ON THE NATURE AND PROPERTIES OF THE ACONIC FUNCTION OF SIX VECTORS

By

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(Proceedings of the Royal Irish Academy, 5 (1853), p. 177–186.)

Edited by David R. Wilkins

2000

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Communicated June 23, 1851.

[*Proceedings of the Royal Irish Academy*, vol. 5 (1853), p. 177–186.]

Sir William Rowan Hamilton entered into some explanatory details respecting the nature and properties of that ACONIC FUNCTION of six vectors, of which he had spoken in a recent communication with reference to a certain generalization or extension of Pascal's theorem, conducting to a relation between ten points on a surface of the second order.

In the Proceedings of the Royal Irish Academy for July 20, 1846, it was remarked by Sir W. Rowan Hamilton, that the theorem of Pascal might, in the calculus of quaternions, be expressed by the following general equation of cones of the second degree:

$$\mathrm{S} \,.\, \beta\beta'\beta'' = 0,$$

where

$$\begin{aligned}\beta &= \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha^{\mathrm{I}} \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{III}}\alpha^{\mathrm{IV}}),\\ \beta' &= \mathrm{V}(\mathrm{V} \,.\, \alpha^{\mathrm{I}}\alpha^{\mathrm{II}} \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}),\\ \beta'' &= \mathrm{V}(\mathrm{V} \,.\, \alpha^{\mathrm{II}}\alpha^{\mathrm{III}} \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{V}}\alpha);\end{aligned}$$

α, α^{I}, α^{II}, α^{III}, α^{IV}, α^{V} being any six homoconic vectors, and the letters S and V being the characteristics of the operations of taking respectively the scalar and vector parts of a quaternion. Now it is precisely *that function* of six vectors $\alpha \ldots \alpha^{\mathrm{V}}$, which was denoted in that communication of 1846, by $\mathrm{S} \,.\, \beta\beta'\beta''$, to which it has since appeared to Sir W. Rowan Hamilton convenient to give the name of the ACONIC (or *heteroconic*) *function* of those six vectors; because in the more general case, when they are *not* sides of any common cone of the second degree, this function no longer *vanishes*, but acquires some positive or negative value.

One of the most important properties of this *aconic* function is, that it *changes its sign without otherwise changing its value*, whenever *any two* of the *six* vectors on which it depends *change places* among themselves. Admitting this property, which there are many ways of easily proving by the general rules of quaternions, and observing that the following function of *four* vectors, α^{VI}, α^{VII}, α^{VIII}, α^{IX}, namely

$$\mathrm{S} \,.\, (\alpha^{\mathrm{VI}} - \alpha^{\mathrm{VII}})(\alpha^{\mathrm{VII}} - \alpha^{\mathrm{VIII}})(\alpha^{\mathrm{VIII}} - \alpha^{\mathrm{IX}}),$$

can be shewn to change sign in like manner, for any binary interchange among the vectors on which it depends, and to vanish when any two of them are equal; denoting also, for conciseness, the former function by 012345, the latter by 6789, and their product by

$$012345 \,.\, 6789;$$

Sir W. Rowan Hamilton proceeds to form, by binary transpositions of these figures, or of the vectors which they denote, from one factor of each product to the other, accompanied with a change of the algebraic sign prefixed to each such product as a term, for every such binary interchange, a system of 210 terms, namely

$$\begin{aligned}
&+ 012345 \,.\, 6789 - 012346 \,.\, 5789\\
&+ 012347 \,.\, 5689 - 012348 \,.\, 5679\\
&+ 012349 \,.\, 5678 - 012359 \,.\, 4678\\
&+ 012358 \,.\, 4679 - 012357 \,.\, 4689\\
&+ 012356 \,.\, 4789 - 012376 \,.\, 4589\\
&+ (\text{a hundred other products}) - (\text{a hundred other products});
\end{aligned}$$

these remaining terms being easily formed in succession, according to the lately mentioned law. And to the algebraic sum of all these 210 terms, of which each separately is a positive or negative number,—its positive or negative character depending of course not alone on the prefixed sign + or −, but also on the positive or negative characters of the *factors* of the product, which enters with that sign prefixed into the term,—Sir W. Rowan Hamilton proposes to give the name of the *heterodeuteric*, or (more shortly) the ADEUTERIC FUNCTION of the ten vectors $\alpha \ldots \alpha^{\mathrm{IX}}$, for a reason which will presently appear.

To make the formation of this function of *ten* vectors more completely clear, it may be observed, that the function of *four* vectors, which has been above denoted by the symbol 6789, is easily found to represent the sextupled volume of the *pyramid*, whose corners are the terminations of the four vectors (all drawn from one common origin); this form being regarded as positive or negative, according to the character (as right handed or left handed) of a certain *rotation*; which character or direction is *reversed* when *any two* of the four vectors, and therefore, also, their terminations, are made to change places with each other. On this account the lately mentioned function of four vectors may be called their PYRAMIDAL FUNCTION; and then the foregoing *rule* for the composition of the *adeuteric function* may be expressed in words as follows:—Starting with *any one set* of *four* vectors, form *their* pyramidal function, and multiply it by the aconic function of the remaining *six*, out of the proposed *ten* vectors, arranging the vectors of each set in any one selected *order*. Choose any vector of the four, and any other of the six, and interchange these *two* vectors, without altering the arrangement of the rest, so as to form a new group of four vectors, and another new group of six; and multiply the pyramidal function of the former group by the aconic function of the latter. Proceeding thus, we can gradually and successively form all the 210 possible groups or sets of four vectors, accompanied each with another set of six; and the four or the six vectors in each set will have an arrangement among themselves, determined by the foregoing process; so that the 210 pyramidal and the 210 aconic functions have each a determined value, *including* a known positive or negative sign or character. Each of the 210 *products*, thus obtained, is therefore itself also *determinate*, as being equal to some one positive or negative number, of which the *sign* as well as the absolute *value* can be definitely found, and may be considered as being *known, before* we introduce or employ any rule for *combining* or incorporating these various products among themselves, by any *additions* or *subtractions*. But if we *now* employ, for such incorporation, the rule that all those products which have been formed by any *even* number of binary interchanges, from the product first assumed, which we may still suppose to be

$$012345\,.\,6789,$$

are to be *algebraically added* thereto; while, on the contrary, all which are formed from that original product by any *odd* number of binary interchanges are to be *algebraically subtracted* from it: we shall complete (as was before more briefly stated) the determination of that *function of* TEN *vectors*, 0 to 9, which was lately called the ADEUTERIC.

Indeed, it may for a moment still appear that this function is in some degree *indeterminate*, because there may be many different ways of passing, by successive binary interchanges, from one given set of six, and a companion set of four vectors, to a second given set of six, with its own companion set of four. For example, we passed from the first to the tenth of the products already written, by a succession of *nine* binary interchanges, which may be indicated thus:

$$56, \quad 67, \quad 78, \quad 89, \quad 45, \quad 98, \quad 87, \quad 76, \quad 57.$$

But we might also have passed from the same first product,

$$+012345\,.\,6789$$

by the *two* binary interchanges 47, 56, to this other product and sign,

$$+012376\,.\,5489,$$

where the sign + is prefixed, on account of their being now an *even* number (two) of such changes. On the other hand, the *odd* number (nine), of binary interchanges above described, had given the term

$$-012376\,.\,4589.$$

But because, by the properties of the pyramidal function of four vectors above referred to, we have

$$+5489 = -4589,$$

the two terms thus obtained differ only in appearance from each other. And similar reductions will in every other case hold good, in virtue of the properties of the pyramidal and aconic functions, combined with a principle respecting transpositions of symbols (which probably is well known): namely, that if a set of n symbols (as here the ten figures from 0 to 9) be brought in any two different ways, by any two numbers l and m of binary interchanges, to any one other arrangement, the *difference* $m - l$ of these two *numbers* is *even*.

The VALUE (including sign) of the foregoing *adeuteric* function, of any ten determined vectors, is therefore itself completely *determined*, if we fix (as before) the *arrangement* of the ten vectors in the *first* of the 210 terms from which the others are to be derived: because the *value* of *each* separate term becomes then fixed, although the *forms* of these various terms may undergo considerable variations, by interchanges conducted as above. If then we choose *any two* of the ten vectors, suppose those numbered 4 and 7, we may *prepare* the expression of the *adeuteric* function as follows. We may first collect into one group the 70 terms in which these two vectors both enter into one common aconic function; and may call the sum of all these terms, Polynome I. We may next collect into a second group all those other terms, in number 28, for each of which the two selected vectors both enter into the composition of one common pyramidal function; and may call the sum of these 28 terms, Polynome II. And finally, we may arrange (after certain permitted transpositions) the remaining 112 terms into 56 pairs, such as

$$+012345\,.\,6789 - 012375\,.\,6489,$$

and

$$-012346\,.\,5789 + 012376\,.\,5489.$$

and may call the sum of these 56 pairs of terms, Polynome III; the rule of pairing being here, that the two selected vectors (in the present case 4 and 7) shall be interchanged in passing from any one term of the pair to the other, with a change of sign as before. But when the expression of the *adeuteric* has been thus prepared, it becomes clear that *each* of its *three* partial polynomes is changed to its own *negative*, when the two selected vectors are interchanged. In fact, *each term* of the first polynome changes sign, by this interchange, in virtue of the properties of the *aconic* function of six vectors. Again, *each* term of the *second* polynome in like manner changes sign, on account of the properties of the *pyramidal* function of four vectors. And finally, *each pair* of terms in the third polynome changes sign, from the manner in which that pair is composed. On the whole then we must infer, that the sum of these three polynomes, or the function above is called the ADEUTERIC, CHANGES SIGN *without otherwise changing value, when* ANY TWO *of the* TEN *vectors on which it depends are made to* CHANGE PLACES *with each other*: whence it is very easy to infer, that *this adeuteric function* VANISHES, *when any two of its ten vectors become* EQUAL.

Now the aconic function is of the *second* degree, with respect to each of the six vectors on which it depends; while the pyramidal function is easily shewn to be only of the *first* degree, with respect to each of the four other vectors which enter into its composition. Hence each of the 210 terms of the adeuteric rises no higher than the *second degree*; and *if we equate this adeuteric function to zero, we thereby oblige any one of the ten vectors to terminate on a given surface of the second order, if the other nine vectors be given.* But it has been seen, that the adeuteric vanishes, when *any two* of its ten vectors are made equal to each other; the surface which is thus the *locus* of the extremity of the *tenth* vector, must, therefore, pass *through the nine points* in which the *nine other* vectors respectively terminate. On this account the ten vectors, or their extremities, may be said to be, under this condition, HOMODEUTERIC, as belonging all to *one common surface of the second order.* And thus we at once justify, by contrast, the foregoing appellation of the ADEUTERIC function, and also see that to equate (as above) this adeuteric to zero, is to establish what may be called the EQUATION OF HOMODEUTERISM, as in fact it was so called in a recent communication to the Academy; while, as an abbreviation of the recent notation, we may now write that equation as follows:

$$\Sigma(\pm 012345\,.\,6789) = 0;$$

where the sum in the left-hand member represents the adeuteric function.

What has been shewn respecting the composition of this *adeuteric*, may naturally produce a wish to possess some *geometrical rule for constructing the aconic function* (012345), of any *six* given vectors; and the *quaternion expression* for that function enables us easily to assign such a *rule*. For this purpose, let A, B, C, D, E, F be the six points at which the six vectors lately numbered as 0, 1, 2, 3, 4, 5 terminate, being supposed to be all drawn from some assumed and common *origin* O; while G, H, I, K may denote the four other points, through which the surface of the second order passes, when the equation of homodeuterism is satisfied, and which are the terminations of the four other vectors above numbered as 6, 7, 8, 9. The aconic function, above denoted by 012345, of the six vectors OA, OB, OC, OD, OE, OF, which terminate generally at the six corners of a gauche hexagon ABCDEF, may now be concisely expressed by the symbol

$$\mathrm{O}\,.\,\mathrm{ABCDEF};$$

or even simply by ABCDEF, the reference to an origin being understood. To construct it, Sir W. Rowan Hamilton constructs first the six vectors

$$\mathrm{V}\,.\,\alpha\alpha^{\mathrm{I}},\quad \mathrm{V}\,.\,\alpha^{\mathrm{I}}\alpha^{\mathrm{II}},\quad \mathrm{V}\,.\,\alpha^{\mathrm{II}}\alpha^{\mathrm{III}},\quad \mathrm{V}\,.\,\alpha^{\mathrm{III}}\alpha^{\mathrm{IV}},\quad \mathrm{V}\,.\,\alpha^{\mathrm{IV}}\alpha^{\mathrm{V}},\quad \mathrm{V}\,.\,\alpha^{\mathrm{V}}\alpha,$$

and then the three other vectors β, β', β'', which depend on these, in order to form thence that scalar $\mathrm{S}\,.\,\beta\beta'\beta''$, which, by what was stated near the commencement of the present Abstract, is the *aconic* function required. It will be seen that all the steps of the following construction of that function are in this way obvious consequences from the quaternion expression above given. The construction itself was communicated to a few scientific friends of his about the end of August and beginning of September, 1849, and has since been publicly stated at the Edinburgh Meeting of the British Association in 1850, although it has not hitherto been printed.

Regarding the given and gauche hexagon, ABCDEF, as a sort of *base* of a *hexahedral angle*, of which the *vertex* is the assumed point O, Sir W. Rowan Hamilton *represents* the *doubled areas* of the six plane and triangular faces of this angle, namely,

AOB, BOC, COD, DOE, EOF, FOA,

by *six right lines* from the vertex,

OL, OM, ON, OL′, OM′, ON′,

which are respectively *normals* to the six faces, and are distinguished from their own opposites by a simple and uniform rule of *rotation*: for example, the line OL contains as many linear units as the doubled area of the triangle AOB (to the plane of which it is perpendicular) contains units of area; and the notation round OL from OA to OB is right-handed. The doubled areas of the three new triangles,

LOL′, MOM′, NON′,

are next to be *represented*, on the same general plan, by *three new lines* from the vertex,

OL″, OM″, ON″;

which three lines will thus be the intersection of the three pairs of opposite faces of the hexahedral angle, and consequently will, by Pascal's theorem, be situated in one common plane, if the given hexagon ABCDEF can be inscribed in a cone of the second degree, with the point O for its vertex. But in the more *general* case, when the given hexagon *cannot* be so inscribed, in any such cone with that assumed point for vertex, we can construct a parallelepipedon with the three last lines, OL'', OM'', ON'', for three adjacent edges: and the *volume of this solid* is the geometrical representation which Sir W. Rowan Hamilton's method assigns for what he calls (as above) the *aconic function* of the six given vectors, or of the six given points A, B, C, D, E, F, in which those vectors terminate, or of the (generally gauche) hexagon of which those points are corners. And with respect to the *sign* of this function, it is to be regarded as being positive or negative, according as the rotation round ON'', from OM'' towards OL'', is to the right hand or to the left.

Such then is the construction of the *aconic* function, 012345, or ABCDEF; and it is still more easy to construct the *pyramidal* function 6789, which may also be denoted by the symbol GHIK; since the absolute value of this function is constructed (as above remarked) by the *sextuple volume of the pyramid*, which has the four points G, H, I, K for corners, or by the volume of the *parallelepipedon* which has GH, GI, GK, for edges; while the quaternion expression assigned near the commencement of this Abstract, admits of being thus written,

$$\mathrm{S}\,.\,(\alpha^{\mathrm{IX}} - \alpha^{\mathrm{VI}})(\alpha^{\mathrm{VIII}} - \alpha^{\mathrm{VI}})(\alpha^{\mathrm{VII}} - \alpha^{\mathrm{VI}}),$$

and conducts to the regarding this volume, or the function 6789, or GHIK, as being positive when the rotation round GH from GI towards GK is right-handed, but negative in the contrary case. And the aconic and pyramidal functions having thus been *separately* constructed, they have only to be *combined* with each other, according to the law already stated, in order to assign a *geometrical signification* to each term of the *adeuteric function*, namely, the sum,

$$\Sigma(\pm\mathrm{ABCDEF}\,.\,\mathrm{GHIK});$$

and also to the *equation of homodeuterism*, which may now be written thus (as in a recent communication to the Academy),

$$\Sigma(\pm\mathrm{ABCDEF}\,.\,\mathrm{GHIK}) = 0,$$

and which expresses that the *ten points*, A, B, …, K, are situated *upon one common surface of the second order*. And if we place the arbitrary origin O at one of the ten points, the *number of terms* in the adeuteric function, or in the equation of homodeuterism, is easily seen to *reduce* itself, then, from 210 to 84.

If the thirty *co-ordinates* of the ten points were substituted in the function above called the *adeuteric*, the resulting expression could doubtless only differ by some numerical coefficient from that *determinant* which might otherwise be found, as the result of the elimination of the nine coefficients A, B, C, D, E, F, G, H, I, between the equations,

$$\begin{aligned}
&Ax_0^2 + By_0^2 + Cz_0^2 + Dy_0z_0 + Ez_0x_0 + Fx_0y_0 + Gx_0 + Hy_0 + Iz_0 + 1 = 0,\\
&\cdots\cdots\cdots\\
&Ax_9^2 + By_9^2 + Cz_9^2 + Dy_9z_9 + Ez_9x_9 + Fx_9y_9 + Gx_9 + Hy_9 + Iz_9 + 1 = 0.
\end{aligned}$$

And Sir W. Rowan Hamilton has much pleasure in referring to a paper by Mr. Cayley, printed near the commencement of the Fourth Volume of the Cambridge Mathematical Journal, on Pascal's Theorem considered in connexion with determinants, which paper had not been noticed by the present writer till his attention was called to it by a friend to whom he had communicated the above-stated construction. But while gladly acknowledging the great mathematical learning and originality exhibited in that and every paper by Mr. Cayley, Sir W. Rowan Hamilton thinks it right to state, that he was led to his own results, respecting the *relation* (above assigned) between *ten points on the surface of the second order*, not by any system of *co-ordinates*, but by the considerations of *vectors*, and by seeking to extend to *ellipsoids* the results respecting *cones*, which he had submitted to the Academy in July, 1846, and had also published in the Philosophical Magazine for the following month, as derived from the Calculus of Quaternions.