and thence, by the same inverse law, to R, Q, P, &c., *this* process will produce an *indefinite tendency* to, and an *ultimate coincidence* with, the *other* of the two fixed points, namely, A. IVth. The common *law* of these two tendencies, direct and inverse, is contained in the formula

$$\frac{\text{QB . PA}}{\text{QA . PB}} = \frac{\text{CB}}{\text{CA}} = \text{constant};$$

which may be variously transformed, and in which the constant is independent of the position of P. Vth. the *alternate points*, P, R, T, &c., are all contained on one *common circular segment* APB; and the *other system of alternate points*, Q, S, &c., has for its locus *another circular segment*, AQB, on the *same fixed base*, AB. VIth. The *relation* between these *two segments* is expressed by this other formula, connecting the *angles* in them,

$$\text{APB} + \text{AQB} = \text{ACB};$$

the angles being here supposed to *change signs*, when their vertices cross the fixed line AB.

Observatory, December 30, 1852.

5. Let us now consider the continued fraction,

$$u_x = \left(\frac{\beta}{\alpha +}\right)^x u_0,$$

where u_0 and u_x are quaternions, and α, β are two rectangular vectors, connected by the relation,

$$\alpha^4 + 4\beta^2 = 0;$$

and, as a sufficient exemplification of the question, let it be supposed that α, β have the values

$$\alpha = i - k, \quad \beta = j.$$

It may easily be shown, by the rules of the present Calculus, that the expression,

$$u_1 = \frac{j}{i - k + u_0},$$

gives the relations,

$$(u_1 - k)^{-1} = k + i(u_0 - k)^{-1}k,$$
$$\text{S} \,.\, (i \pm k)\{(u_1 - k)^{-1} \pm (u_0 - k)^{-1}\} = \mp 1,$$
$$(u_2 - k)^{-1} - (u_0 - k)^{-1} = k - i;$$

and generally, by an indefinite repetition of the last process,

$$(u_{2n+x} - k)^{-1} - (u_x - k)^{-1} = n(k - i).$$

There is no difficulty in hence inferring that

$$\left(\frac{j}{i-k+}\right)^{\infty} u_0 = u_\infty = k,$$

whatever arbitrary *quaternion* (u_0) may be assumed as the original subject of the operation, which is thus indefinitely repeated. By assuming for this original operand a *vector* ρ_0 in the plane of *ik*, some geometrical* theorems arise, less general indeed in their import than the foregoing results respecting quaternions, yet perhaps not uninteresting, as belonging to a somewhat novel class, and coming fitly to be stated here, because they bear a sort of *limiting* relation to the results recently published in the Philosophical Magazine as part of the present paper.

6. Let C and D be the extremities, and E the summit of a semicircle. Assume any point P in the same plane, and draw CQ perpendicular to DP, so that the rectangle CQ . DP may be equal to the given square CE^2. Then it is clear, Ist, that if the *hand* (or direction of rotation) be duly attended to, in thus drawing CQ $\perp$ DP, the point Q will *coincide* with P, when the latter point P is so assumed as to coincide with the given summit E. But I say also, IInd, that if the point P be taken *anywhere else* in the same plane, and if, after deriving Q from it as above, we derive R from Q, &c., by repeating the same process, these *new* or derivative points Q, R, S, &c., will *tend*, successively and indefinitely, to coincide with the point E. I add, IIIrd, that if, from an arbitrarily assumed point S, we go *back*, on the same plan, to other points R, Q, P, &c., these new points, thus inversely derived, will *also* tend indefinitely to coincide with the same fixed summit E. IVth. The *alternate* points P, R, T, ... are all contained on one common circular circumference; and the *other* alternate system of derived points Q, S, U, ... are all contained on *another circular locus*. Vth. These two new circles *touch* each other and the given semicircle at the given summit E; and their *centres* are *harmonic conjugates* with respect to the completed circle CED. (The same harmonic conjugation of the centres of the two loci might easily have been derived for the more general case considered in an earlier part of this paper, from the last formula of art. 4; I have found that it holds good also in another equally general case, hereafter to be considered, when the given area of the rectangle under CQ and DP is *greater* than the square on the quadrantal chord CE, in which case there

* Note added during printing.—Since the foregoing communication was forwarded, I have perceived that the theorem VIII. of art. 6, which presented itself to me as an interpretation of the expression for $(u_1 - k)^{-1}$, when k = CE, i = DE, u_0 = CP, u_1 = CQ, may be very simply proved by means of the two similar triangles QEC, ECP$''$: and may be then employed to deduce *geometrically* all the other theorems of that article. (Each of these two triangles is similar to EP$_{\prime}$C, if P$_{\prime}$ be on EP$''$, and PP$_{\prime}$ $\parallel$ DC.) I see also that the lately published results of art. 4 may all be deduced geometrically, from the consideration of the two pairs of similar triangles, ADP, QCA, and BDP, QCB. These geometrical simplifications have only recently occurred to me; but it may have been perceived that, on the present occasion, *geometry* has been employed merely to illustrate and exemplify the signification and validity of certain new symbolical expressions, and methods of calculation; some account of which expressions and methods I hope to be permitted to continue. (March 15, 1853.)

can be *no convergence* to a limiting position, but there *may* be, under certain conditions, *circulation.*) VIth. If the chords PE, RE, TE, ... of the one circular locus, and also the chords QE, SE, UE, ... of the second locus, be prolonged through the point of contact E, so as to render the following rectangles equal to the given square or area,

$$\text{PEP}' = \text{RER}' = \text{TET}' = \cdots = \text{QEQ}' = \text{SES}' = \text{UEU}' = \cdots = \text{CE}^2,$$

then not only will the points $\text{P}'\,\text{R}'\,\text{T}', \ldots$ be ranged on *one straight line*, and the points $\text{Q}'\,\text{S}'\,\text{U}', \ldots$ on another, but also the *intervals* $\text{P}'\text{R}'$, $\text{R}'\text{T}', \ldots$ $\text{Q}'\text{S}'$, $\text{S}'\text{U}', \ldots$ will all be *equal* to each other and to the given diameter CD; and will have the same *direction* as that diameter. Thus the four points EPRT, or the four points EQSU, form what may be called an *harmonic group*, on the one or on the other circular locus: and if, as in some modern methods, the *directions* (and not merely the *lengths*) of lines be attended to, the chords EP, ER, ET, ... or EQ, ES, EU, ... may be said to form, each set within the circle to which they belong, a species of *harmonical progression.* VIIth. The orthogonal projection of $\text{P}'\text{Q}'$ or $\text{Q}'\text{R}'$, &c., on CD, is equal in length and direction to the *half* of that given diameter. VIIIth. If $\text{P}'\text{P}''$ be so drawn as to be perpendicularly bisected by the common tangent to the three circles, the line $\text{P}''\text{Q}'$ will be equal in length and direction to the given quadrantal *chord* CE.

Observatory, February 19, 1853.

7. The geometrical theorems stated in recent articles of this paper, although perhaps not inelegant, cannot pretend to be important: indeed a hint has been given (in a note) of a quite elementary way, in which they may be geometrically demonstrated. But I think that the analytical process, by which I was led to the formulæ of art. 5, whereof the geometrical statements of art. 6 are in part an interpretation, may deserve to be considered with attention, on account of the novelty of the *method* employed: and especially for the examples which it supplies of calculation with *biquaternions.*

8. After obtaining the result already published in this Magazine (compare the number for May, 1852), for a certain continued fraction in quaternions, namely that if*

$$u_x = \left(\frac{b}{a+}\right)^x c, \tag{1}$$

then†

$$\frac{u_x - u''}{u_x - u'} = u''^x \left(\frac{c - u''}{c - u'}\right) u'^{-x}, \tag{2}$$

u', u'' being roots of the quadratic equation

$$u^2 + ua = b; \tag{3}$$

* The numbering of the equations commences here anew.

† This result holds good also in ordinary algebra, and even in *arithmetic*: but in applying it to quaternions, the *order of the factors* must be attended to.

and after hence deducing the theorem (given in page 303 of the Philosophical Magazine for October, 1852 [article 3]), that for the case of *real quaternions*, and of *unequal tensors*,

$$u_\infty = u'', \text{ if } Tu'' < Tu'; \tag{4}$$

it was obvious, as a particular application, that by changing a, b, c, u', u'', u_x to α, β, ρ_0, ρ', ρ'', ρ_x, and by supposing these last to be six *real vectors*, among which β is perpendicular to all the rest, I might write

$$\frac{\rho_x - \rho''}{\rho_x - \rho'} = \rho''^x \frac{\rho_0 - \rho''}{\rho_0 - \rho'} \rho'^{-x}; \tag{5}$$

and ultimately,

$$\rho_\infty = \rho'', \text{ if } T\rho'' < T\rho', \tag{6}$$

the vectors ρ', ρ'' being roots of the quadratic,

$$\rho^2 + \rho\alpha = \beta. \tag{7}$$

This last equation gave, by taking separately the scalar and vector parts,

$$\rho^2 + \mathrm{S} \,.\, \rho\alpha = 0; \tag{8}$$

$$\mathrm{V} \,.\, \rho\alpha = \beta; \tag{9}$$

whereof the former (8) expressed that ρ terminated on a *spheric surface*, passing through the origin, and having the vector $-\alpha$ for its diameter; while the latter (9) expressed that ρ terminated on a *right line*, which was drawn through the extremity of the vector $\beta\alpha^{-1}$, in a direction parallel to that diameter. Thus (9) gave, by the rules of the present calculus,

$$\rho = \beta\alpha^{-1} + x\alpha, \quad \rho^2 = -\beta^2\alpha^{-2} + x^2\alpha^2, \quad \mathrm{S} \,.\, \rho\alpha = x\alpha^2; \tag{10}$$

and therefore, by (8), I had the ordinary quadratic equation,

$$x^2 + x = \beta^2\alpha^{-4}, \quad \text{or} \quad (2x+1)^2\alpha^4 = \alpha^4 + 4\beta^2 > 0, \tag{11}$$

as in art. 4 (Phil. Mag. for February, 1853): the two values of the vector ρ, which answer to the two values of the scalar coefficient x, being here supposed to be geometrically real and unequal; or the right line (9) being supposed to meet the spheric surface (8), in two distinct and real points, A, B. Hence by assuming

$$\rho' = \mathrm{CA}, \quad \rho'' = \mathrm{CB}, \quad \rho_0 = \mathrm{CP}, \quad \rho_1 = \mathrm{CQ}, \quad \alpha = \mathrm{DC}, \quad \beta = \mathrm{CA} \,.\, \mathrm{DA}, \tag{12}$$

I was conducted with the greatest ease to the theorems of the last-cited article.

9. But in the case of art. 5, namely when

$$\alpha^4 + 4\beta^2 = 0, \tag{13}$$

and when consequently

$$x = -\frac{1}{2}, \quad \rho'' = \rho' = \beta\alpha^{-1} - \frac{1}{2}\alpha, \tag{14}$$

the *equality* of the *two roots* of the quadratic (11) in x, or of the two real and vector roots of the equation (7) in ρ, appeared to reduce the formula (5) to an *identity*: and the simple process of the article last cited did not immediately occur to me. I therefore had recourse to certain *imaginary* or purely *symbolical solutions*, of that quadratic equation (7), or rather of the following, by which we may here conveniently replace it,

$$u^2 + u(i - k) = j; \tag{15}$$

the continued fraction to be studied being now,

$$u_x = \left(\frac{j}{i - k+}\right)^x u_0, \tag{16}$$

where $i\,j\,k$ are the usual symbols of this calculus, and u_0 may denote any arbitrarily assumed quaternion. By an application of a general process (described in art. 649 of my unpublished Lectures on Quaternions), I found that the quadratic (15) might be symbolically satisfied by the two following *imaginary quaternions*, or *biquaternion expressions*:

$$u' = -i - h(1 - j); \quad u'' = -i + h(1 - j); \tag{17}$$

where h is used as a temporary and abridged symbol for the *old and ordinary imaginary* of common algebra, denoted usually by $\sqrt{-1}$, and regarded as being always a *free* or *commutative factor* in any multiplication: so that

$$h^2 = -1, \quad hi = ih, \quad hj = jh, \quad hk = kh, \tag{18}$$

although $ji = -ij$ &c. In fact the first of these expressions (17) gives,

$$\begin{aligned} u'(u' + i - k) &= \{i + h(1 - j)\}\{k + h(1 - j)\} \\ &= ik + h\{i(1 - j) + (1 - j)k\} + h^2(1 - j)^2 \\ &= -j + h(i - k + k - i) + h^2(1 - 2j - 1) \\ &= -j + 0h + 2j = j; \end{aligned} \tag{19}$$

and the second expression (17) gives, in like manner,

$$u''(u'' + i - k) = j: \tag{20}$$

so that, without entering at present into any account of the process which enabled me to *find* the biquaternions (17), it has been now *proved, à posteriori*, by actual substitution, that those expressions do in fact *symbolically satisfy* the quadratic equation (15). And because they are *unequal roots* of that equation, as differing by the sign of h, I saw that they might be employed in the general formula (2), without being liable to the practical objection that lay against the employment of the two *real* by *equal* roots ρ', ρ'' of the equation (7).

10. Introducing therefore into the formula (2), or into the following, which is a transformation thereof,

$$\frac{u_x - u''}{u_x - u'} = \frac{u''^x(u_0 - u'')}{u'^x(u_0 - u')}, \tag{21}$$

the values (17), or these which are equivalent,

$$u' = -h(1 - j - hi), \quad u'' = h(1 - j + hi); \tag{22}$$

and observing that

$$(j \pm hi)^2 = j^2 \pm h(ji + ij) - i^2 = 0, \tag{23}$$

and that therefore*

$$(1 - j \mp hi)^x = 1 - xj \mp xhi; \tag{24}$$

we see that

$$\frac{u_x + i - h(1 - j)}{u_x + i + h(1 - j)} = (-1)^x \frac{\mathrm{A}_x - h\mathrm{B}_x}{\mathrm{A}_x + h\mathrm{B}_x}, \tag{25}$$

where A_x, B_x are *two real quaternions*, namely,

$$\left.\begin{aligned} \mathrm{A}_x &= (1 - xj)(u_0 + i) + xi(1 - j), \\ \mathrm{B}_x &= (1 - xj)(1 - j) - xi(u_0 + i); \end{aligned}\right\} \tag{26}$$

or, as we may also write them,

$$\left.\begin{aligned} \mathrm{A}_x &= (1 - xj)(u_0 - k) + i + k, \\ \mathrm{B}_x &= -xi(u_0 - k) + 1 - j. \end{aligned}\right\} \tag{27}$$

In this manner I found it possible to *eliminate the symbol* h, or to return from imaginary to real quaternions, and so perceived that

$$\frac{u_{2n} + i}{1 - j} = \frac{\mathrm{A}_{2n}}{\mathrm{B}_{2n}}; \quad \frac{u_{2n+1} + i}{1 - j} = -\frac{\mathrm{B}_{2n+1}}{\mathrm{A}_{2n+1}}. \tag{28}$$

Both of these two last formulæ agree in giving, as a limit,

$$\frac{u_\infty + i}{1 - j} = \frac{j}{i} = \frac{-i}{j} = +k; \tag{29}$$

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* More generally, with these rules of combination of the symbols $h\,i\,j\,k$, if f be any algebraic function, and f' the derived function,

$$f(1 + tj \pm thi) = f(1) + tf'(1)(j \pm hi);$$

because $(tj \pm thi)^2 = 0$, if t be any scalar coefficient.

and therefore (as in art. 5),

$$\left(\frac{j}{i-k+}\right)^{\infty} u_0 = u_\infty = -i + k(1-j) = k, \tag{30}$$

whatever *real* quaternion may be assumed for u_0. This last restriction becomes *here* necessary, from the generality of the analysis employed: because, for the very reason that u', u'' are admitted as being at least *symbolical* (or imaginary) *roots* of the equation (15), therefore we must here say that

$$\text{if } u_0 = u', \text{ then } u_x = u', \ \ u_\infty = u'; \tag{31}$$

and in like manner,

$$\text{if } u_0 = u'', \text{ then } u_x = u'', \ \ u_\infty = u''. \tag{32}$$

11. By the first of the two real quaternion equations (28), we have,

$$u_{2n} - k = -i - k + \mathrm{A}_{2n}\mathrm{B}_{2n}^{-1}(1-j); \tag{33}$$

but also, by the latter of the two values (27),

$$\mathrm{B}_{2n}^{-1}(1-j) = \{(1-j)^{-1}\mathrm{B}_{2n}\}^{-1} = \left(\frac{1+j}{2} \,.\, \mathrm{B}_{2n}\right)^{-1} = \{1 + n(k-i)(u_0 - k)\}^{-1}; \tag{34}$$

again, by the former of the same two values (27),

$$\mathrm{A}_{2n} - (k+i)\{1 + n(k-i)(u_0-k)\} = \mathrm{A}_{2n} - (k+i) + 2nj(u_0 - k) = u_0 - k; \tag{35}$$

therefore

$$u_{2n} - k = (u_0 - k)\{1 + n(k-i)(u_0-k)\}^{-1} = \{(u_0-k)^{-1} + n(k-i)\}^{-1}; \tag{36}$$

or more simply,

$$(u_{2n} - k)^{-1} - (u_0 - k)^{-1} = n(k-i). \tag{37}$$

It was in this way that I was originally led to the formula of art. 5, namely,

$$(u_{2n+x} - k)^{-1} - (u_x - k)^{-1} = n(k-i); \tag{38}$$

but having once come to *see* that this result held good, it was easy then to pass to a much more simple *proof*, such as that given in the last-cited article, which was entirely independent of the imaginary symbol here called h, and employed only *real quaternions.*

12. It may be regarded as still more remarkable, that the *same real results* are obtained, when we *combine a a real root with an imaginary one*, instead of combining *two* real roots, or *two* imaginary ones. Thus the quadratic* equation (15) has *one* root, namely k, which must be considered as *real* in this theory, whether by contrast to the symbol h (or to the old imaginary of algebra), or because in the geometrical interpretation it is constructed by a *real line*, namely by the *chord* CE drawn to the point of *contact* E of the spheric surface (8) with the right line (9), under the condition (13); α and β being then for convenience replaced, as in art. 5, by the more special symbols $i - k$ and j. Now if we adopt this *real root* k as the value of u', but retain the second of the two imaginary or biquaternion roots (17), as being still the expression for u'', the numerators of the formula (21) will remain unchanged, but the denominators will be altered; and instead of (25) we shall have this other formula,

$$\frac{u_x + i - h(1-j)}{u_x - k} = \frac{h^x(\mathrm{A}_x - h\mathrm{B}_x)}{k^x(u_0 - k)}, \tag{39}$$

with the significations (26) or (27) of A_x, B_x, and therefore with the relations (34) (35). Observing that

$$h^{2n} = (-1)^n = k^{2n}, \tag{40}$$

we find that the formula (39), by comparing separately the real and imaginary parts, in the two cases of x even and x odd, gives these four others, not involving the symbol h:

$$\frac{u_{2n} + i}{u_{2n} - k} = \frac{\mathrm{A}_{2n}}{u_0 - k}; \quad \frac{1 - j}{u_{2n} - k} = \frac{\mathrm{B}_{2n}}{u_0 - k}; \tag{41}$$

$$\frac{u_{2n+1} + i}{u_{2n+1} - k} = \frac{\mathrm{B}_{2n+1}}{k(u_0 - k)}; \quad \frac{1 - j}{u_{2n+1} - k} = \frac{-\mathrm{A}_{2n+1}}{k(u_0 - k)}; \tag{42}$$

of which the consistency with (28) is evident, and which are found to agree in all other respects with conclusions otherwise obtained. Thus all these different processes of calculation conduct

* An equation of the nth dimension in quaternions has generally n^4 *roots*, real or imaginary; because it may be generally resolved into a system of *four* ordinary and algebraical equations, which are *each* of the nth degree. However, it is shown in my Lectures that for the particular form (3), $u^2 + ua = b$ (or $q^2 = qa + b$), which occurs in the present investigation, *only six* (out of the sixteen) roots are *finite*; and that of these six, *two* are generally *real*, and *four imaginary*. In the particular case of the equation (15), k is by this theory a *quadruple root*, representing at once two real and two imaginary solutions, which have all become *equal* to each other, by the vanishing of certain radicals. Thus there remain in this case *only three distinct roots* of the quadratic (15), namely the one real root k, and the two imaginary roots (17): and what appears to me remarkable in the analysis of the present article 12, although otherwise exemplified in my Lectures, is the *mixture of these two classes* of solution of an equation in quaternions, a root of one kind being *combined* with a root of the other kind, so as to conduct to a correct determination of the value of a certain continued fraction, regarded as a real quaternion, which admits (as in art. 6) of being geometrically interpreted.

to consistent and interpretable results, although the method of the present article appears to depart even more than those of former ones from the ordinary analogies of algebra.

Observatory of T. C. D.
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[To be continued.]