Stability of AdS and Phase Transitions

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Abstract

This is a transcript of the lecture given by Professor S. W. Hawking for the Strings '99 conference. Professor Hawking is the Lucasian Professor at the University of Cambridge, England.

Black holes are often thought of as completely dead classically. That is, they can absorb radiation and energy, but not give any radiation or energy out. However, way back in 1970, Penrose devised a classical process to extract energy from a rotating black hole. This is possible because the horizon is rotating faster than light, with respect to the stationary frame at infinity. In other words, the killing vector, that is time like at infinity, is space like on the horizon. The energy flux vector, defined relative to this killing vector, can therefore also be space like, even for matter obeying the dominant energy condition. Thus, the energy flow across the future horizon of a rotating black hole can be negative. What is happening is that one is extracting the rotational energy of the black hole, and slowing its spin.

A nice way of extracting rotational energy, is to scatter an incoming wave of frequency ω, and axial quantum number, m.

$$dE = \frac{\kappa}{8\pi} dA = \Omega dJ + \Psi dQ \tag{1}$$

First Law of Black Hole Mechanics

$$dA < 0 \tag{2}$$

Second Law of Black Hole Mechanics

$$\begin{gathered} \text{Superradiance } dE < 0 \text{ if:} \\ \text{Rotating black holes: } \omega < m\Omega \\ \text{Charged black holes: } \omega < q\Psi \end{gathered} \tag{3}$$

By the first law of black hole mechanics, these determine the change in area of the horizon. For classical scattering, with fields whose energy momentum obeys the dominant energy condition, the area increase will be positive. For $\omega < m\Omega$, the angular velocity of the black hole, the change of area being positive, implies that the change of mass of the black hole must be negative. In other words, instead of part of the incident wave being absorbed by the black hole, and part reflected back, the

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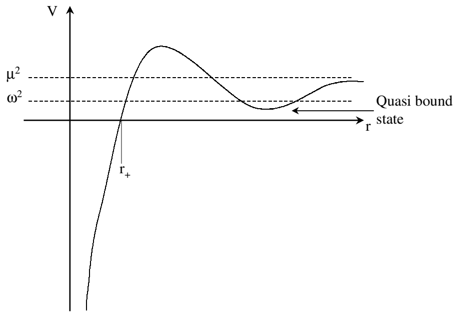
reflected wave would actually be stronger than the original incoming wave. Such amplified scattering is called superradiance.

It occurs for integer spin fields, but not for half integer. At one level, this is because the classical energy momentum tensors of spin half fields do not satisfy the dominant energy condition. At another level, it can be seen as a consequence of the exclusion principle, which prevents more than one particle being in each out going mode.

In quantum language, superradiance is stimulated emission by the black hole. One would therefore expect spontaneous quantum emission in the superradiant modes. Indeed, it was investigating this that led me to discover quantum thermal radiation, even from non-rotating holes. But in a purely classical theory, a black hole can remain rotating forever, as far as mass-less fields like gravity are concerned. It is a different story, however, with massive fields. A mass term, μ, for a scalar field, will prevent waves of frequency, $\omega < \mu$, from escaping to infinity.



Effective Potential

(4)

Instead, they will be reflected by a potential barrier at R, back into the hole. If they satisfy the condition for superradiance, the waves will be amplified by scattering off the hole. Their amplitude will increase exponentially, and the black hole will lose its rotation by a classical process. One can understand this instability in the following way: In the WKB limit, a mode, with frequency less than rest mass, corresponds to a particle orbiting the black hole. If the orbital angular velocity is less than the angular velocity of the black hole, angular momentum and energy will flow from the black hole, to the particle. Because orbits in asymptotically flat space can have arbitrarily long periods, rotating flat space black holes will always lose angular momentum to massive fields, although in practice the rate is very low. But it is important for ADS black holes, because of the ADS, CFT duality.

Charged Black Holes

$$\begin{gathered}\text{Superradiance is } \omega < q\Psi \\ \text{Bogomolny} \Rightarrow \Psi \leq 1 \\ \text{BPS} \Rightarrow \nu \geq q\end{gathered} \tag{5}$$

Charged fields scattering off an electrically charged black hole, have similar superradiant amplification. The condition is now $\omega < q\Phi$, where q is the charge of the field, and Φ is the electro static potential of the horizon. There is, however, an important difference from the rotating case. Black holes with regular horizons obey a Bogolomony bound, that their charges are not greater than their masses, with equality only in the BPS extreme state. This bound implies that the electro static repulsion between charged black holes can never be greater than their mutual gravitational attraction. The best one can do is to have a static equilibrium of extreme charged black holes, a situation Jim Hartle and I investigated, way back in 1971.

The Bogomolny bound on the charge of a black hole, implies that the potential difference, Φ, between the horizon, and infinity in asymptotically flat space, can never be greater than one. In a Kaluza Klein or super symmetric theory, the charges of fields will generally obey the same BPS bound as black holes, with respect to their rest mass. This means the inequality for superradiance, can never be satisfied. One can think of this as a consequence of the fact that the BPS bound implies that gauge repulsions never dominate over gravitational attraction. It means that charged black holes, in super symmetric and Kaluza Klein theories, are classically stable. The black hole can't lose charge by sending out a charged particle, while maintaining the area of the horizon, as it must in a classical process. Of course, the quantum process of thermal emission allows the area to decrease, but that is another story.

So far, I have been talking about superradiance for black holes in asymptotically flat space. But it is interesting to study it for holes in Anti de Sitter space, because their behaviour will be related to conformal field theory, on the boundary of the Anti de Sitter space. For simplicity, I will consider AdS 5 cross S5, although AdS 4 cross S7 is very similar.

For thermodynamic purposes, one is interested in asymptotically AdS spaces, that are periodic in the Euclidean regime, under a Euclidean time translation in the AdS part, plus a rotation. The rotation can either be on the five sphere, or on the three sphere at infinity in Anti de Sitter. The rotation on five sphere has three independent angular velocities, corresponding to the Cartan sub algebra of SO6. On dimensional reduction to five dimensional gauged supergravity, this Cartan sub algebra corresponds to three U1 gauge fields. Black hole solutions with these charges have been found by various authors.

S. W. Hawking, C. J. Hunter, M. M. Taylor-Robinson: hep-th/9811056
M. Cvetic, S. S. Gubser: hep-th/9902195 (6)
A. Chamblin, R. Emparan, C. V. Johnson, R. C. Myers: hep-th/9902170

I shall concentrate on the simplest case, in which the three charges are equal. In this case, the solutions are solutions of Einstein Maxwell theory, with a negative cosmological constant. I shall refer to these as Reissner Nordstrom Anti de Sitter.

On the other hand, when the rotation is in the three sphere in Anti de Sitter space, there are no Kaluza Klein gauge fields, and the five dimensional reduced theory has Kerr Anti de Sitter black hole solutions. The most general solution has two independent angular velocities.

Given these rotating and charged black holes, one can investigate whether they exhibit superradiance, and instability, and what that implies for the conformal field theories on their boundaries. The short answer is that Kerr AdS holes are classically unstable to lose angular momentum, if they are rotating at more than a certain rate. On the other hand, Reissner Nordstrom AdS holes don't lose charge classically, no matter how rapid the rotation of the S5. The basic reason is the different action of rotations on the AdS, and S5, parts of AdS 5 cross S5. The boundary of AdS 5 at infinity can be regarded as a four dimensional Einstein universe.

The Killing vector, partial T, defines the static frame in the Einstein universe, but one can define rotating frames by adding multiples of the Φ and Ψ Killing vectors on the S3. If the absolute magnitudes of the two angular velocities, Ω_1 and Ω_2 are less than one, the Killing vector K, will be time like everywhere in the Einstein universe.

$$
\begin{gathered}
ds^2 = -(1+\tfrac{y^2}{l^2})dT^2 + \frac{dy^2}{1+\frac{y^2}{l^2}} \\
+y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2
\end{gathered}
\tag{7}
$$

Co-rotating Killing vector field

$$
K = \frac{\partial}{\partial T} + \Omega_1 \frac{\partial}{\partial \Psi} + \Omega_2 \frac{\partial}{\partial \Phi}
$$

It can also be extended to a Killing vector of Anti de Sitter space, that is time like everywhere. In other words, one can have a rotating frame in Anti de Sitter space, that is moving at less than the speed of light, all the way out to infinity. This is in contrast to the situation in flat space, where any rigidly rotating frame will, necessarily, move faster than light far from the axis of rotation.

The horizon of the five dimensional Kerr AdS black hole, with two angular velocities, will be rotating with angular velocities Ω_1 and Ω_2, with respect to a frame that is itself rotating with respect to the static frame, in the Einstein universe at infinity.

$$
\begin{aligned}
ds^2 = &-\frac{\Delta}{\rho^2}(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi)^2 \\
&+\frac{\Delta_\theta \cos^2\theta}{\rho^2}(adt - \frac{(r^2+a^2)}{\Xi_a}d\phi)^2 \\
&+\frac{\Delta_\theta \cos^2\theta}{\rho^2}(bdt - \frac{(r^2+b^2)}{\Xi_b}d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\
&+\frac{(1+r^2/l^2)}{r^2\rho^2}(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi)^2
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2/l^2) - 2M \\
\Delta_\theta &= (1 - (a^2/l^2)\cos^2\theta - (b^2/l^2)\sin^2\theta) \\
\rho^2 &= (r^2 + a^2\cos^2\theta + b^2\sin^2\theta) \\
\Xi_a &= (1-a^2/l^2) \qquad \Xi_b = (1-b^2/l^2)
\end{aligned}
\qquad
\boxed{
\begin{aligned}
\Omega_1' &= \Xi_a \frac{a}{r_+^2+a^2} & \Omega_1 &= \frac{a(1+r_+^2/l^2)}{r_+^2+a^2} \\
\Omega_2' &= \Xi_b \frac{b}{r_+^2+b^2} & \Omega_2 &= \frac{b(1+r_+^2/l^2)}{r_+^2+b^2}
\end{aligned}
}
$$

***AdS*$_5$**

(8)

This means there will be an ergo-sphere outside the horizon, where the static Killing vector is space like. Particles in this region can have negative energy, and can fall into the black hole, reducing its mass and angular momentum. Corresponding to this, one would expect to be able to satisfy the superradiance condition, $\omega < m\Omega$.

In Anti de Sitter space, the normalizable modes of quantum fields, have a discrete spectrum of frequencies, with ω, which are strictly greater than m, the axial quantum number. Such modes could satisfy the superradiance condition, if only Ω, the angular velocity of the black hole, was greater than one. In the units I'm using, an angular velocity greater than one means: the frame that co-rotates with the black hole is rotating faster than light at infinity. One might therefore think that Kerr AdS black holes are classically stable, if and only if their angular velocities are less than one.

However, one has to be careful about this conclusion. The presence of a black hole, in Anti de Sitter space, changes the spectrum of any field on the background, and makes it continuous, with no lower bound. Perturbations to fields in the graviton super multiplet will propagate out to infinity, in anti de Sitter space, and will be reflected back by the boundary conditions there. Modes of massive fields will get reflected back before they reach infinity.

Thus one might expect that all Kerr ADS black holes, would be classically unstable, and would lose angular momentum to a cloud of particles orbiting the hole. However, holes with an angular velocity less than one will have a co-rotating Killing vector that is time like everywhere outside the horizon. This enables one to prove the holes are classically stable.

One can contrast this result with the stability analysis of Cvetic and Gubser. Their criterion is that the entropy should be sub-additive, with respect to the extensive variables, energy and angular momentum. This would prevent the black hole from dividing into two black holes, with the same total energy and angular momentum. But even if the sub-additivity condition is not satisfied, it does not mean that a black hole will necessarily be unstable.

Black holes in Anti de Sitter space, have to be moving relative to each other. So, the total energy and angular momentum will not be the sum of the individual values. Large Kerr AdS black holes, with angular velocities slightly less than one, will not satisfy the sub-additivity condition. Nevertheless, they are stable, because they have an everywhere time like Killing vector. Similarly, as I shall describe, sub-additivity is not a good guide to the stability of charged black holes in ADS.

There are Kerr ADS black holes, with angular velocities greater than one. These solutions will be classically unstable. However, all Kerr AdS with negative action relative to Anti de Sitter, have angular velocities that are less than one. Thus they are classically stable. This is important because under the conjectured AdS, CFT duality, the action of the Kerr AdS solution gives the partition function for Super Yang Mills theory, in a rotating Einstein universe. On the supergravity side, the action of the hole is straightforward to calculate, if a bit messy. It diverges when one or both angular velocities approach one. But on the conformal field theory side, we don't know how to calculate the partition function of a strongly coupled Yang Mills theory.

The best we can do, is calculate the partition function for a free theory, with zero coupling. As is well known, in the non-rotating case, the log of the free field partition function at high temperature, is four-thirds the action of Schwarzschild AdS. The conventional wisdom, is that strong coupling effects reduce the partition function by three-quarters. So far, this has to be a matter of faith, because we don't have a reliable method of calculating the partition function for a strongly coupled theory. However, the Kerr AdS solutions provide another limit in which to compare actions and partition functions, that of rotation at the speed of light in the Einstein universe at infinity. In the high temperature limit, the log of the partition function of the free theory, is again four-thirds the action of the black hole.

$$\begin{gathered}\text{For } \Omega_1 \to \pm 1, \text{ with } |\Omega_2| < 1,\\ logZ \simeq \frac{2N^2}{\beta(1-{\Omega_1}^2)}\sum_{n=1}^{\infty}\frac{(\cosh(\beta n\Omega_2/2)+(-)^{n+1})^2}{n^2\sinh(\beta n(1-\Omega_2)/2)\sinh(\beta n(1+\Omega_2)/2)}\\ \text{If now } \Omega_2 \to \pm 1, \text{ then}\\ logZ \simeq \frac{8N^2}{\beta^2(1-{\Omega_1}^2)(1-{\Omega_2}^2)}\sum_{n=1}^{\infty}\frac{(\cosh(\beta n/2)+(-)^{n+1})^2}{n^3\sinh\beta n}\\ \text{in units such that } S^3 \text{ has unit radius}\end{gathered} \tag{9}$$

But, if one takes the angular velocity tending to one limit, at finite temperature, the mode that co-rotates with the hole, dominates the partition fuction. This makes the conformal field theory behave like it is in 1+1 or 2+1 dimensions, and the relation between the free field partition function, and the action of the black hole, is not just a factor four-thirds.

The discussion of the Reissner Nordstrom AdS black holes is along similar lines, but there's an important difference: the five sphere tends to a finite size at infinity, but the proper time interval between surfaces of constant cordinate time in AdS becomes infinite. Thus, no matter how rapidly the five spheres rotate, they will be rotating slower than light at infinity. On the other hand, the proper time interval will have a minimum value of one, at the origin of Anti de Sitter space. Thus if the angular velocity of the five sphere is greater than one, it will be rotating faster than light, in a region near the origin.

This would signal an instability of the Kaluza Klein theory, with the creation of charged particles near the origin. However, for angular velocities greater than one, the Reissner Nordstrom AdS solutions have lower action than Anti de Sitter, and so will be favoured. The Reissner Nordstrom AdS solutions can have arbitrarily large angular velocity of the five sphere, and arbitrarily large electrostatic potential, Φ. Thus, one might expect that superradiance would lead to an instability, in which the hole loses charge to a cloud of particles. The Sub-Additivity Condition would also lead to the conclusion that: some Reissner Nordstrom AdS solutions are unstable. However, in all these solutions, the five sphere is rotating everywhere slower than light. This means that there is an everywhere time like killing vector, which rules out superradiance, like in Kerr AdS. So, one can conclude that Reissner Nordstrom AdS is stable for any angular velocity of the five sphere at infinity.

By the AdS, CFT duality conjecture, the action of the Reissner Nordstrom AdS solutions will give the partition function for Super Yang Mills theory, with a chemical potential lying in a U(1) sub group of the global SO6 R symmetry. The chemical potential would cause Bose Einstein condensation, in a free Yang Mills theory. It is significant that this would correspond to the angular velocity, $\Omega = 1$, at which the co-rotating Killing vector becomes null at the origin of Anti de Sitter space. For $\Omega < 1$, the chemical potential would be so large, that the partition function of the free Yang Mills theory, would diverge. Presumably, this would indicate a phase transition in the theory at low coupling, and large chemical potential. However, the theory on the Einstein universe, has high coupling, which means that we do not know how to calculate the partition function directly. Nevertheless, if we believe the AdS, CFT duality, we can determine it from the actions of Reissner Nordstrom Anti de Sitter solutions. These are finite for all values of the potential, or rate of rotation. Thus, one could conclude that strongly coupled Yang Mills theory would not have the phase transition at high chemical potential, that the weakly coupled theory showed.

To sum up; Anti de Sitter space cross a sphere can have two kinds of rotation: Rotation in the anti de Sitter part, and rotation of the sphere. The first leads to Kerr AdS black holes, and the second, to charged AdS holes. There is an important difference between the two kinds of rotation. Rotation in Anti de Sitter can be faster than light at infinity, but rotation of the sphere, can never be faster than light at infinity. This means that the partition function of the boundary CFT will diverge in the first case, but not in the second. This is born out by the fact that the action of the Kerr AdS, and charged AdS black hole solutions, have that behaviour. This tells us how super Yang Mills theory behaves, under rotation, and chemical potential. But that is only if we believe the ADS, CFT duality. We have no independent confirmation, so far. Maybe we will have at Strings 2000.

That is all. Thank you for listening.

Acknowledgments

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